Discrete signals and their frequency analysis.

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- recapitulation – fundamentals on discrete signals.
- periodic and harmonic sequences
- discrete signal processing
- convolution
- Fourier transform with discrete time
- Discrete Fourier Transform

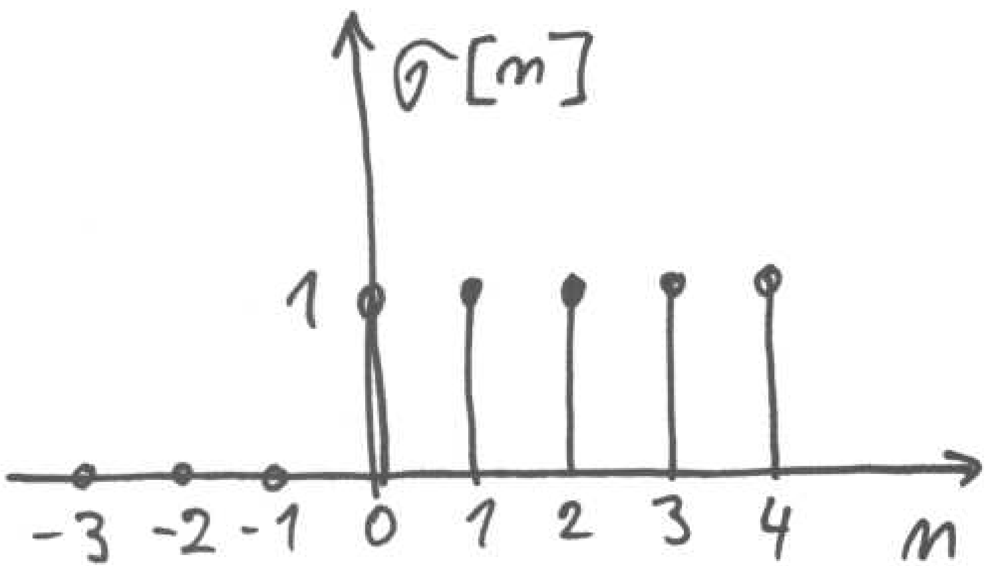
Sampled signal $\Rightarrow$ discrete signal

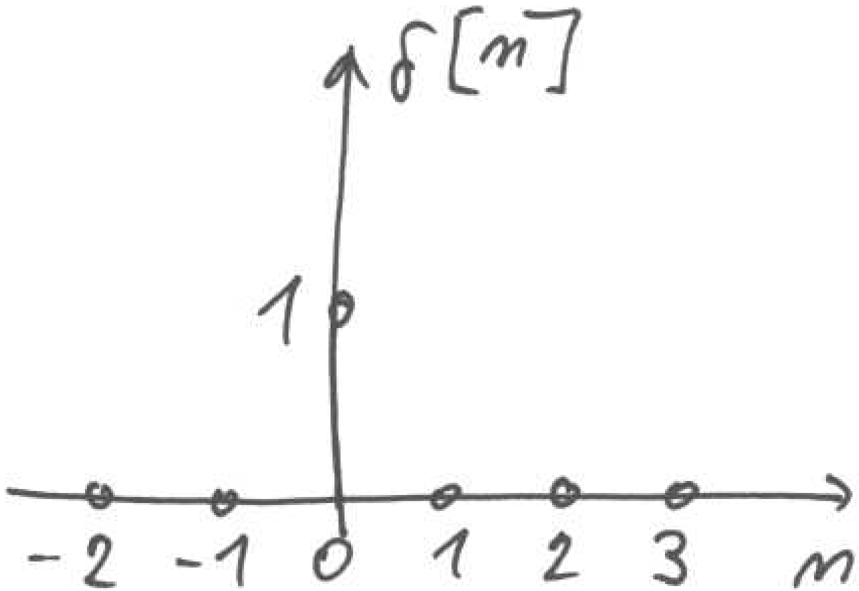
During sampling we consider only values of the signal at sampling period multiplies T:
$x(nT), \quad nT = \ldots - 2T, -T, 0, T, 2T, 3T, \ldots$ For a discrete signal, we forget about real time and simply count the samples. Discrete time becomes :
$x[n], \quad n = \ldots - 2, -1, 0, 1, 2, 3, \ldots$
Thus we often call discrete signals just **sequences**.

Important discrete signals

Unit step and unit impulse:

$$\sigma[n] = \begin{cases} 1 & \text{for} \quad n \geq 0 \\ 0 & \text{elsewhere} \end{cases} \qquad \delta[n] = \begin{cases} 1 & \text{for} \quad n = 0 \\ 0 & \text{elsewhere} \end{cases}$$





Periodic discrete signals

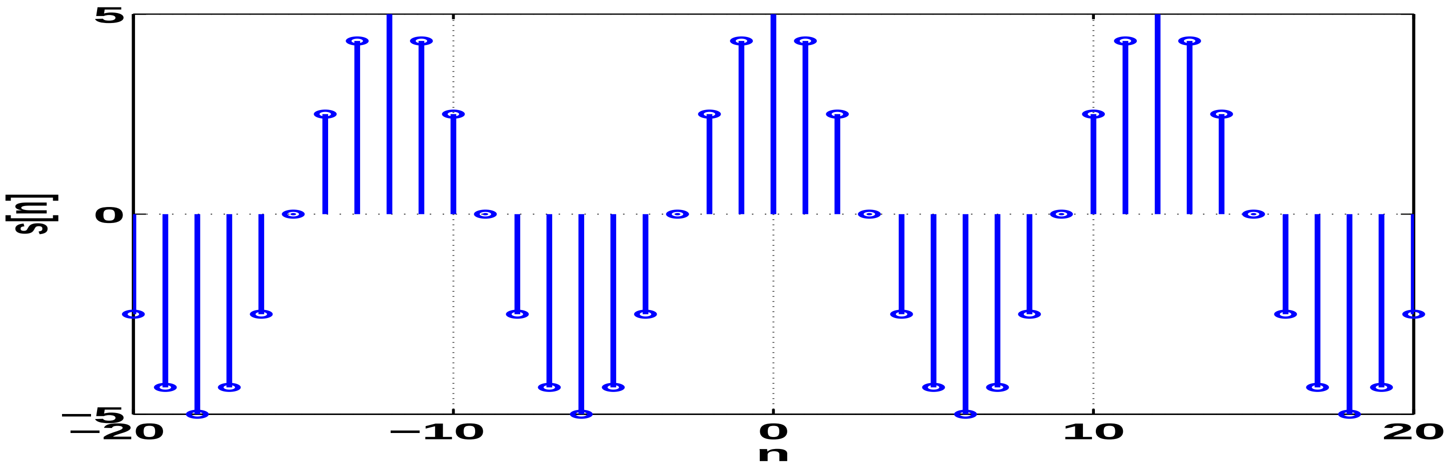
their behaviour repeats after N samples, the smallest possible N is denoted as N_1 and is called **fundamental period**.

Harmonic discrete signals (harmonic sequences)

$$x[n] = C_1 \cos(\omega_1 n + \phi_1) \tag{1}$$

- C_1 is a positive constant – magnitude.
- ω_1 is a spositive constant – **normalized** angular frequency. As n is just a number, the unit of ω_1 is [rad]. Note, that in the previous lecture we denoted with the same simbol an angular frequency of continuous signals. Although in the last lecture we used symbol ω_1' for discrete time, we will not do it any longer. You will recognize continuous time frequency if there is real time associated with it (for instance $\cos(\omega_1 t)$). If you see discrete time n (for instance $\cos(\omega_1 n)$) you should know we are talking about normalized angular frequency.
- ϕ_1 is an initial phase [rad]. Value of a signal in time $n = 0$ is $x[0] = C_1 \cos \phi_1$.

Example: $x[n] = 5\cos(2\pi n/12)$, $\omega_1 = \pi/6$.



We have a small trouble with fundamental period of a harmonic sequence: it cannot be computed similarly as for continuous time signals using : $N_1 = \frac{2\pi}{\omega_1}$ as can result into a real number and N_1 has to be an integer. Thus we have to find such N_1 that satisfy:

$$\cos\left[\omega_1(n + N_1)\right] = \cos\omega_1 n.$$

We know that the fundamental period of a cosine function is 2π, thus :

$$\omega_1(n + N_1) - \omega_1 n = \omega_1 N_1 = k2\pi,$$

where k is such an integer that N_1 is the smallest possible.

Properties of harmonic sequences

- equation for a **sampled signal** $x[n] = C_1 \cos(\omega_1' n + \phi_1)$ corresponds to a signal with real time: $x(nT) = C_1 \cos(\omega_1 nT + \phi_1)$, where ω_1' again denotes normalized frequency and ω_1 denotes corresponding real frequency. As the corresponding values of signals have to be equal, arguments of the signals have to equal too. Thus we can calculate normalized frequency from real frequency as:

$$\omega_1' = \omega_1 T \quad \text{thus} \quad \omega_1' = \frac{\omega_1}{Fs}.$$

