Periodic signals - Fourier series

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- Why do we like exponentials.
- Fourier series.
- Basis.
- FS of square signals.
- Example from real world.
- Hints.
- Mean power and Parseval theorem.

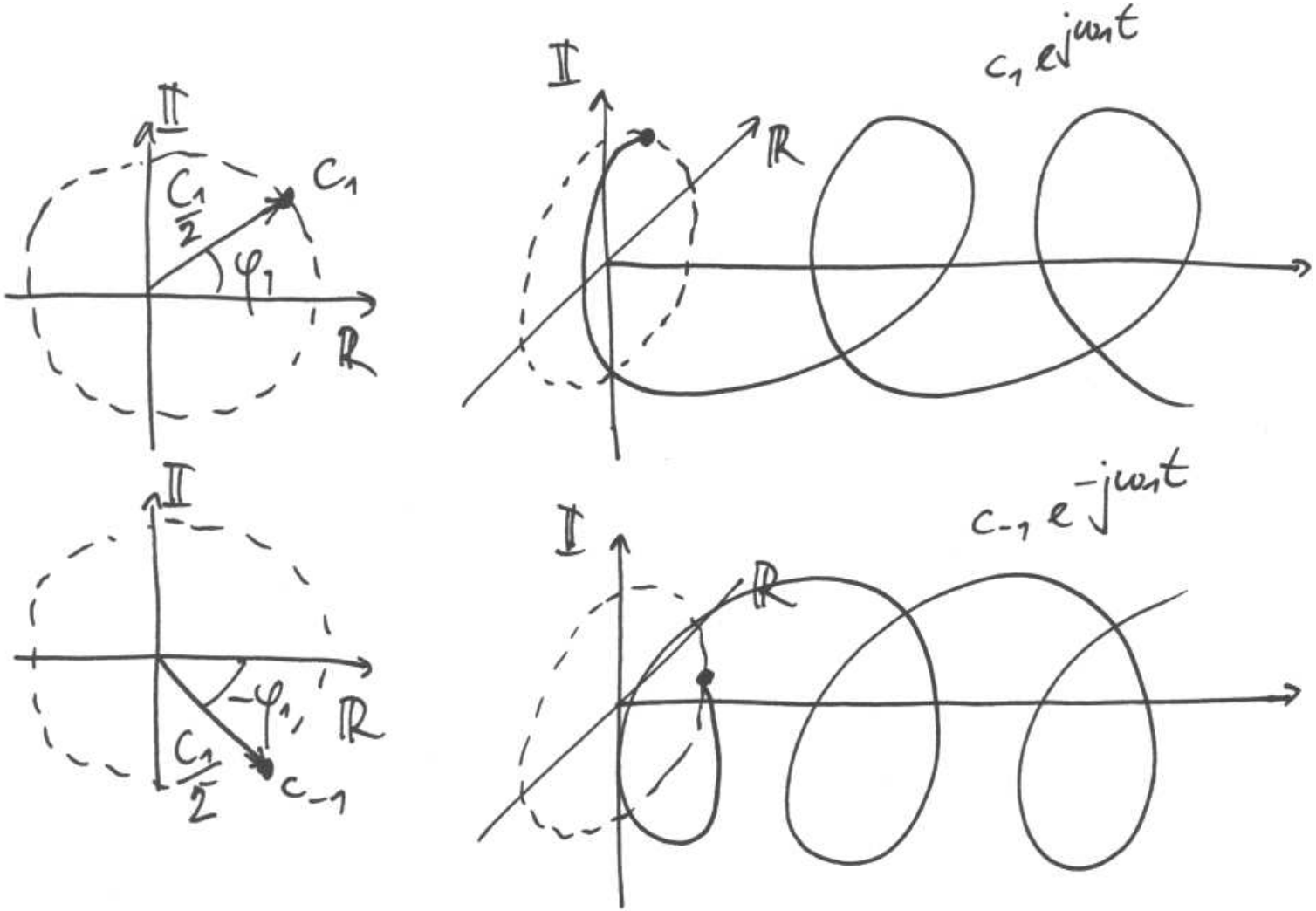
By means of FS, we can decompose a periodic signal into a series of complex exponentials.
$\Rightarrow$ **Any cosine function with an arbitrary phase can be expressed as sum of exponentials** $\cos(x) = \dfrac{e^{jx} + e^{-jx}}{2}$:

$$C_1 \cos(\omega_1 t + \phi_1) =$$

$$= \tfrac{C_1}{2} e^{j(\omega_1 t + \phi_1)} + \tfrac{C_1}{2} e^{-j(\omega_1 t + \phi_1)} =$$

$$= \tfrac{C_1}{2} e^{j\phi_1} e^{j\omega_1 t} + \tfrac{C_1}{2} e^{-j\phi_1} e^{-j\omega_1 t},$$

where $c_1 = \frac{C_1}{2} e^{j\phi_1}$, $c_{-1} = \frac{C_1}{2} e^{-j\phi_1}$ are complex constants.



