This practical lab covers fundamental work with random signals, in particular ensemble estimation of parameters on the set of a number of realizations of a random process. We will be working with random processes exclusively in discrete time domain. The theory for this lab can be found in the lecture on random processes. (see http://www.fit.vutbr.cz/study/courses/ISS/public/en/nah_e.pdf).

1 Generation of a Random Process

The random process will be the Gaussian noise with zero mean value and standard deviation equal to 5, which will be passed through the filter with the transmission function defined as:

\[ H(z) = \frac{1}{1 - 1.1314z^{-1} + 0.6400z^{-2}} \]

We will generate \( \Omega = 10000 \) realizations of the given random process of the length of \( N = 200 \) samples. The generated processes will be stored in the matrix \( ksi \).

```plaintext
Om = 10000; N = 200; % Om je zkratka Omega
b=[1]; a = [1.0000 -1.1314 0.6400];
nn = 0:N-1;
ksi = zeros(Om,N);
for ii=1:Om,
x = randn(1,N) * 5 + 0;
y = filter(b,a,x);
ksi(ii,:) = y;
% plot(nn,x,nn,y); pause
end
```

First, uncomment the line `plot(nn,x,nn,y); pause` and look at a few realizations. Consequently, comment the line and let the cycle run.

Q: What is the difference between the random signal generated by `randn` alone and the signal passed through the filter?

2 Estimation of the Distribution Function

The estimations will be done for the fixed time \( n \), for instance:

```plaintext
n = 50;
```

Look up the definition and the theory of the estimation in the lecture slides. The distribution function essentially answers the question “how likely is that the value of the signal of the sample \( n \) will be less than some value \( x \)”. First, we have to determine the span of the auxiliary variable \( x \), for which \( F(x,n) \) is to be estimated. We can for instance use the minimum and the maximum values of the data and put 50 values between them:

```plaintext
xmin = min(min(ksi)); xmax = max(max(ksi));
kolik = 50;
x = linspace(xmin,xmax,kolik);
```
now we can do the estimation:

```matlab
Fx = zeros(size(x));
for ii = 1:kolik,
    thisx = x(ii);
    % vybereme vsechna pozorovani pro prislusny cas:
    ksin = ksi(:,ii);
    % a pocitame odhad Fx jako pomer tech, co jsou pod x a vsech:
    Fxn(ii) = sum(ksin < thisx) / Om;
end
% vysledek
subplot(211); plot (x,Fxn);
```

Q: What does the following line of code mean \( F_{x,i} = \frac{\sum(\text{ksi} < \text{thisx})}{\text{Om}}; \) If this is not clear, try to display the result of the condition \( \text{ksi} < \text{thisx} \)

Q: Does the distribution function satisfy the theoretical assumption (0 for \(-\infty\), 1 for \(+\infty\))?

Q: How would you calculate the probability of the random variable an time \( n \) lying in the interval \([-10, 5]\)?

Q: Calculate \( F(x,n) \) at different time \( n \) (e.g. 100). Is the value the same? Can we state that the signal is stationary?

3 Estimation of the Probability Density Function

\( p(x,n) \) will be realized using histogram, again for the fixed time \( n \):

```matlab
deltax = x(2) - x(1);
pxn = hist(ksin,x) / Om / deltax;
subplot(212); plot (x,pxn);
```

Q: Verify that \( \int_{-\infty}^{\infty} p(x,n)dx = 1 \).

Q: Why do the histogram has to be devided by \( \Delta x \) and \( \Omega \)?

Q: How would you cancelate the probability that the random variable at time \( n \) lie within the interval \([-10, 5]\)?

Q: Calculate \( p(x,n) \) for a different time \( n \) (e.g.. 100). Is the probability the same? Can we state that the sinal is stationary?

4 Mean Value and Standard Deviation

For the fixed time \( n \) the values are:

```matlab
an = mean(ksin);
stdn = std(ksin);
```

For the computation of the mean value and the standard deviation at all time points \( n \) we will use the fact that the functions \( \text{mean} \) and \( \text{std} \) calculate through columns:

```matlab
aalln = mean(ksi);
stdalln = std(ksi);
subplot(211); plot (nn,aalln);
subplot(212); plot (nn,stdalln);
```

Q: According to you, is the signal stationary?