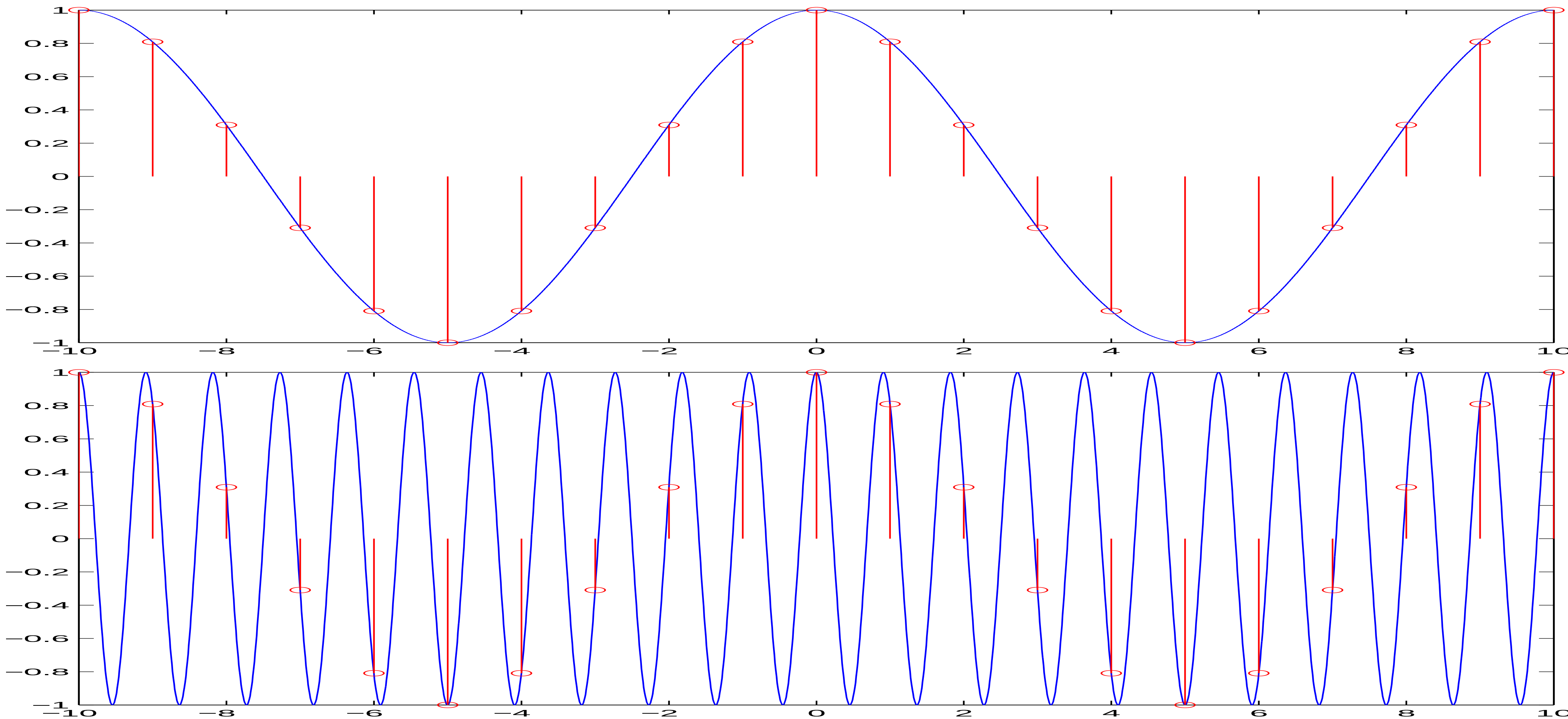
Normalization of sampling frequency.

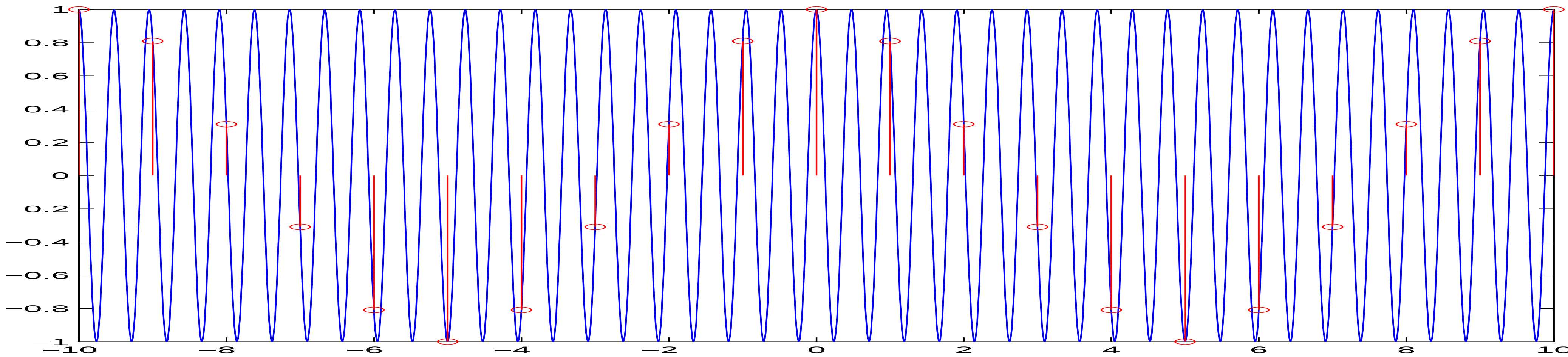
- In continuout time domain, we assumed that two different angular frequencies generate two different cosine functions. How is it with discrete cosines? We know that cosine is a periodic function with the basic period 2π (from now on we again use simple ω to denote normalized frequencies).

$$\cos[(\omega_1 + 2k\pi)n + \phi_1] = \cos[\omega_1 n + 2k\pi n + \phi_1].$$

$2k\pi n$ is again a multiple of 2π, thus the cosine for $\omega_1 + 2k\pi$ is the same as for ω_1.

How is that possible ? :





- as $\cos$ is an **even function**, the following holds:

$$\cos(\omega_1 n) = \cos(-\omega_1 n),$$

and thus:

$$\cos(\omega_1 n) = \cos[(-\omega_1 + k2\pi)n].$$

Exponencial sequence

$$x[n] = e^{j\omega_1 n}$$

Looks the same for all angular frequencies $\omega_1 + 2k\pi$, because:

$$x[n] = e^{j(\omega_1 + 2k\pi)n} = e^{j(\omega_1 n + \phi_1 + 2k\pi n)} = e^{j(\omega_1 n + \phi_1)}e^{2k\pi n} = e^{j\omega_1 n}$$

as $e^{j2k\pi n}$ is always equal to 1.

Decomposition of a harmonic sequence into exponentials

Similar as for continuous signals, using the equation:

$$\cos x = \frac{e^{jx} + e^{-jx}}{2}$$

we can decompose a harmonic sequence as a sum of two exponentials:

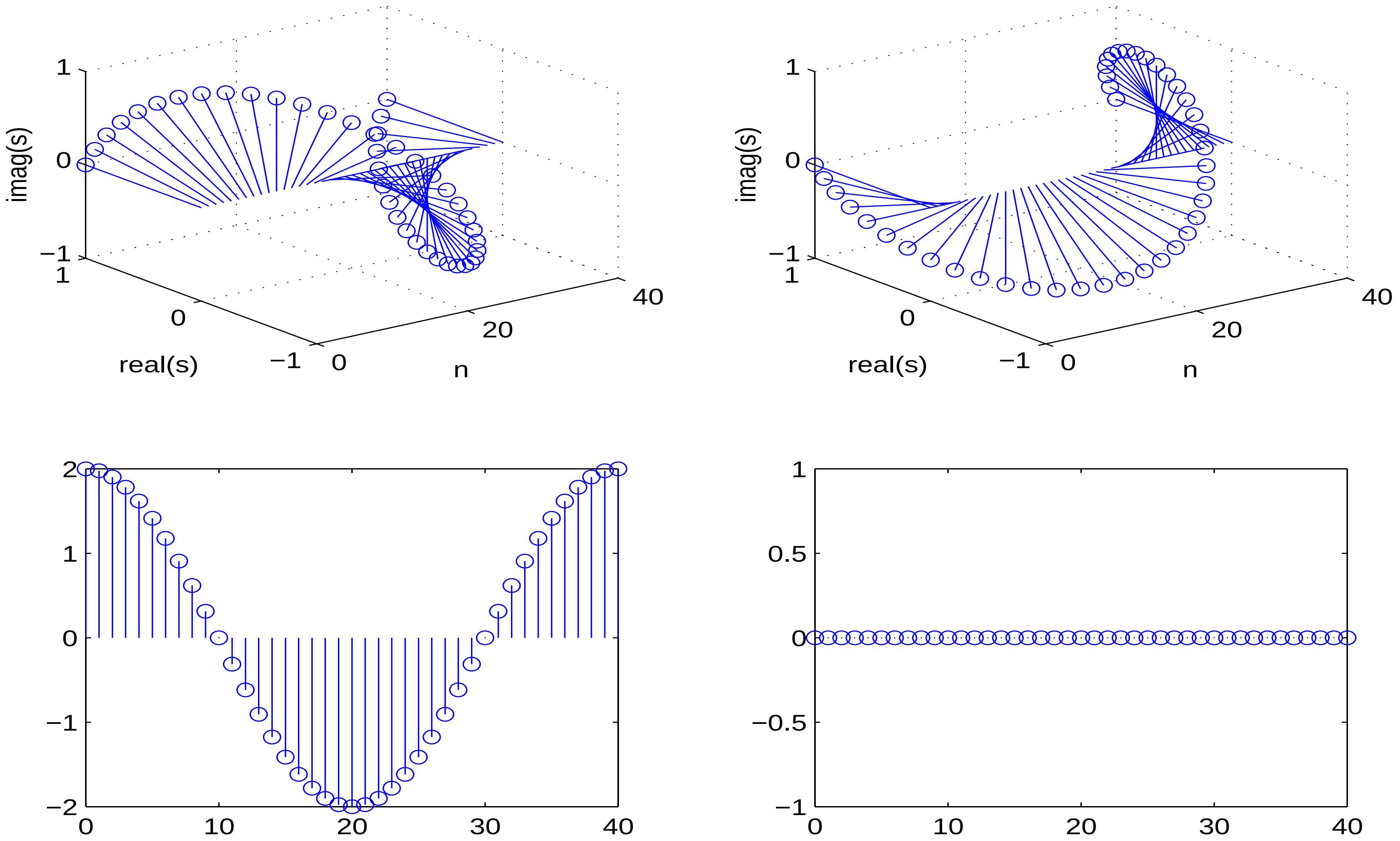
$$x[n] = C_1 \cos(\omega_1 n + \phi_1) = c_1 e^{j\omega_1 n} + c_{-1} e^{-j\omega_1 n}$$

where the coefficients c_1 are expressed similarly as for continuous time signals:

