Lattice structures for bisimilar Probabilistic Automata

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Outline

- Probabilistic Automata
- Bisimulations, quotients, isomorphisms, rescaledness
- Intersections in the finite case
- Infinite case: counterexample and results
- Conclusion

Example

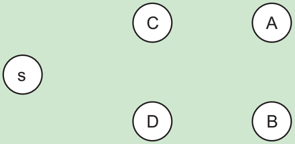
Definition

A probabilistic automaton (PA) P consists of

- a countable set of states S
- a countable set of actions $Act = \{\tau\} \dot{\cup} E$
- a possibly uncountable set of transitions $T \subseteq S \times Act \times Dist(S)$
- an initial state s_0

Probabilistic Automata (PA)

Example



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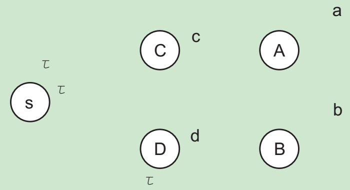

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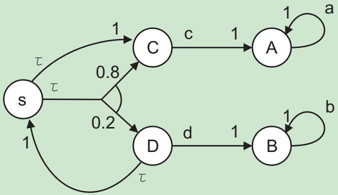

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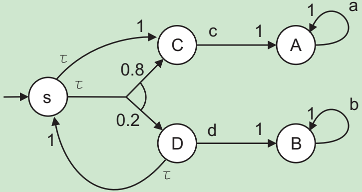

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For $(s, a, \mu) \in T$ we also write $s \xrightarrow{a} \mu$

different types

	strong	weak
non-combined (deterministic schedulers)	$\xrightarrow{a}$	$\underbrace{\overbrace{\xrightarrow{\tau} \dots \xrightarrow{\tau}}^{0\text{–}n \text{ times}} \xrightarrow{a} \overbrace{\xrightarrow{\tau} \dots \xrightarrow{\tau}}^{0\text{–}m \text{ times}}}_{\overset{a}{\Rightarrow}}$
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Bisimulations

Definition (Strong bisimulation)

An equivalence relation R is called *strong (probabilistic) bisimulation* if for all actions $a \in Act$ it holds that sRt implies that for every $s \xrightarrow{a} \mu$ we find $t \xrightarrow{a}_c \mu'$, such that μ and μ' coincide on equivalence classes. We write $P \sim P'$ if the initial states are strongly probabilistic bisimilar.

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Two automata P and P' are called isomorphic if they coincide after relabelling their states. For isomorphic automata we write $P \equiv_{iso} P'$.

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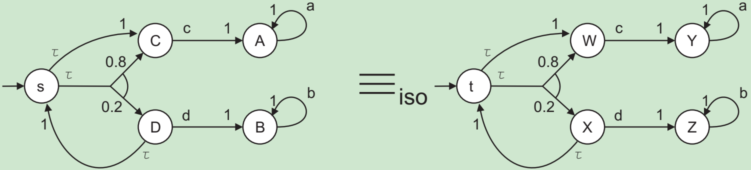

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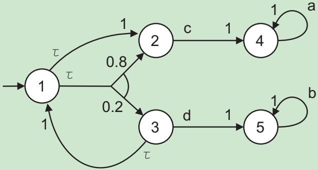

Definition (Quotient automaton)

Let $P = (S, Act, T, s_0)$ be a PA and R an equivalence relation over S. We write ${}^{P}/_{R}$ to denote the quotient automaton of P wrt. R, that is

$$ {}^{P}/_{R} = ({}^{S}/_{R}, Act, {}^{T}/_{R}, [s_0]_R). $$

We call an automaton a *quotient wrt.* R if it holds that $P \equiv_{iso} {}^{P}/_{R}$.