Q: Why do we see odd values of the first few samples of the standard deviation?
5 Probability Density Function within a Time Interval, Autocorrelation Coefficients

Autocorrelation coefficient $R(n_1, n_2)$ indicates, how “the signal at time $n_1$ is similar with the same signal at time $n_2$”. The coefficient is calculated as:

$$R(n_1, n_2) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} x_1 x_2 p(x_1, x_2, n_1, n_2) dx_1 dx_2,$$

thus we need to calculate 2-dimensional probability density function in the time interval from $n_1$ to $n_2$ which will be estimated using a 2-D histogram. Calculation of the histogram, approximation to the 2-D probability density function and calculation of the autocorrelation coefficients is implemented in the function hist2opt (don’t use the function hist2 – reference in the lecture slides). An example for $n_1 = 50$ and $n_2 = n_1 + 0...+ 20$:

```matlab
n1 = 50;
for n2 = n1:n1+20;
    [h,p,r] = hist2opt(ksi(:,n1),ksi(:,n2),x);
    imagesc (x,x,p); axis xy; colorbar; xlabel('x2'); ylabel('x1');
    [n1 n2 r]
    pause
end
```

Q: Look at the shape of the 2-D probability density function and try to figure how the signal at times $n_1$ and $n_2$ is correlated. Look at the value of the autocorrelation coefficient. Does it correspond to your assumption?

Q: Why is the value $R(50, 50)$ the highest?

Q: Store the autocorrelation coefficients for all $n_2 = n_1 + 0...+ 20$: to a vector and display it. Describe what you see.

Q: Display the impulse response of the filter the random signal was passed through:

```matlab
plot(filter(b,a,[1 zeros(1,255)]))
```

compare the course of the autocorrelation coefficients. What is your impression?

Q: Calculate the autocorrelation coefficients for different $n_1$, e.g. $n_1 = 100$ and again with $n_2 = n_1 + 0...+ 20$. Is the signal stationary?

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Time Estimation, Spectra

This part of the exercise covers fundamental operations on random signals, in particular time estimation of the parameters within one realization of the random process. We will be working with random processes exclusively in discrete time domain. The theory for this lab can be found in the lecture on random processes II. (http://www.fit.vutbr.cz/study/courses/ISS/public/en/nah2_e.pdf).
6 Random Process Generation

We will use the same random process as in the first part of the exercise, thus Gaussian noise passed through the filter

\[ H(z) = \frac{1}{1 + -1.1314z^{-1} + 0.6400z^{-2}}. \]

We will be using only one realization of it which will be much longer though: 10000 samples:

```
N = 10000;
b=[1]; a = [1.0000 -1.1314 0.6400];
nn = 0:N-1;
aux = randn(1,N) * 5 + 0;
x = filter(b,a,aux);
plot (nn,x);
```

7 Estimation of Mean Value and Standard Deviation

...is very simple:

```
a = mean(x)
sigma = std(x)
```

Q: Did you get the same (approximatively) values as in the first part of the exercise? If so, the signal is called ergodic, where the ensemble estimates can be replaced by temporal estimates (“what can be estimated from a set of signals can be estimated on a single example of the signal”)

8 Calculation of Distribution Function and Probability Density Function

similar as in the first part, we will be using one available sample of the signal. The signal is denoted as \( x \), the auxiliary variable is denoted as \( g \):

```
gmin = min(min(x)); gmax = max(max(x));
% budeme chtit 50 bodu:
number_of_samples = 50;
g = linspace(gmin,gmax,number_of_samples);
% a jedeme
Fg = zeros(size(g));
for ii = 1:number_of_samples,
    % a pocitame odhad Fg jako pomer tech, co jsou pod g a vsech:
    thisg = g(ii);
    Fg(ii) = sum(x < thisg) / N;
end
% vysledek
subplot(211); plot (g,Fg);
```

Q: Does the distribution function look similar as in the previous example?

Along with the distribution function we will try to generate a Gaussian distribution with the pre-calculated mean value and standard deviation. Does the Gaussian correspond to our distribution? The Gaussian is defined as:

\[ p(g) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(g-a)^2}{2\sigma^2}} \]
\[
deltag = g(2) - g(1); \\
p_g = \frac{\text{hist}(x,g)}{N} / \deltag; \\
\%
\text{teoreticka Gaussovka} \\
pggaus = \frac{1}{\sqrt{2\pi} \sigma} \exp\left(-\frac{(g - a)^2}{2\sigma^2}\right); \\
\%
\text{subplot}(212); plot (g,p_g,p_ggaus); \\
\]

Q: Can we state that the signal is Gaussian?