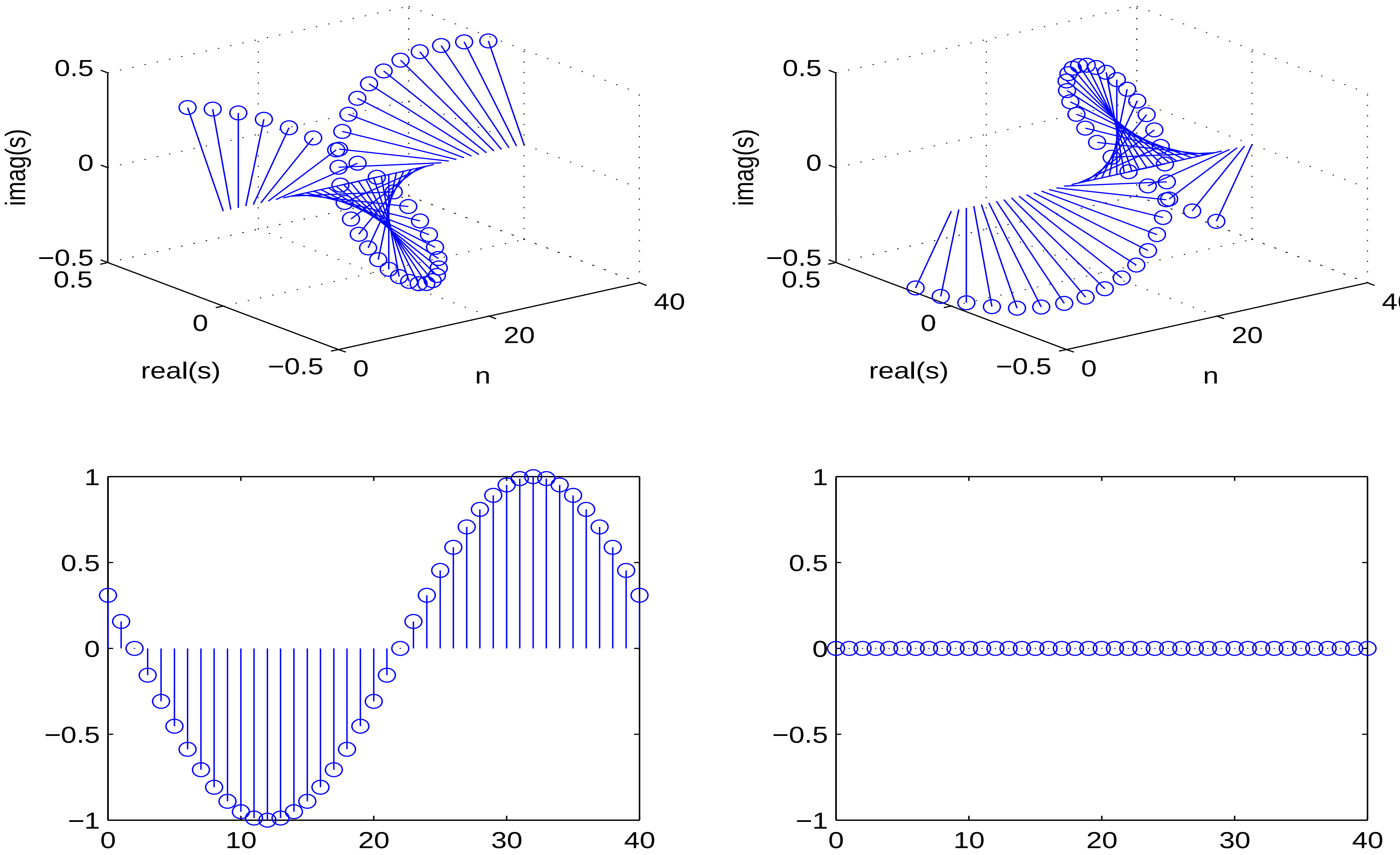
$$|c_1| = |c_{-1}| = \frac{C_1}{2} \qquad \arg c_1 = -\arg c_{-1} = \phi_1,$$

thus c_1 and c_{-1} are complex conjugate (same magnitudes and reverse phases).

Example 1 : $\omega_1 = \frac{2\pi}{40}$, $c_1 = c_{-1} = 1$, $x[n] = 2\cos\frac{2\pi}{40}n$.



Example 2 : $\omega_1 = \frac{2\pi}{40}$, $c_1 = 0.5e^{j0.4\pi}$, $c_{-1} = 0.5e^{-j0.4\pi}$, $x[n] = \cos(\frac{2\pi}{40}n + 0.4\pi)$.



Operations with discrete signals

Sequence of length N

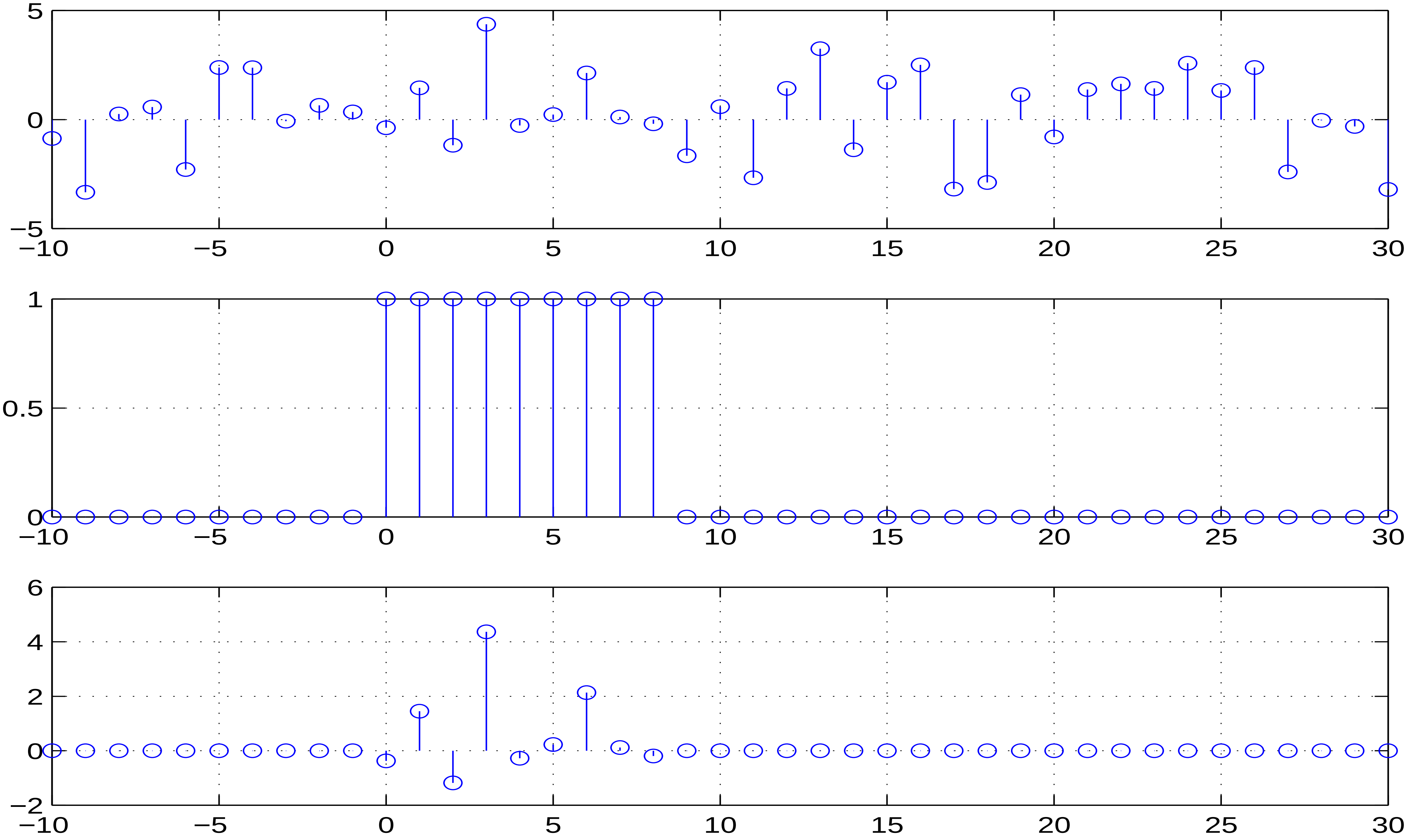
The sequence has nonzero values for time: $n \in [0, N-1]$, and zero elsewhere.

Extraction of a sequence of length N

We multiply a signal by a windowing function of length N:

$$R_N[n] = \begin{cases} 1 & \text{for} \quad n \in [0, N-1] \\ 0 & \text{elsewhere} \end{cases}$$

$$y[n] = x[n] R_N[n]$$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Periodization of a sequence of length N

Having a sequence $x[n]$ of length N with non-zero samples for time 0 to $N-1$, we repeat this sequence infinite number of times. Mathematicly, we use function **modulo**, that returns reminder of integer division:

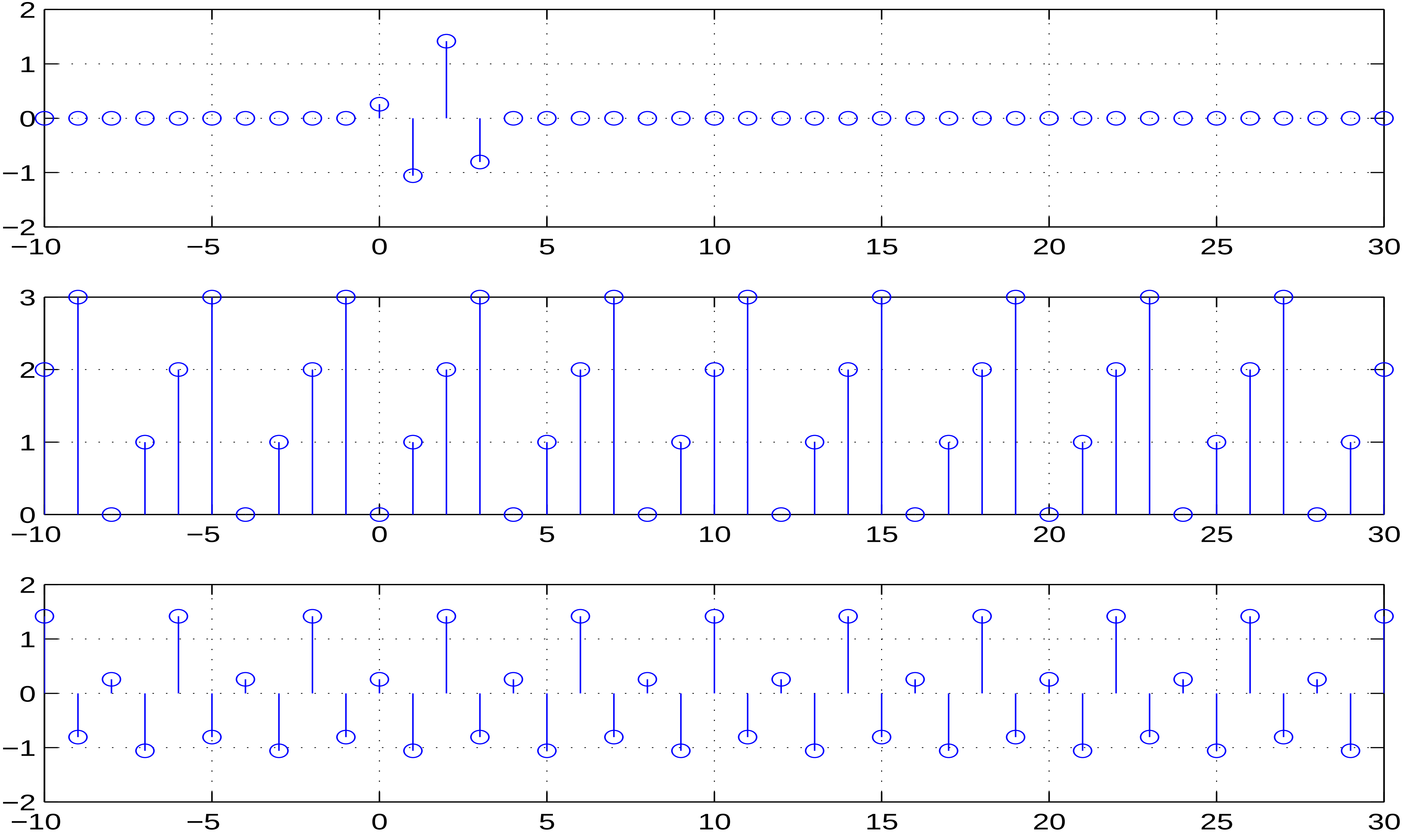
$$\tilde{x}[n] = x[\mod_N n]$$

How does it work?