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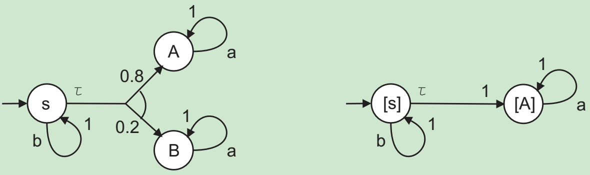

P $\qquad\qquad$ $P/\sim$

Rescaled automata

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An automaton P is called rescaled if for all its transitions $s \xrightarrow{\tau} \mu$ it holds that $\mu(s) = 0$ or $\mu(s) = 1$.

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For every automaton P there is a rescaled automaton P' such that $P \approx P'$

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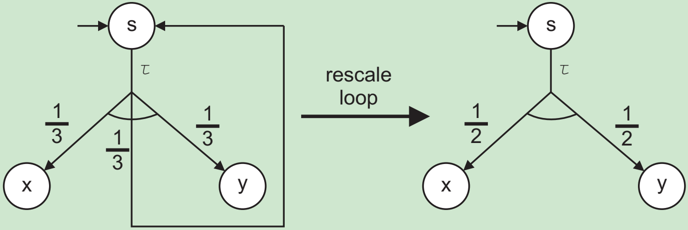

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Known facts on lattice structures for *finite* automata

The following lemma is our interpretation of a result from Segala/Cattani '02.

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The intersection of strongly bisimilar finite (finitely many states and transitions) quotient automata is again bisimilar.

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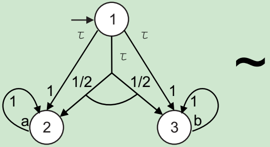

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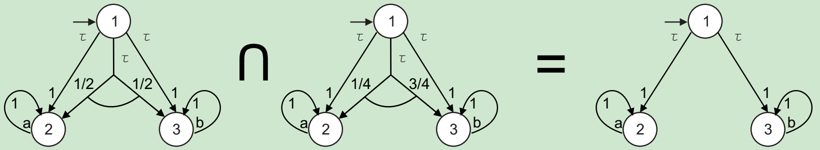
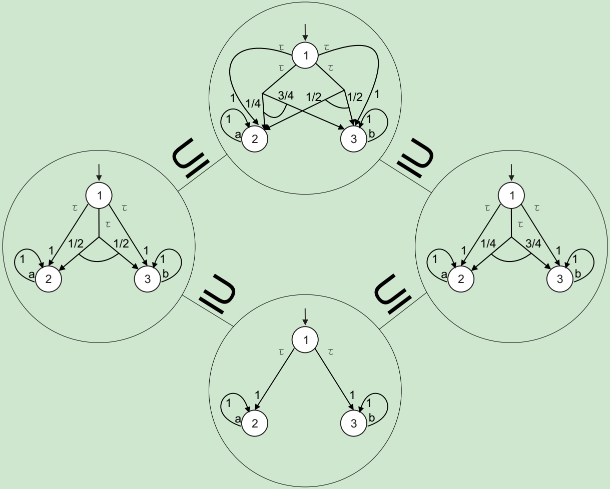


This gives rise to a lattice structure on bisimilar quotients:

Example (Underlying partial order of previous example)



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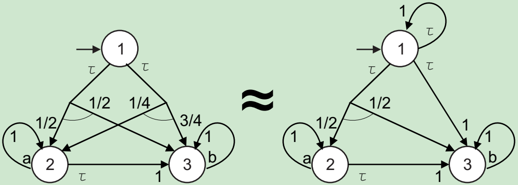

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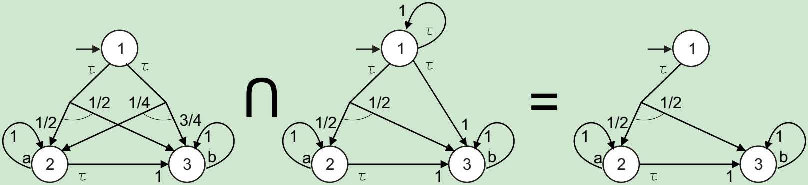