$\Rightarrow$ **Transfer of an exponential through an LTI system results in the original exponential scaled (multiplied) by a constant:** the input signal is defined as $x(t) = e^{st}$, where s is a complex number. Mainly, we will be interested in the case when s is purely imaginary, $s = j\omega$. However, we will demonstrate the computation on a general complex s.

LTI system output is given by convolution of the input signal and the impulse response:

$$y(t) = h(t) \star x(t) = \int_{-\infty}^{+\infty} h(\tau)x(t-\tau)d\tau = \int_{-\infty}^{+\infty} h(\tau)e^{s(t-\tau)}d\tau$$

The expression $e^{s(t-\tau)}$ can be decomposed (using equation $e^{a+b} = e^a e^b$) and the component independent of τ is moved in front of the integral

$$y(t) = \int_{-\infty}^{+\infty} h(\tau)e^{st}e^{-s\tau}d\tau = e^{st}\int_{-\infty}^{+\infty} h(\tau)e^{-s\tau}d\tau.$$

The integral becomes now a function of the complex number s and the system's impulse response, but NOT a function of the input! Let us denote the integral as a complex function $H(s)$. Factor e^{st} is the input, thus we can write:

$$y(t) = e^{st}H(s),$$

Here we see that the output is the same exponential multiplied by the constant defined as:

$$H(s) = \int_{-\infty}^{+\infty} h(\tau)e^{-s\tau}d\tau.$$

... what happens when we multiply complex exp. $c_1 e^{j\omega_1 t}$ by complex $H(s)$?

Short recap:

- Every complex number can be decomposed into a module and an agument (phase): $H(s) = |H(s)|e^{j \arg H(s)}$.
- Multiplication rule: $z_1 z_2 = r_1 r_2 e^{j(\phi_1+\phi_2)}$.

How will it be? We start off with:

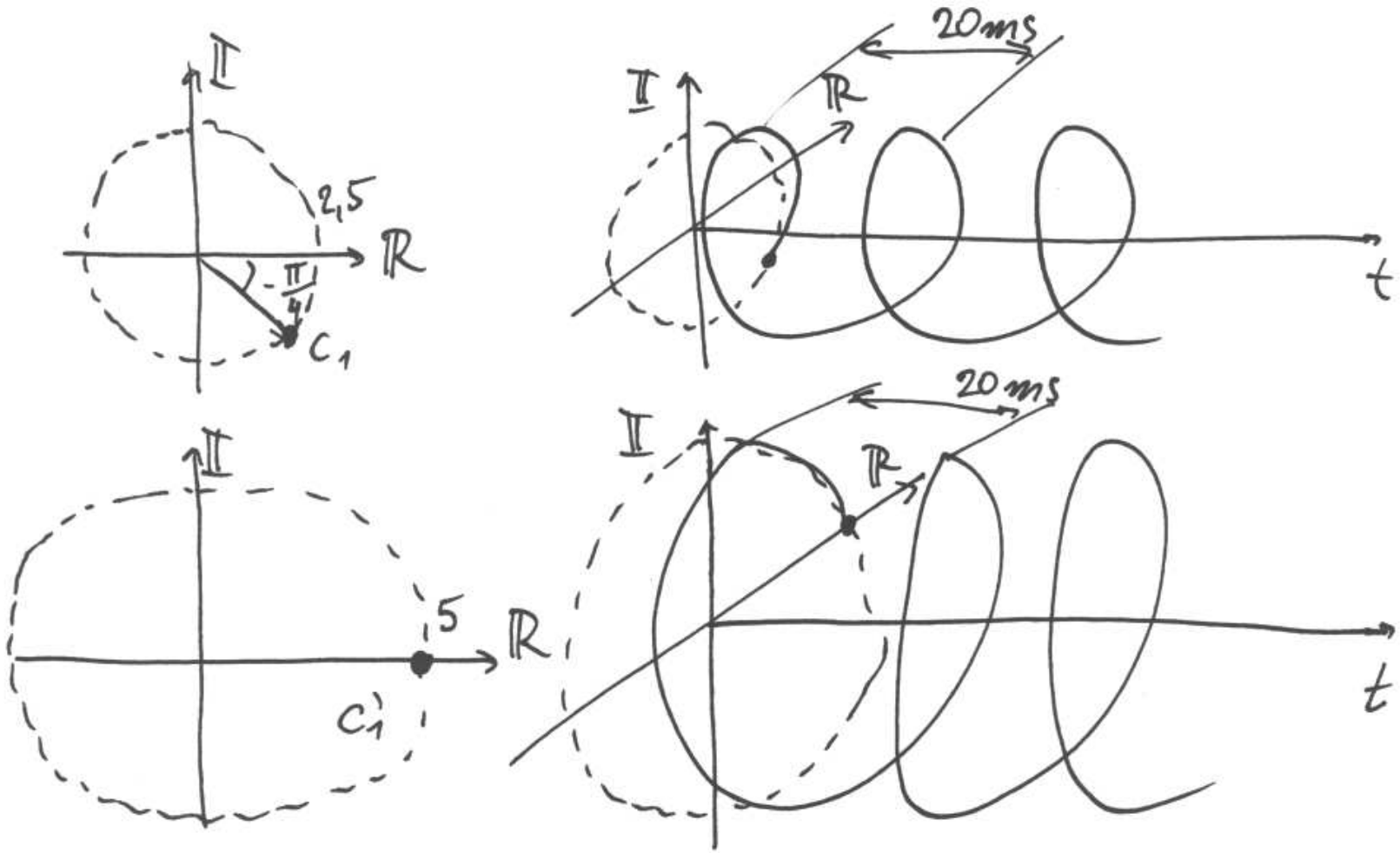
$$H(s)c_1 e^{j\omega_1 t} = \dots$$

We do nothing with $e^{j\omega_1 t}$ and define a new complex constant where $c_1' = H(s)c_1$.

As both $H(s)$ and c_1 are complex, during multiplication we multiply the modules and add up the arguments. The output of LTI system is thus a complex exponential that is scaled (wider or thinner) and has a different phase. The "speed" of the exponential does not change relative the original input signal.

Example: complex exponential $c_1 e^{j\omega_1 t} = 2.5 e^{-j\frac{\pi}{4}} e^{j100\pi t}$ is multiplied by $H(s) = 2e^{j\frac{\pi}{4}}$.

$$c_1' = c_1 H(s) = 2.5 e^{-j\frac{\pi}{4}} 2 e^{j\frac{\pi}{4}} = 2.5 \times 2 e^{-j\frac{\pi}{4} + j\frac{\pi}{4}} = 5$$