Q: Try to verify (similarly as in the previous task) that \(\int_{-\infty}^{\infty} p(g)dg = 1\).

9 Estimation of Autocorrelation Coefficients

Depending on whether divided by the overall number of samples or only by the number of overlapping samples, we define biased or unbiased estimation, respectively:

\[
\hat{R}_v(k) = \frac{1}{N} \sum_{n=0}^{N-1} x[n]x[n+k], \quad \hat{R}_{nv}(k) = \frac{1}{N - |k|} \sum_{n=0}^{N-1} x[n]x[n+k],
\]

In Matlab, we call function \text{xcorr} with option 'biased' or 'unbiased':

\[
\text{Rv} = \text{xcorr}(x, \text{'biased'}); \text{ k } = \text{-N+1:N-1}; \%
\text{subplot}(211); plot(k,Rv) \\
\text{Rnv} = \text{xcorr}(x, \text{'unbiased'}); \text{ k } = \text{-N+1:N-1}; \%
\text{subplot}(212); plot(k,Rnv); \\
\]

Q: Why the boundary values of the unbiased signal look 'strange'?

Q: Why the boundary values of the biased signal are small?

Q: Check whether the value of the zero'th autocorrelation coefficient (in vectors \text{Rv}, \text{Rnv} it is in the middle) is equal to the average power of the signal:

\[
P_s = \frac{1}{N} \sum_{n=0}^{N-1} x^2[n]
\]

10 Power Spectral Density (PSD)

will be estimated using DFT either from one samples or as a average from a set of samples. Averaging should be leading to a smoother estimation.

10.1 Estimation from One Sample

\[
N = 10000; \\
Gdft = 1/N * abs(fft(x)).^2; \\
\text{om} = (0:N/2-1)/N * 2*pi; \text{Gdft} = \text{Gdft}(1:N/2); \\
\%
\text{subplot}(211); plot (om,Gdft); grid; \\
\]

Q: What do you think about the estimate? Does it look reasonable?

10.2 Estimation from a Set of Samples

will be done by function \text{gprum.m} (you will also need function \text{trame.m} \footnote{Segments are sometimes called frames}, which will divide the signal to segments with length of 100 and estimate PSD for each of the segments and then average them:

\[
\text{[Gprum,om] = gprum(x); \%
\text{subplot}(212); plot (om,Gprum); grid;}
\]

Q: What do you think about this estimate?
Passing of a random signal through a linear system

Pass the signal through the filter with the transmission function:

\[ H(z) = 1 - 1.1314z^{-1} + 0.6400z^{-2} \]

(notice that the filter is the inverse to the filter we generated the signal with). PWD is:

\[ G_y(e^{j\omega}) = |H(e^{j\omega})|^2 G_x(e^{j\omega}), \]

which we will verify by:

\[
\begin{align*}
\% & \text{ puvodni PSD - tu mame, ale pro uplnost ...} \\
& [Gx, om] = gprum(x); \\
& subplot(131); plot(om, Gx); grid; \\
\% & \text{ filtr - freq. char na druhou:} \\
& a = [1]; b = [1.0000 -1.1314 0.6400]; \\
& [h, om] = freqz (b, a, 50, 2*pi); \\
& h2 = abs(h).^2; \\
& subplot(132); plot(om, h2); grid; \\
\% & \text{ filtrujeme a zobrazime psd vystupu} \\
\% & \text{ a jeste teoreticky vstupni spek,. hust. vykonu nasobenou} \\
\% & \text{ druhou mocninou H} \\
& y = filter(b, a, x); \\
& [Gy, om] = gprum(y); Gyteor = Gx .* h2; \\
& subplot(133); plot(om, Gy, om, Gyteor); grid;
\end{align*}
\]

Q: Does the equation hold \( G_y(e^{j\omega}) = |H(e^{j\omega})|^2 G_x(e^{j\omega}) \) ?

Q: Why PSD of the output signal is (approximately) constant ?

Q: How is such signal called? (white light has a constant spectrum).

Q: Only the zeroth correlation coefficient of such random should be nonzero. Verify.