Example : Periodize a sequence of length 4.

Evaluate function $\mod_4 n$:

n	...	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	...
$\mod_4 n$	...	3	0	1	2	3	0	1	2	3	0	1	2	3	...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Periodic shift of a sequence of length N

delay of the signal by m samples is defined as:

$$x[n] \longrightarrow x[n-m]$$

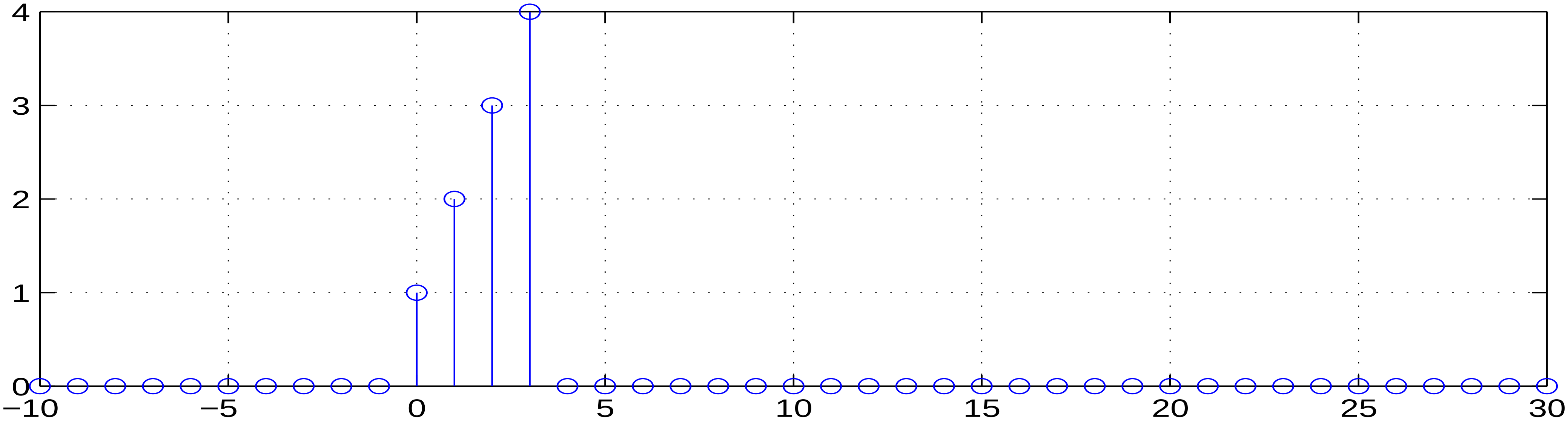
For a **periodic shift** we shift a sequence of length N using funtion $\mod_N$ to derive time indeces:

$$x[n] \longrightarrow x[\mod_N(n-m)]$$

Example : given a sequence of length 4, delay it periodicly by 2 samples:

n	...	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	...
$\mod_4 n$	...	3	0	1	2	3	0	1	2	3	0	1	2	3	...
$\mod_4(n-2)$	...	1	2	3	0	1	2	3	0	1	2	3	0	1	...

we can understand it as a shift of samples in an auxiliary buffer of length N and a consequent repeating of the content of the buffer.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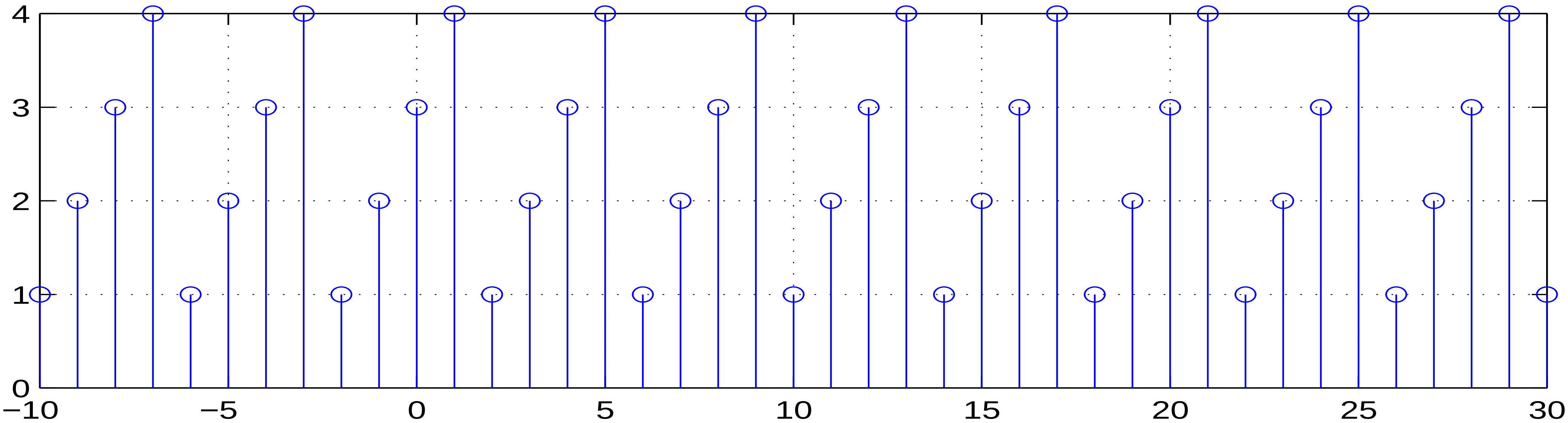
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Circular shift of a sequence of length N

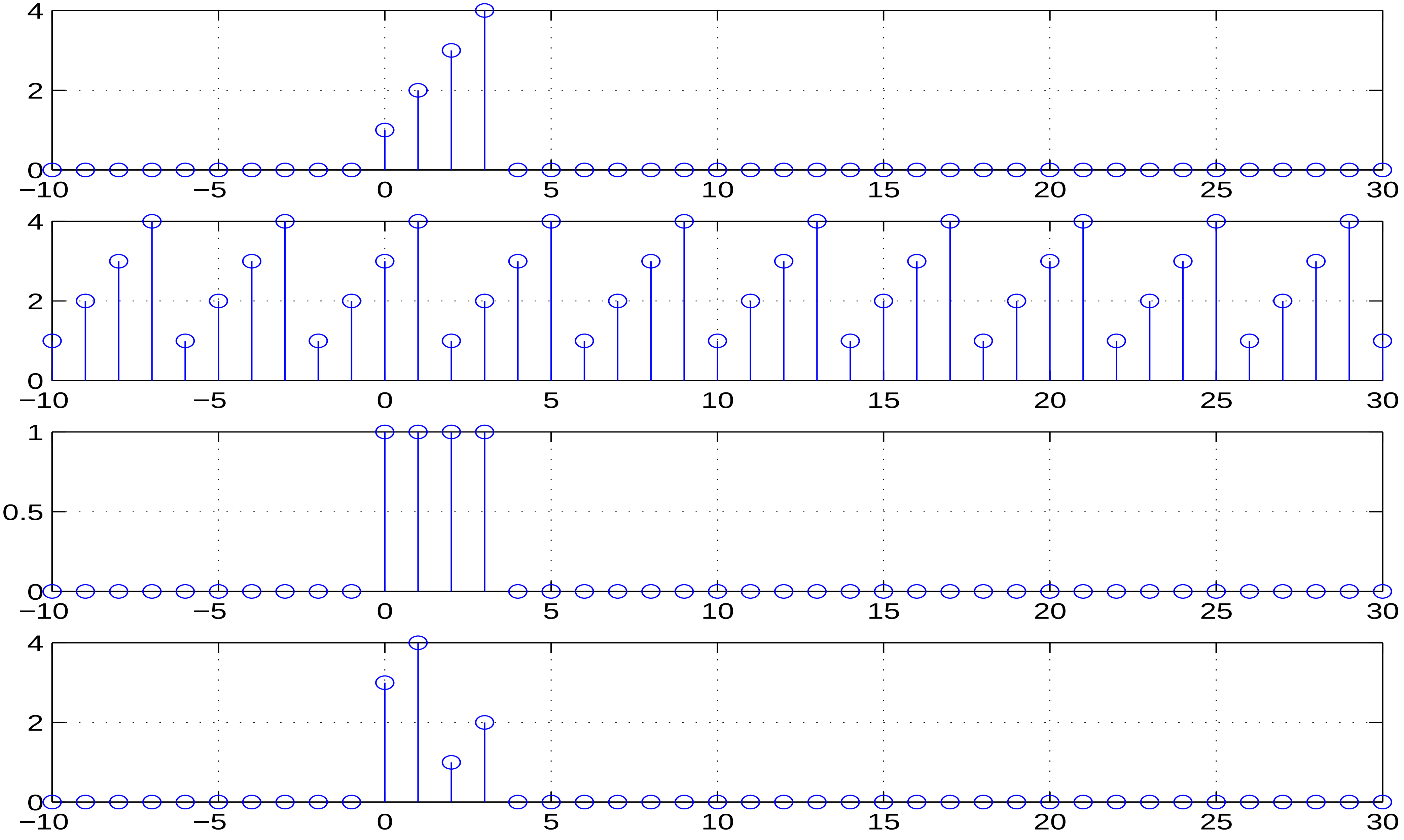
is similar to periodic shift, but the resulting sequence is non-periodic. From the result of periodic shift, we extract just an interval $n \in [0, N-1]$ using windowing function:

$$x[n] \longrightarrow R_N(n)x[\mod_N(n-m)]$$

the operation can be understood as extracting items from the beggining of a queue and placing them to the end. We still have the same number of samples but in a different order.