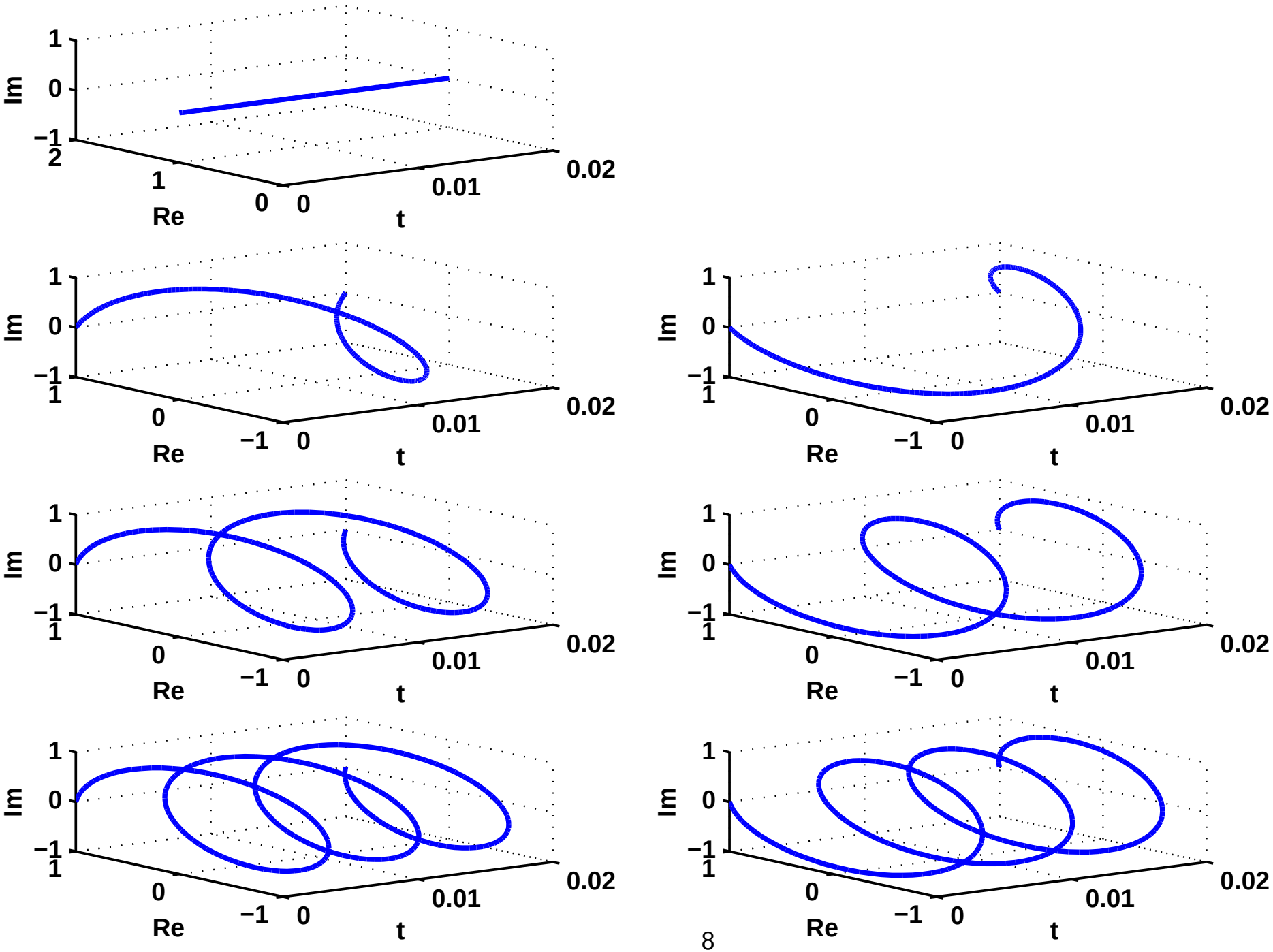


FOURIER SERIES

A periodic signal $x(t) = x(t + T_1)$, with fundamental period T_1, is decomposed into a series of exponentials:

$$\boxed{x(t) = \sum_{k=-\infty}^{+\infty} c_k e^{jk\omega_1 t}}$$

ω_1 is the fundamental period of the signal, $\omega_1 = \frac{2\pi}{T_1}$. Function $e^{jk\omega_1 t}$ for $k = 0, \pm 1, \pm 2 \ldots$ is called **harmonic complex exponential**.


Im
Re
t
1
0
−1
2
1
0
0
0.01
0.02

Basic properties of the coefficients:

- For real signals $x(t)$: c_k and c_{-k} are always **complex conjugate**:

$$c_k = c_{-k}^{\star}$$

- c_0 is complex conjugate with itself, thus is real: $c_0 \in \Re$. As $e^{jk\omega_1 t}$ equals to 1 for $k = 0$, c_0 must be **direct component**.
- We can rewrite the series into the form:

$$s(t) = c_0 + \sum_{k=1}^{\infty} \underbrace{[c_k e^{jk\omega_1 t} + c_{-k} e^{-jk\omega_1 t}]}_{C_k \cos(k\omega_1 t + \phi_k)}$$

- for $k > 0$
  - magnitude of the k-th harmonic component is $C_k = 2|c_k|$.
  - initial phase of the k-th harmonic is $\phi_k = \arg c_k$.

Recapitulation - basis and projection into basis

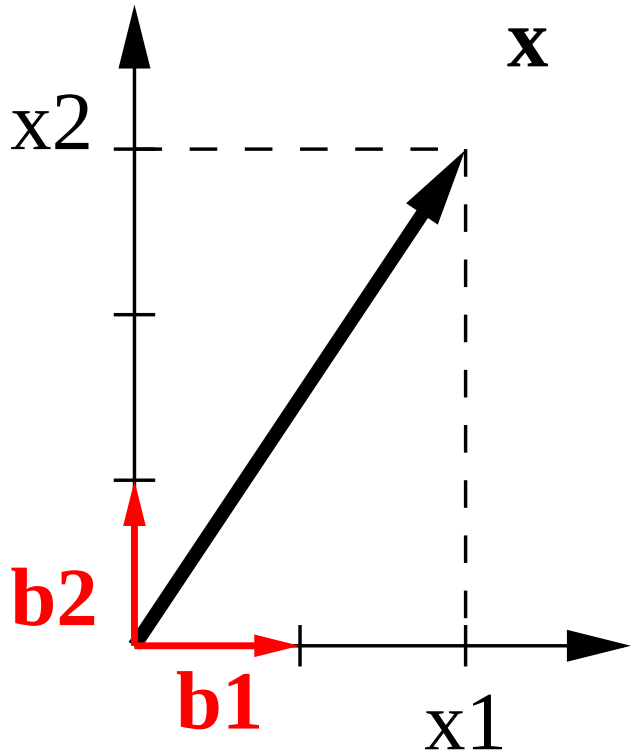
Basis

- we express one 'thing' by means of some other 'thing'.
- how similar is some input (vector, signal, function) to some reference.

$$\mathbf{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\mathbf{b}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \mathbf{b}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$x_1 = [1 \;\; 0] \begin{bmatrix} 2 \\ 3 \end{bmatrix} = 2 \quad x_2 = [0 \;\; 1] \begin{bmatrix} 2 \\ 3 \end{bmatrix} = 3$$