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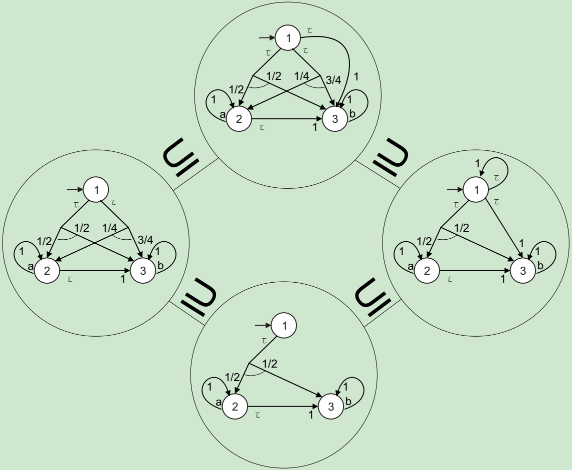

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Counterexample for the infinite case

Lemma (no canonical extension to infinite case)

The intersection of strongly bisimilar infinite quotient automata does not have to be bisimilar.

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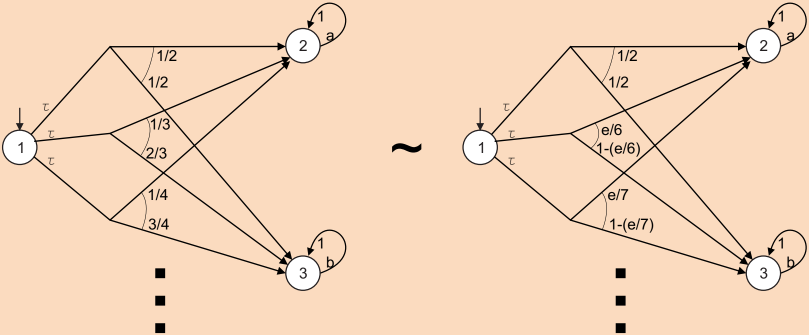

Our goal

Main objective of this work

We search conditions where the intersection behaves well, i.e. we get lattice structures.

What we need

- metrics on distributions
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Segala/Cattani defined convex sets of reachable distributions:

- $S_{\sim}(s,a) := \{\mu \in Dist(S) | s \xrightarrow{a}_C \mu\} / \sim$
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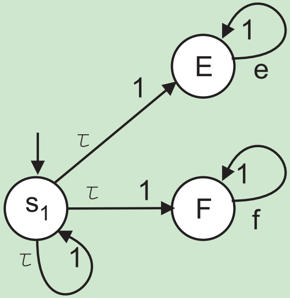

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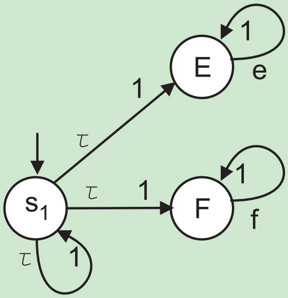

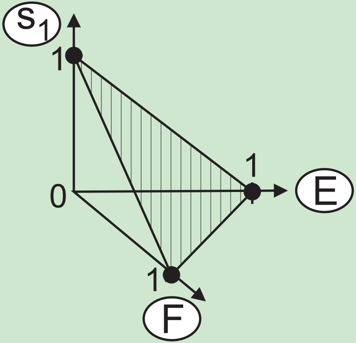

Metric spaces

Definition (Desharnais et al. '10)

$$d(\mu_1, \mu_2) := sup_{A \subseteq S} |\mu_1(A) - \mu_2(A)|$$

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d is a metric on distributions over S

Metric spaces of Segala/Cattani sets

$(S_{\sim}(s, a), d)$ and $(S_{\approx}(s, a), d)$ are metric spaces

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We consider product metric spaces $(\prod_{(s,a)\in S\times Act} S_R(s,a), d)$ where $d(x,y) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{d_i(x_i,y_i)}{1+d_i(x_i,y_i)}$.