Example: given a sequence of length 4, delay it cirularly by 2 samples:

n	...	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	...
$\mod_4 n$	...	3	0	1	2	3	0	1	2	3	0	1	2	3	...
$\mod_4(n-2)$	...	1	2	3	0	1	2	3	0	1	2	3	0	1	...
$R_4[n] \mod_4(n-2)$	...	-	-	-	-	-	2	3	0	1	-	-	-	-	...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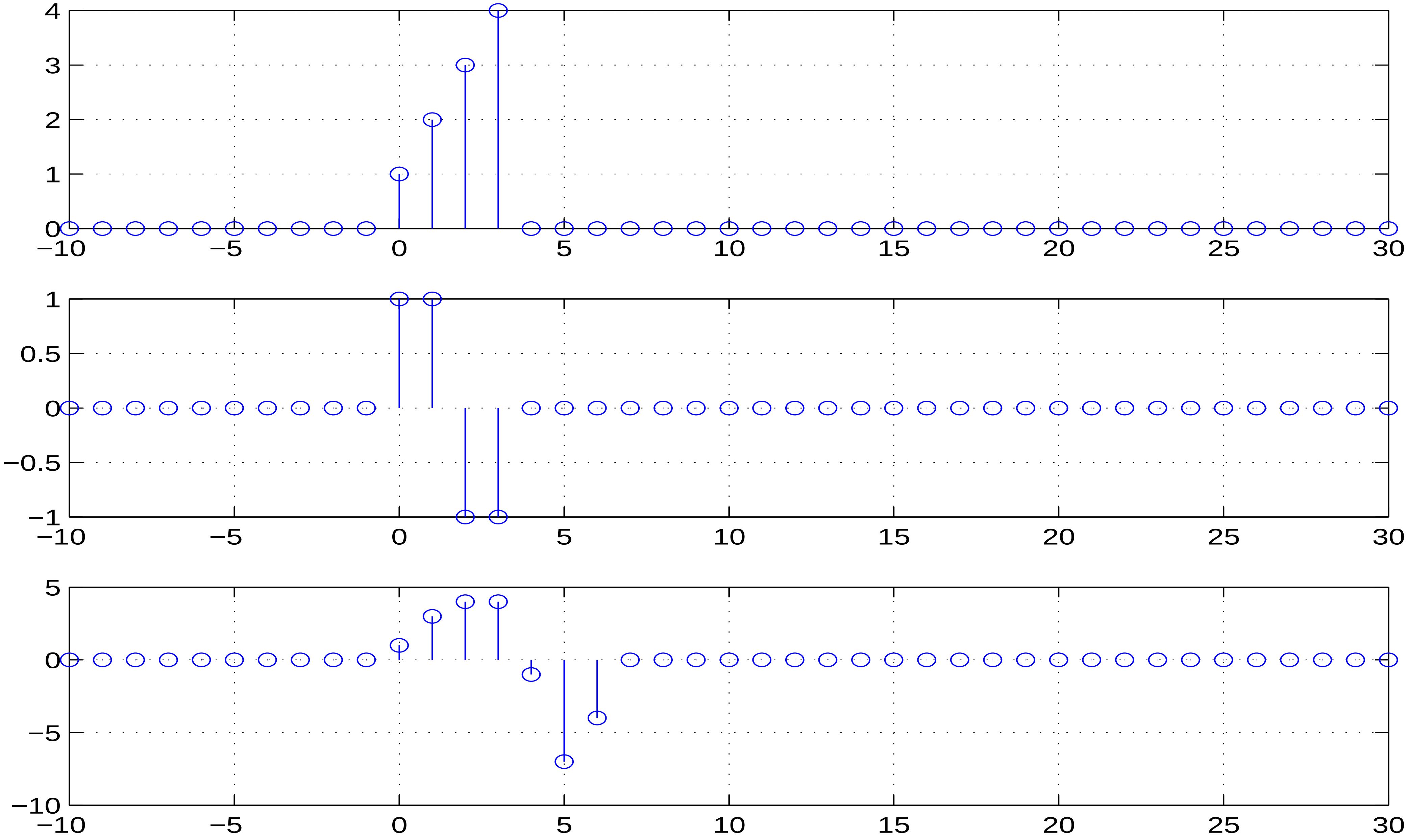
CONVOLUTION

For continuous time signals, we defined one type of convolution. For discrete signals, we have different types of convolution, depending on what type of shift (standard, periodic,or circular) we use in $x[n-m]$.

Linear convolution

Linear convolution is defined as: $x[n] \star y[n] = \displaystyle\sum_{k=-\infty}^{\infty} x[k]y[n-k]$ and for a sequence of length N it becomes: $x[n] \star y[n] = \displaystyle\sum_{k=0}^{N-1} x[k]y[n-k]$

The resulting sequence is of length $2N-1$ (zeros elsewhere on the time axis) as for any shift outside interval $[0, 2N-1]$, signals do not overlap. This type of convolution is the most useful as it gives us means to implement **filtering** (remember that for LTI systems the output is given by convolution of the input and the system's impulse response).



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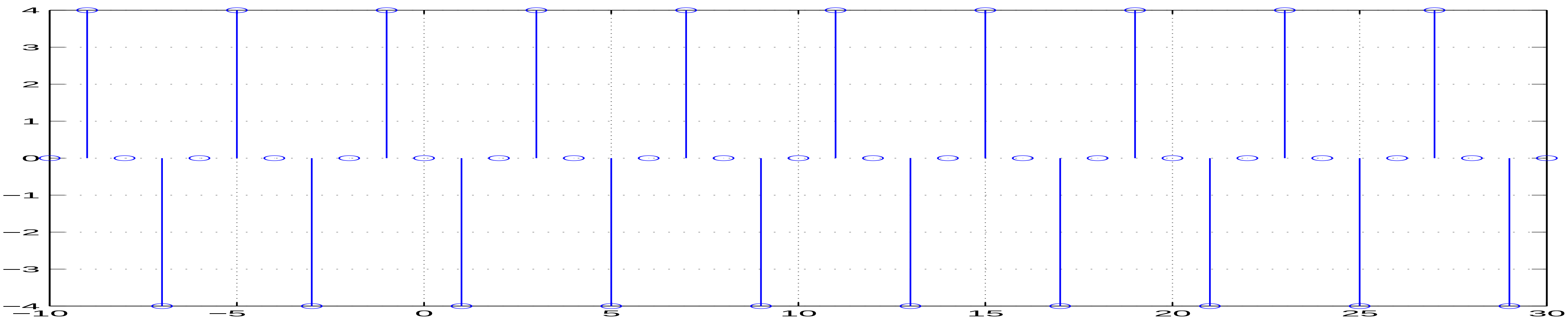
0
−0.5
−1
5
−10

Periodic convolution

In the expression $[n-k]$ we place operation $\mod$ that regresses indeces to the interval $[0, N-1]$. This convolution thus is of the infinite length with repeating basic pattern of length N.

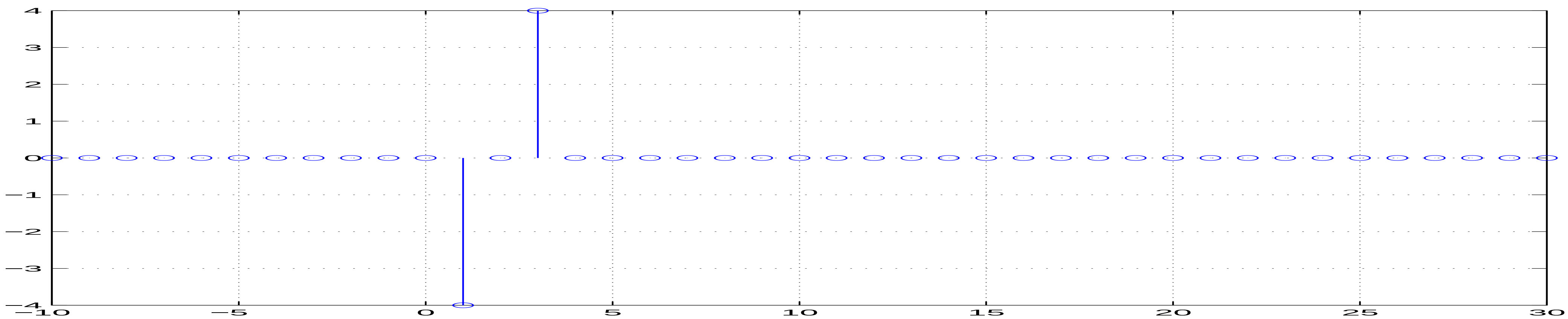
$$x[n]\tilde{\star}y[n] = \sum_{k=0}^{N-1} x[k]y[\mod_N(n-k)]$$



Circular convolution

is similar to periodic but we extract just one period using window $[0, N-1]$.

$$x[n] \circledN y[n] = R_N[n] \sum_{k=0}^{N-1} x[k] y[\mod_N(n-k)]$$



Circular convolution is again well demonstrated on paper stripes. This time, however, we need stepler or glue:

- plot the two sequences on paper stripes and denote zero sample,
- make circles out of the stripes,
- reverse one of the circles,
- to calculate n-th sample of circular convolution, shift the reverted circle by n samples to the right. Multiply everything item by item and add multiplies up to get a scalar.
- calculate samples for n equal to 0 to $N-1$.
- how should we change the algorithn to calculate periodic convolution?

We know that for continuous signals that multiplication in time corresponds to convolution in spectrum. Similarly for discrete signals – multiplication of two signals in time domain correponds to circular convolution of Discrete Fourier transofr (DFT) spectra of the two signals.

Spectral analysis of discrete signals

Fourier transform with discrete time – DTFT

in lecture on sampling we learnd that spectrum of a sampled signal is derived from the original signal spectrum as:

$$X_s(j\omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X(\omega - k\omega_1),$$

where T is sampling period. Often we dispose only of a samples signal so we do not know how the original signal's spectrum looks like.

We can try to derive spectral function of discrete signal from scratch. We know that sampling is done by multiplying the original signal with the Dirac impulse sequence:

$$x_s(t) = x(t)s(t) = x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT)$$

Sampled signal si given by the original signal samples in time nT.