. . . **scalar product**

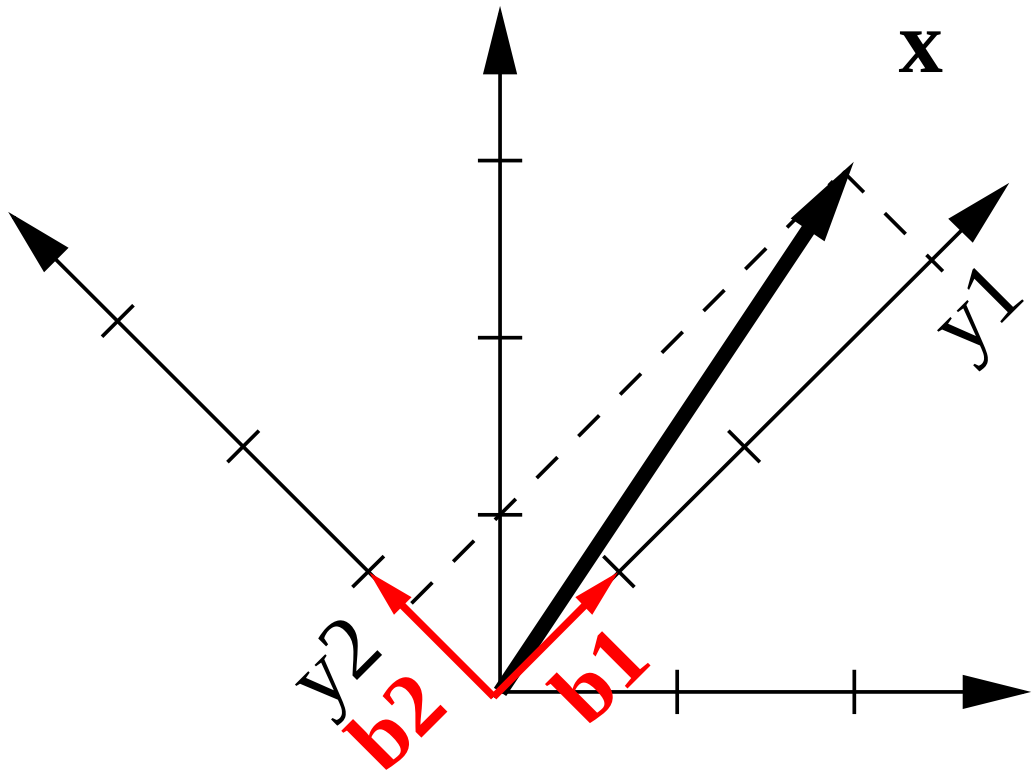
Rotated axes

$$\mathbf{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\mathbf{b}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \quad \mathbf{b}_2 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$y_1 = [\tfrac{1}{\sqrt{2}} \quad \tfrac{1}{\sqrt{2}}] \begin{bmatrix} 2 \\ 3 \end{bmatrix} = 3.53$$

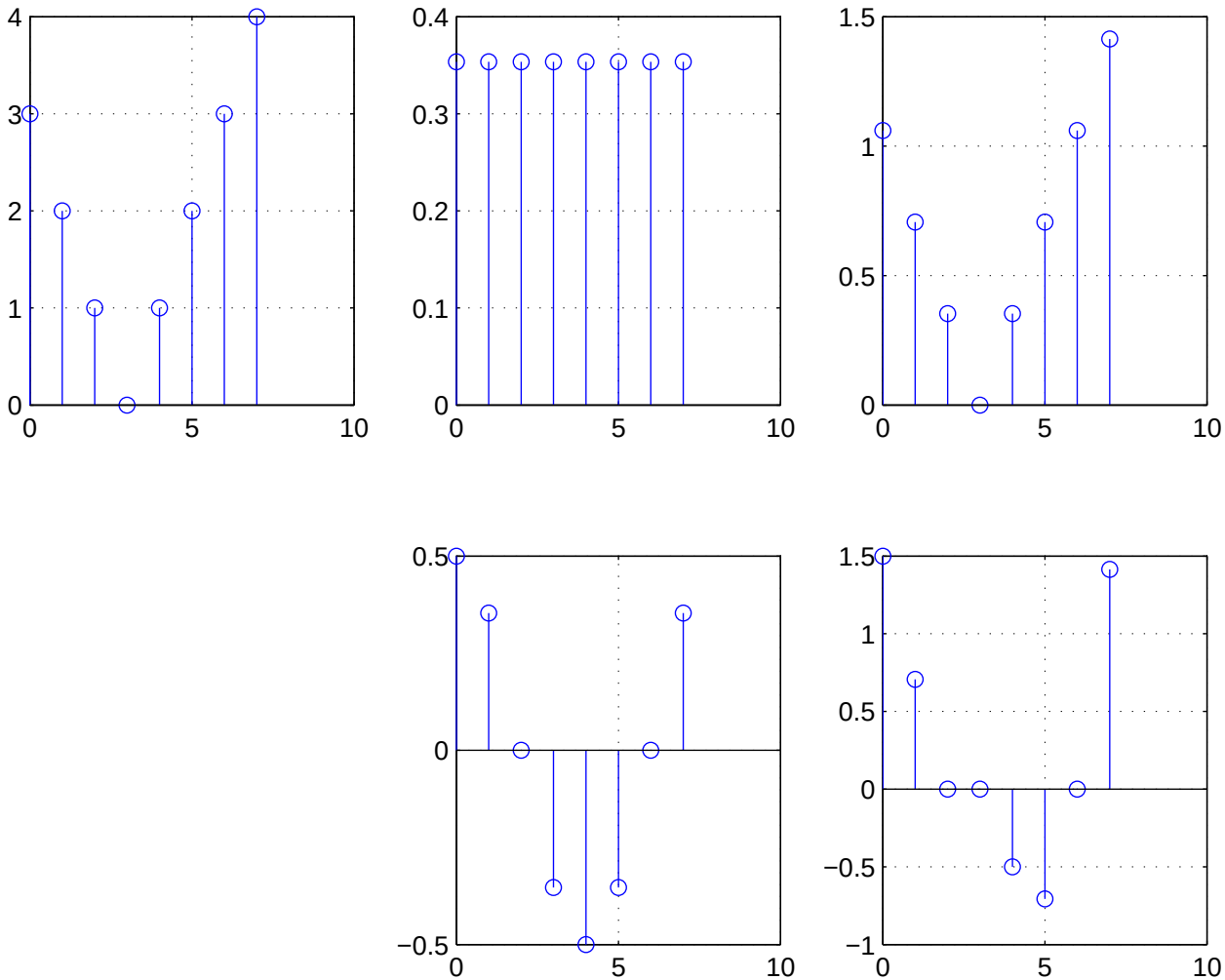
$$y_2 = [-\tfrac{1}{\sqrt{2}} \quad \tfrac{1}{\sqrt{2}}] \begin{bmatrix} 2 \\ 3 \end{bmatrix} = 0.707$$