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The following theorem gives the big picture:

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For compact automata, intersections of strongly bisimilar quotient automata are again bisimilar

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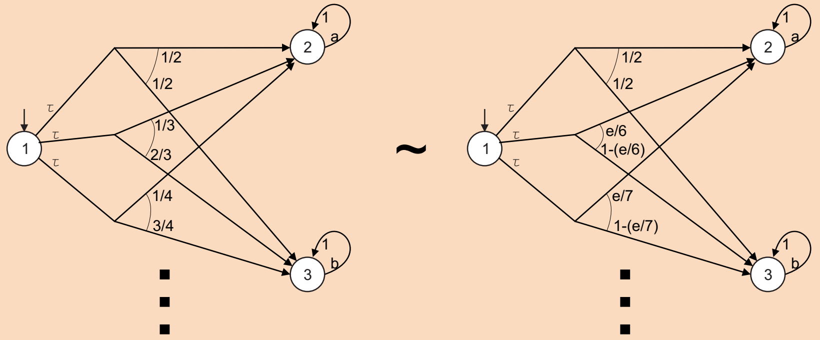
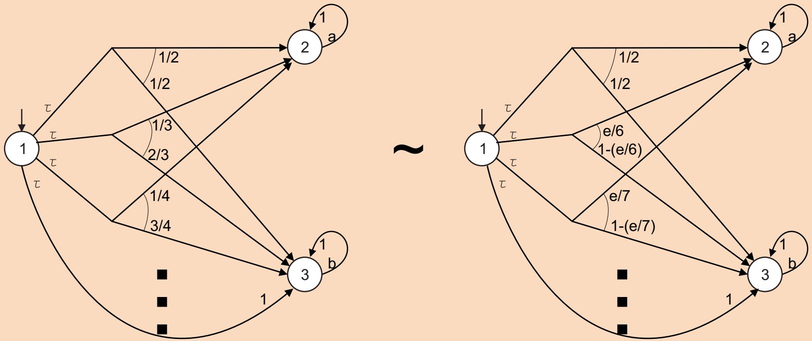
Counterexample - not compact



...and what about the counterexample?

Compactification - intersection again bisimilar



Bounded and unbounded lattices

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Considering also unreachable parts of the state spaces for compact quotient automata leads to unbounded lattices, i.e. no upper bound.

Example (proof idea)

- add a unreachable state s_c with $s_c \xrightarrow{\tau} c\Delta_1 \oplus (1-c)\Delta_2$
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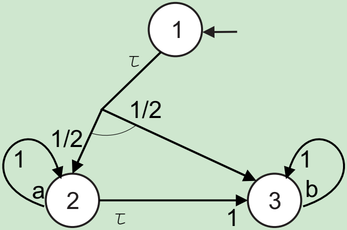

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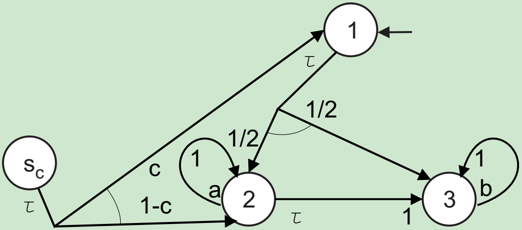

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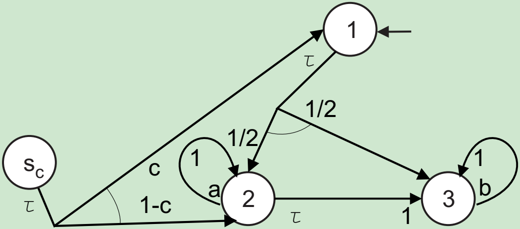

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Theorem

Restriction to reachable state spaces for compact quotient automata leads to bounded lattices.

Summing up

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- new viewpoint to bisimilar quotient automata
- deep structure in quotients of bisimilar automata (up to isomorphism)
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