Fourier transform of such signal is:

$$X_s(j\omega) = \int_{-\infty}^{+\infty} \sum_{n=-\infty}^{\infty} x(nT)\delta(t-nT)e^{-j\omega t}dt$$

For a fixed ω, exponential $e^{-j\omega t}$ becomes a function of time. If we multiply this function by a sequence of Dirac impulses, we obtain a sampled funtion too – so we use samples of exp function only for time nT:

$$X_s(j\omega) = \int_{-\infty}^{+\infty} \sum_{n=-\infty}^{\infty} x(nT)\delta(t-nT)e^{-j\omega nT}dt$$

We swap order of integral and sum and using shift property :

$$\int_{-\infty}^{+\infty} x(t)\delta(t-\tau)dt = x(\tau), \quad \text{and} \quad \int_{-\infty}^{+\infty} x(nT)\delta(t-nT)e^{-j\omega nT}dt = x(nT)e^{-j\omega nT}$$

Spectrum of a sampled signal thus becomes :

$$X_s(j\omega) = \sum_{n=-\infty}^{\infty} x(nT)e^{-j\omega nT}$$

that can be interpreted as a sum of complex exponentials multiplied by samples value for a given ω. We are able to calculate $X_s(j\omega)$ for a given ω as we normally have a finite number of samples. The last step is normalization of frequencies:

$$n = \frac{nT}{T} \qquad \omega' = \frac{\omega}{F_s}$$

and we get:

$$\omega nT = \omega' F_s n \frac{1}{F_s} = \omega' n$$

thus:

$$X_s(j\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

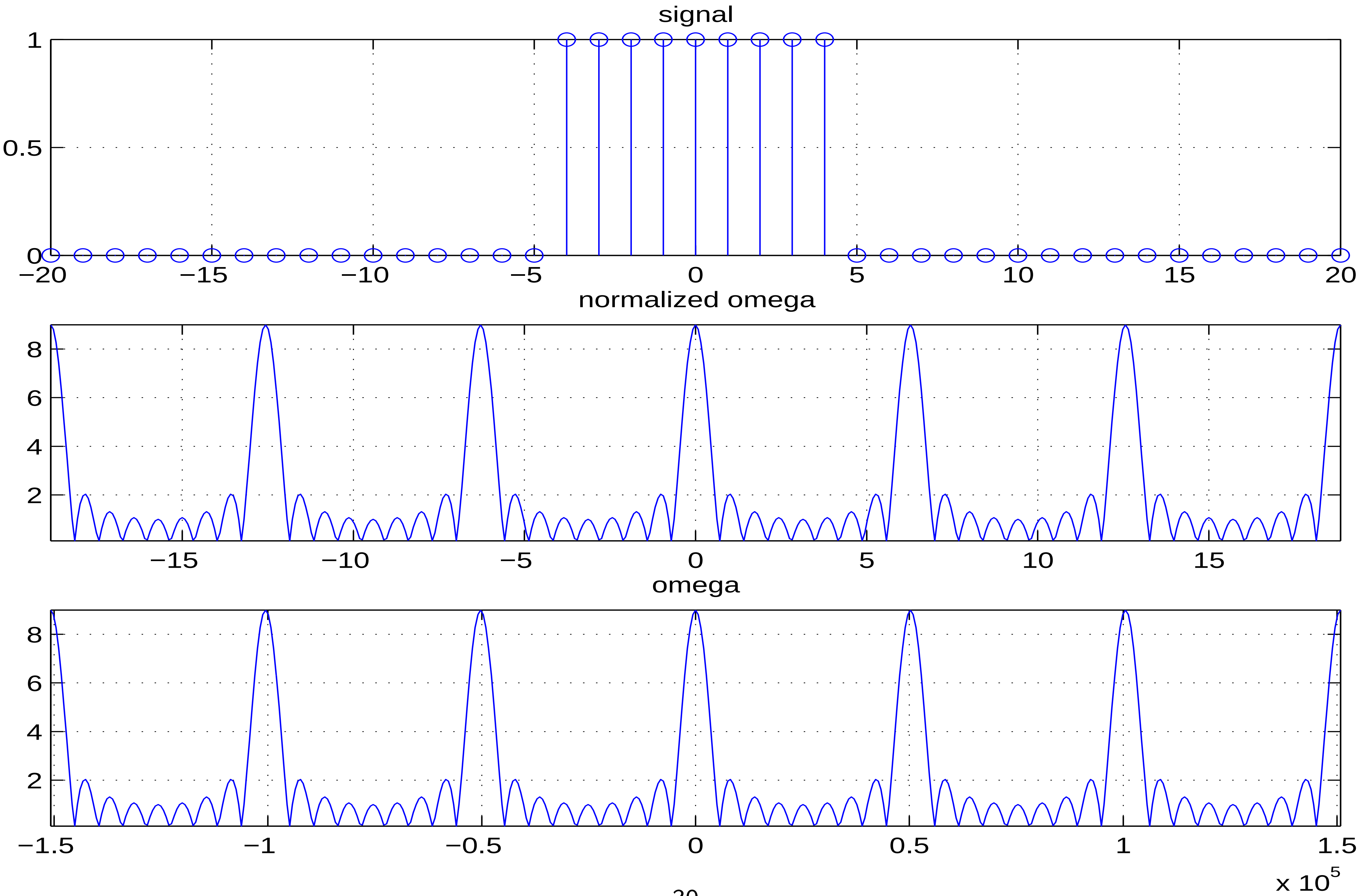
For a discrete signal we introduce notation $X(e^{j\omega})$:

$$\boxed{\tilde{X}(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}}$$

This transform is called **Fourier transform with discrete time**, or **Discrete-time Fourier transform** and short-cut **DTFT**. Tilde above $\tilde{X}$ means that DTFT is periodic function. It can be denoted as: $x[n] \stackrel{DTFT}{\longrightarrow} \tilde{X}(e^{j\omega})$.

When protting spectrum of a discrete signal we should be careful with the time axis. ω in $\tilde{X}(e^{j\omega})$ is normalized (in the equation we use indeces n bu no real time). To obtain real angular frequency, we multiply ω by sampling frequency F_s; to obtain frequency in Herz, we additionally divide real angular frequecy by 2π.

Example : Given discrete square impuls of length 9, sampling frequency $F_s = 8000$ Hz. Width of the square in real time: $\vartheta = 9T$. Maximum value in spectrum for the non-samped signal $D\vartheta$. For the sampled version: $\frac{D\vartheta}{T} = 9$. First intersection of spectral function with ω axis for normal angular frequency: $\frac{2\pi}{\vartheta} = 5585$ rad/s.



signal
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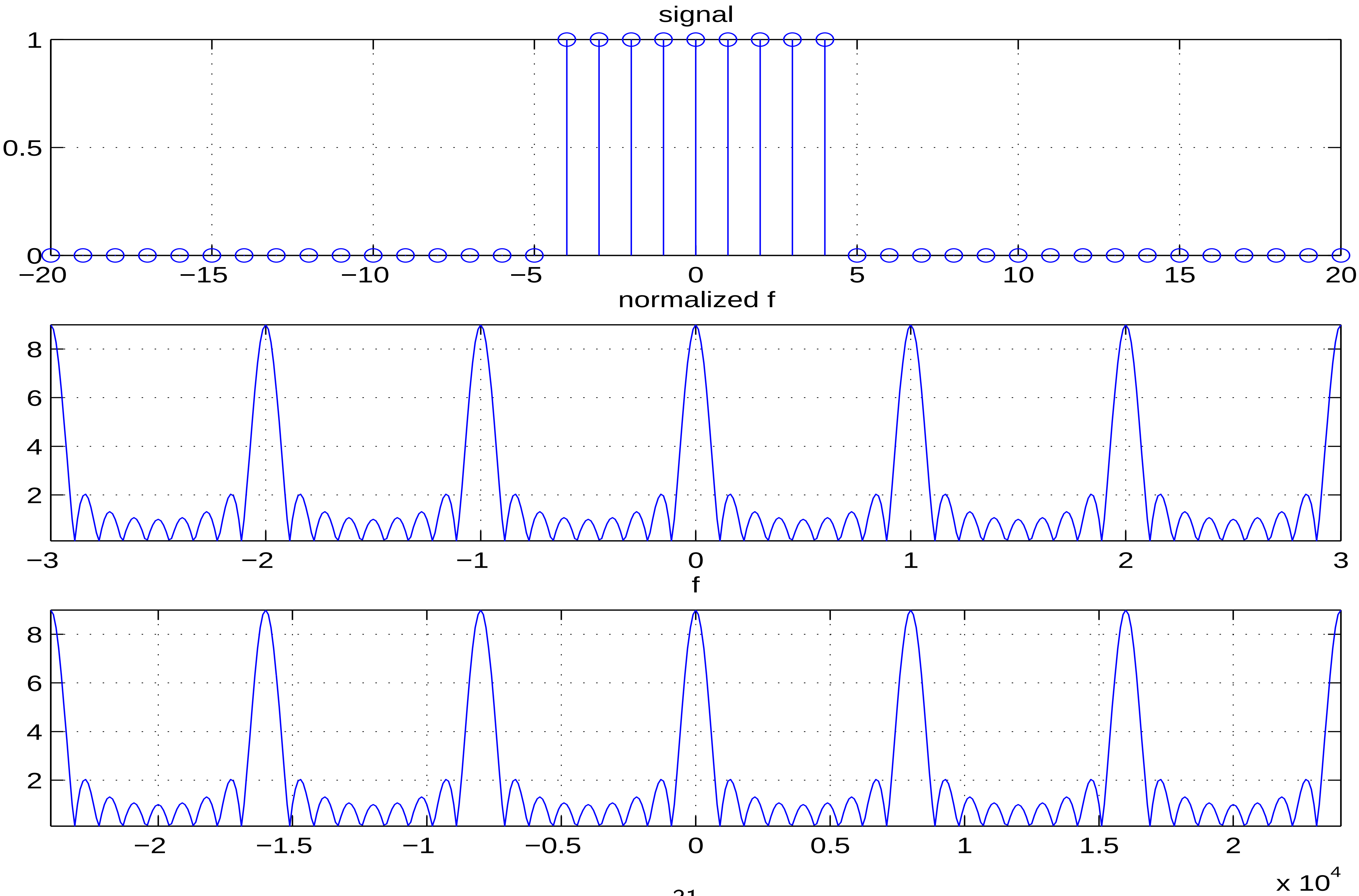
signal
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−2
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−1
−0.5
0
0.5
1
1.5
2
x 10^4

Periodicity of spectrum:

- for normalized circular frequencies: 2π rad
- for circular frequencies: $2\pi F_s$ rad/s
- for normalized frequencies: 1
- for standard frequencies: F_s Hz

Inverse DTFT:

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \tilde{X}(e^{j\omega}) e^{+j\omega n} d\omega$$

…we integrate only over one period in frequency. $[-\pi, +\pi]$ is related to normalized angular frequencies $[-\frac{F_s}{2}, +\frac{F_s}{2}]$.

To remember about DTFT:

- is periodic because the signal is discrete.
- is defined for all ω, because the signal is arbitrary.
- can be plotted in different frequency axis (angular, normalized,...).