How can we plot a higher dimensional space ?

$\mathbf{x} = [3 \;\; 2 \;\; 1 \;\; 0 \;\; 1 \;\; 2 \;\; 3 \;\; 4]$, $\mathbf{b}_1 = \sqrt{\frac{1}{8}}[1 \;\; 1 \;\; 1 \;\; 1 \;\; 1 \;\; 1 \;\; 1 \;\; 1]$, $b_{2n} = \frac{1}{2}\cos(2\pi/8n)$

$y_1 = 5.65$, similarity $y_2 = 2.41$ similarity.



General expression in an arbitrary dimensional space...

vector representation $y_n = \mathbf{b}_n^T \mathbf{x}$,

matrix representation $\mathbf{y} = \mathbf{B}\mathbf{x}$

Properties of basis

- One coefficient should not be dependent on the others - ortogonality:

$$\mathbf{b}_i^T \mathbf{b}_j = 0$$

- All basis are of the same 'importance' - basis magnitude equal to 1

$$|\mathbf{b}_i| = 1$$

- Ortogonality and unit magnitude of basis gives us an **orthonormal system**

Signal and basis can be functions!

- one function can be represented by means of some other function
- how two functions are similar.

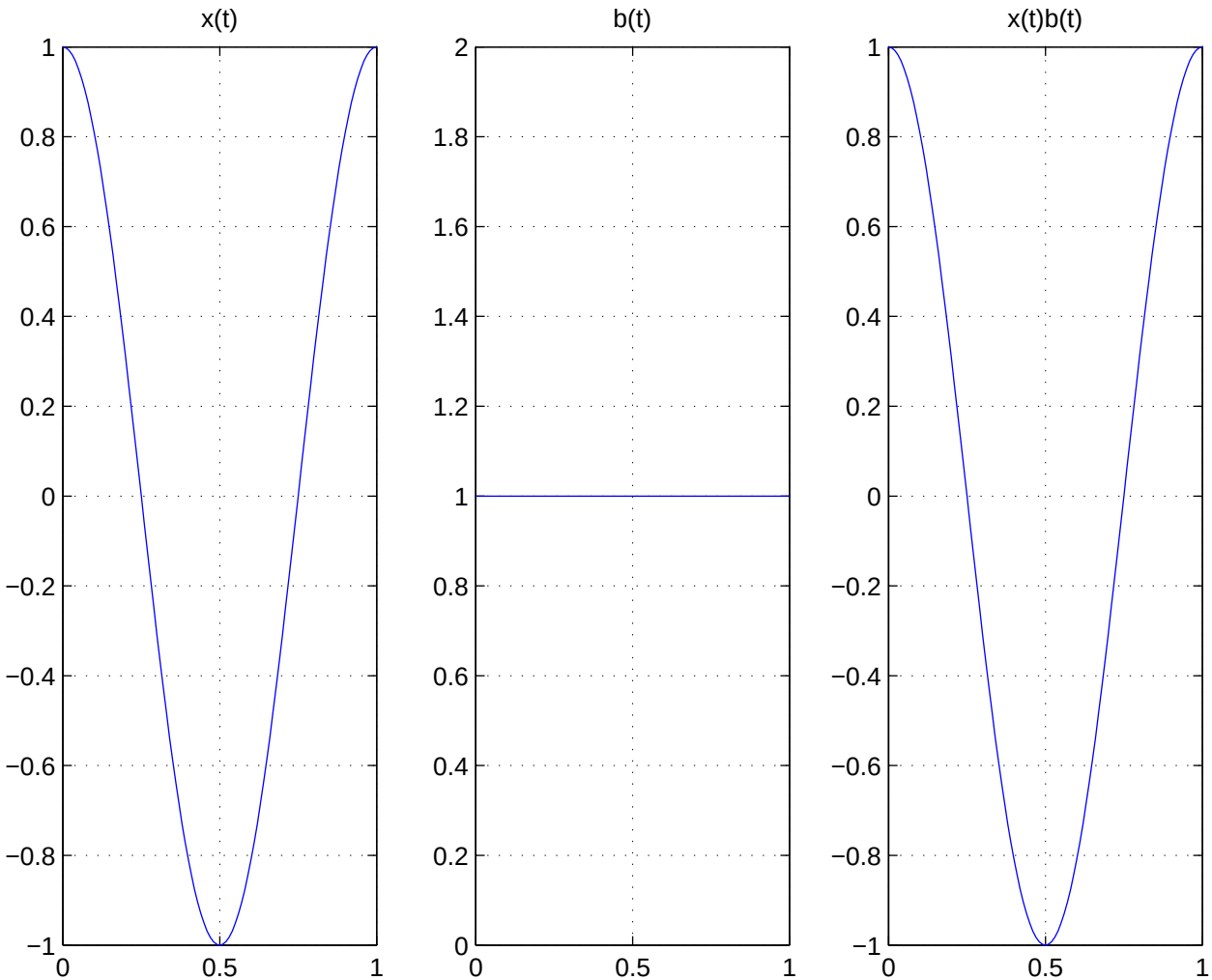
Analogiously to the scalar product:

- **multiplication with the basis**
- **adding up - integral $\int$ for continuous signals or sum $\sum$ for descrete**

Example - what is cosine similar to ?

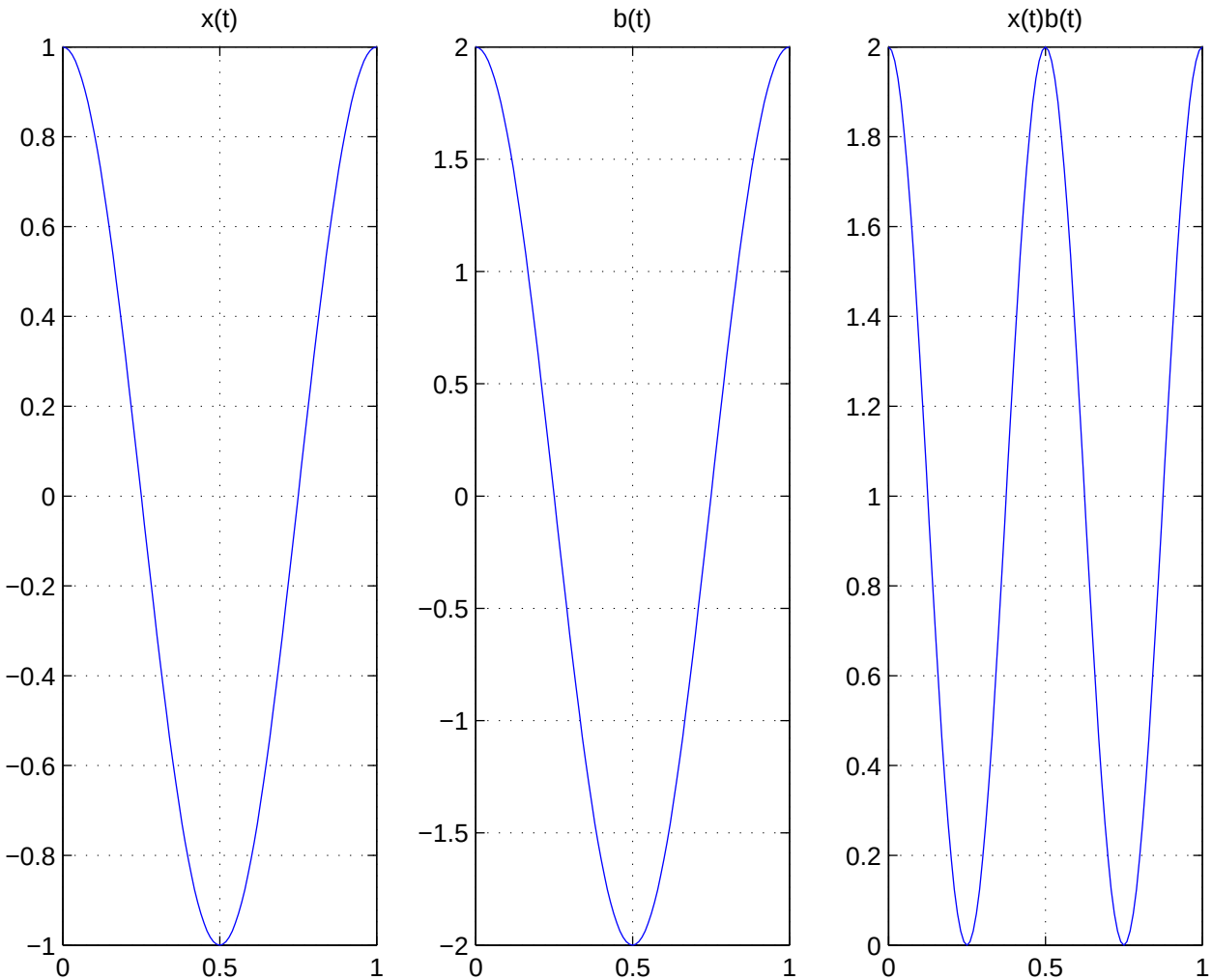
To a direct signal ?