Discrete Fourier Series

is used for frequency analysis of discrete periodic signals:

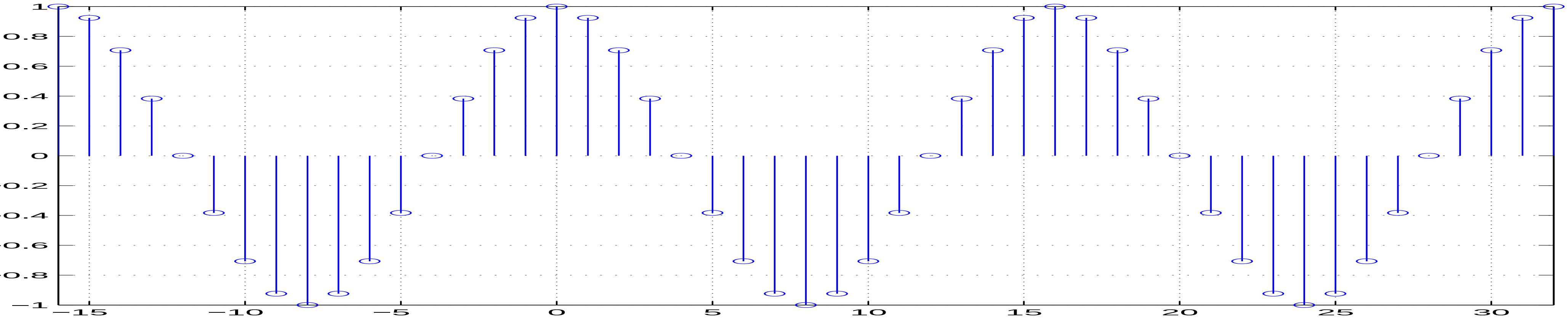
- Siganl is discrete, thus we expect something periodic in spectrum.
- Siganl is periodic, thus we expect discrete spectrum (coefficients, not function).

The two conditions imply that for a periodic discrete signal, spectrum is composed of a finite number of coefficients which means, it is something we can finally compute.

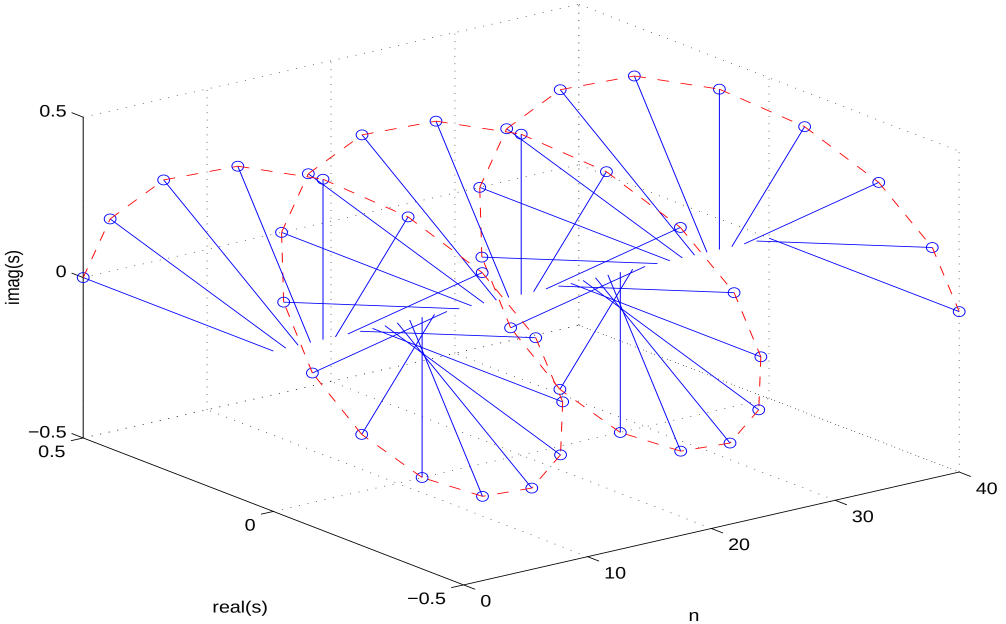
For a periodic sequence $\tilde{x}[n]$ with period N, define fundamental angular frequency:

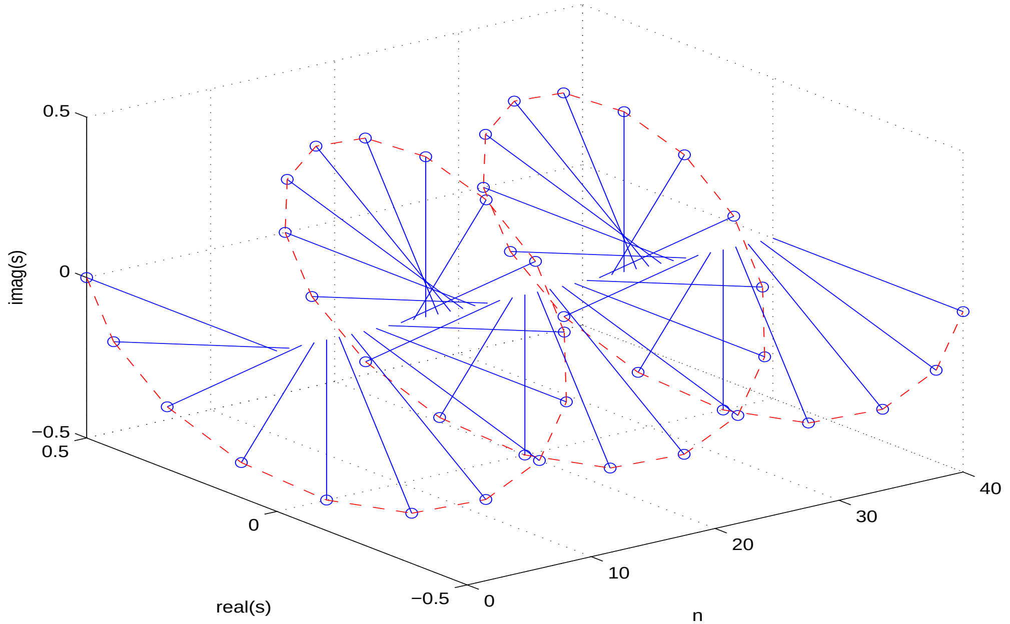
$$\omega_1 = \frac{2\pi}{N}$$

Example: cosine function with period $N = 16$ and angular frequency $\omega_1 = \frac{2\pi}{16} = \frac{\pi}{8}$ is defined as $x[n] = \cos(\frac{\pi}{8}n)$



This cosine can be decomposed into two functions: $x[n] = \cos(\frac{\pi}{8}n) = \frac{1}{2}e^{j\frac{\pi}{8}n} + \frac{1}{2}e^{-j\frac{\pi}{8}n}$





Similarly as for continuous time signals, we can express an arbitrary discrete-time periodic signal as a sum of complex exponentials with $\omega_1 = \frac{2\pi}{N}$:

$$\tilde{x}[n] = \frac{1}{N} \sum_{k=\{N\}} \tilde{X}[k] e^{j\frac{2\pi}{N}kn}$$

The equation is called **Discrete Fourier Series – DFS**. Coefficients of DFS are computed as :

$$\tilde{X}[k] = \sum_{n=\{N\}} \tilde{x}[n] e^{-j\frac{2\pi}{N}kn}$$

Details:

- $\tilde{X}[k]$ is a k-th coefficient of DFS. It is tied with an exponential function with frequency: $\frac{2\pi}{N}k$.
- again we see the sign "-" when go from time to frequency and "$+$" when go back.

- indeces of the sum, $\{N\}$, mean summing over an arbitrary period. In case of continuous FS we were integrating over one period in time domain. Same as for discrete periodic
  signals, spectrum is periodic too. For both n and k are usually integrate over $[0, N-1]$:

$$\boxed{\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n] e^{-j\frac{2\pi}{N}kn}} \qquad \boxed{\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1} \tilde{X}[k] e^{j\frac{2\pi}{N}kn}}$$

Why are DFS coefficients periodic?

because function $e^{-j\frac{2\pi}{N}kn}$ is the same for $k = k + gN$:

$$\frac{2\pi}{N}(k + gN)n = \frac{2\pi}{N}kn + \frac{2\pi}{N}gNn = \frac{2\pi}{N}kn + 2\pi gn$$

and we know, that function $e^{j\cdot}$ is periodic with period 2π.

Consequences:

- if function $e^{-j\frac{2\pi}{N}kn}$ is the same to $e^{-j\frac{2\pi}{N}(k+gN)n}$, then also coefficients are the same:

$$\tilde{X}[k] = \tilde{X}[k+gN]$$

- furthermore, if the signal is real: $\tilde{x}[n]$, positive and negative coefficients are complex conjugate:

$$\tilde{X}[k] = \tilde{X}^\star[-k]$$

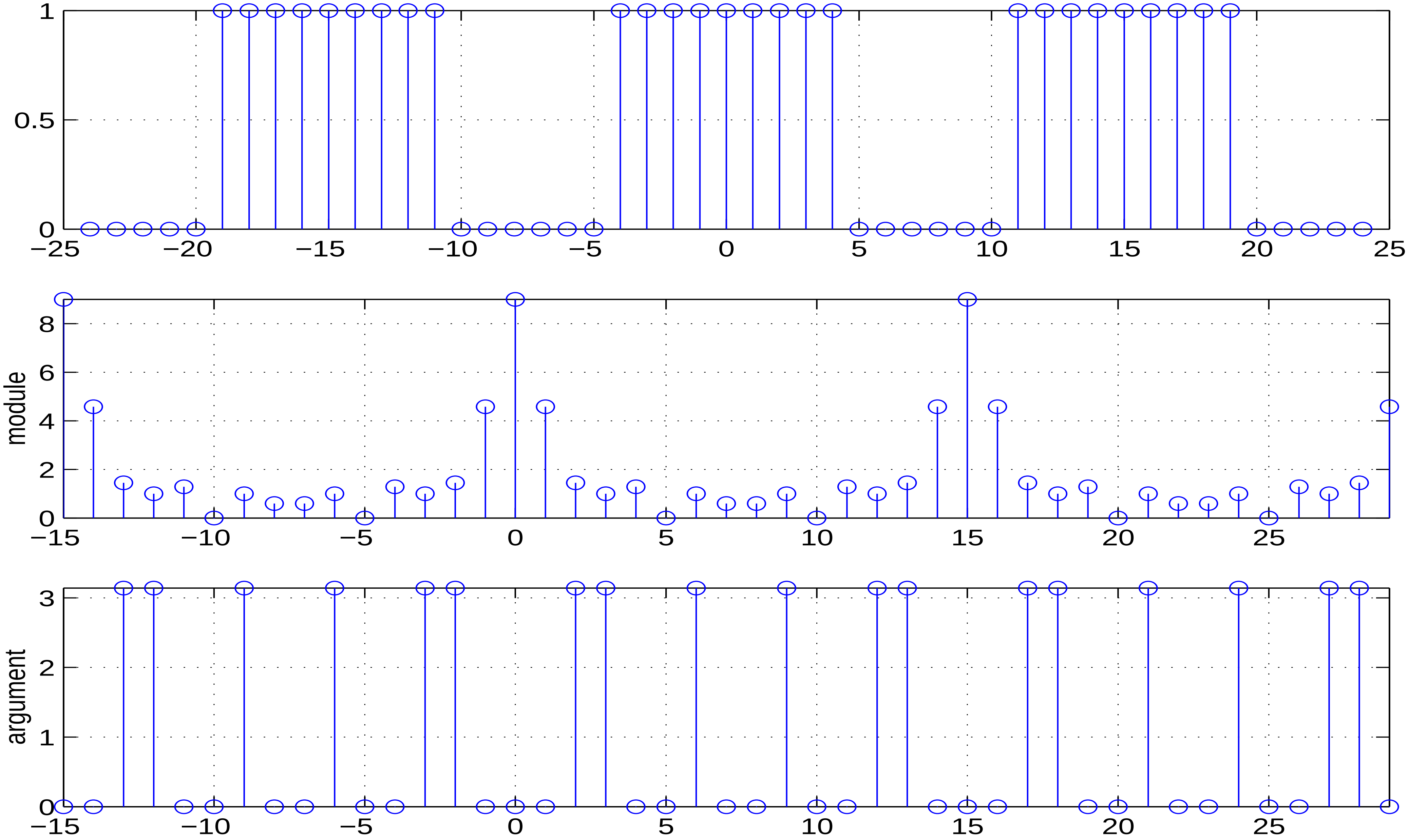
therefore

$$\tilde{X}[k] = \tilde{X}^\star[gN-k]$$

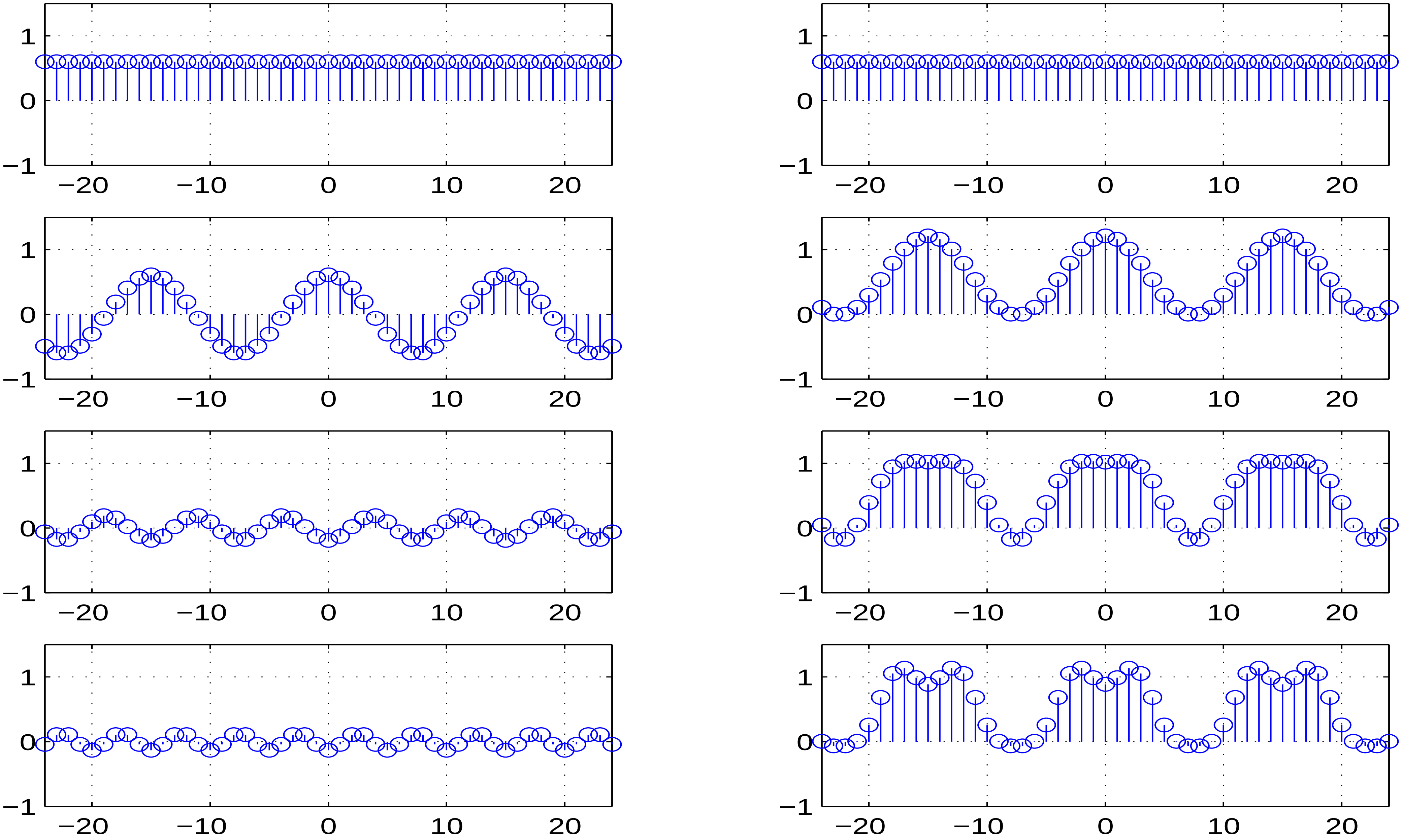
- now we know why for synthesis of the signal we take into account just one period of k:

$\tilde{x}[n] = \dfrac{1}{N}\displaystyle\sum_{k=0}^{N-1} \tilde{X}[k]e^{j\frac{2\pi}{N}kn}$ $k = \{N\}$ – all others are same.

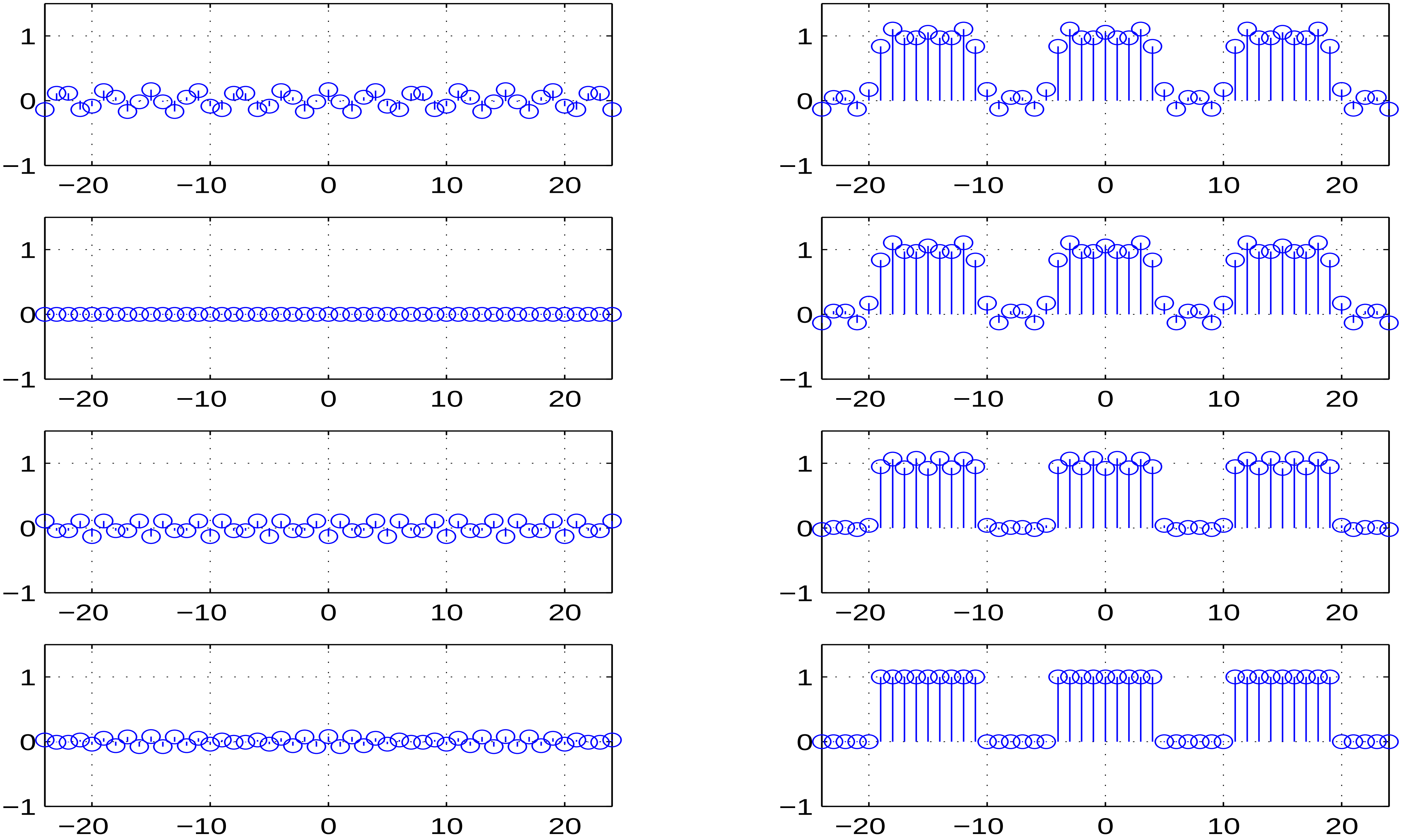
Example: periodic sequence of square impulses, $N = 15$, with length of square is 9.



Synthesis: $k = 0, 1, 2, 3$



Synthesis: $k = 4, 5, 6, 7$, no more, becasue we already used $k = 8......15 - 7 = 8$.



DFS of harmonic signals with period N

a cosine with period N can be written as :

$$x[n] = \cos(\frac{2\pi}{N}n) = \frac{1}{2}e^{j\frac{2\pi}{N}n} + \frac{1}{2}e^{-j\frac{2\pi}{N}n}$$

General cosine can be decomposed as:

$$x[n] = C_1\cos(\frac{2\pi}{N}n + \phi_1) = \frac{C_1}{2}e^{j\frac{2\pi}{N}n + j\phi_1} + \frac{C_1}{2}e^{-j\frac{2\pi}{N}n - j\phi_1}$$

Now compare it to the synthesis formula from DFS: $\tilde{x}[n] = \frac{1}{N}\sum_{n=\{N\}} \tilde{X}[k]e^{j\frac{2\pi}{N}kn}$, we find that only two coefficients are non-zero in one period N:

$$\tilde{X}[1] = \frac{NC_1}{2}e^{j\phi_1} \qquad \tilde{X}[-1] = \frac{NC_1}{2}e^{-j\phi_1}$$

In period $[0, N-1]$ (that we usually use), $\tilde{X}[-1]$ is projected as $\tilde{X}[N-1]$. Thus:

$$|\tilde{X}[1]| = |\tilde{X}[N-1]| = \frac{NC_1}{2} \qquad \arg\tilde{X}[1] = -\arg\tilde{X}[N-1] = \phi$$

Example: $N = 15$, $C_1 = 1$, $\phi = \frac{\pi}{4}$.