$b(t) = 1 \quad \int_0^1 s(t)b(t)dt = 0$ ... is not similar



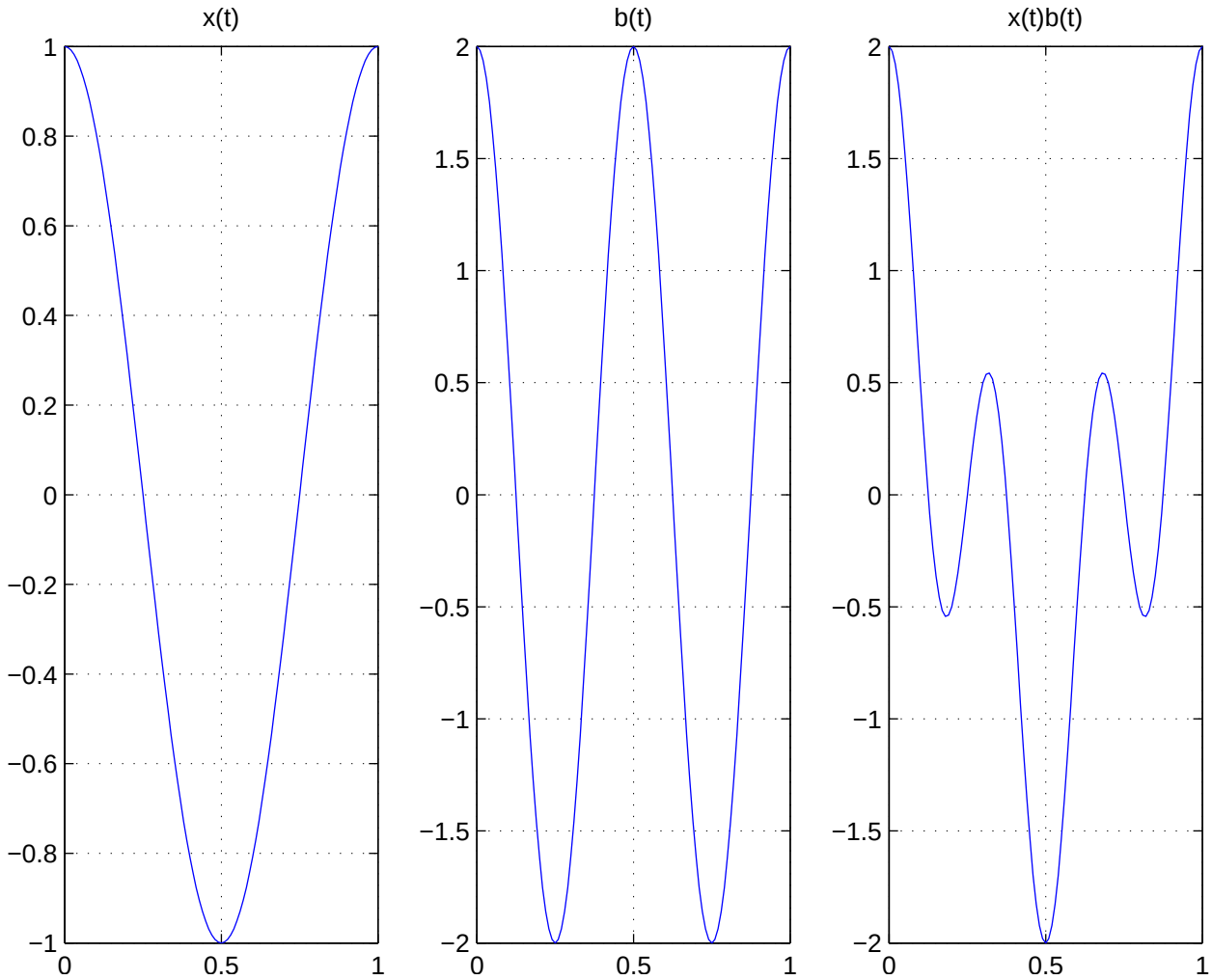
Maybe to a cosine function ?

$b(t) = 2\cos(2\pi t)$ $\quad \int_0^1 s(t)b(t)dt = 1$ ... is similar



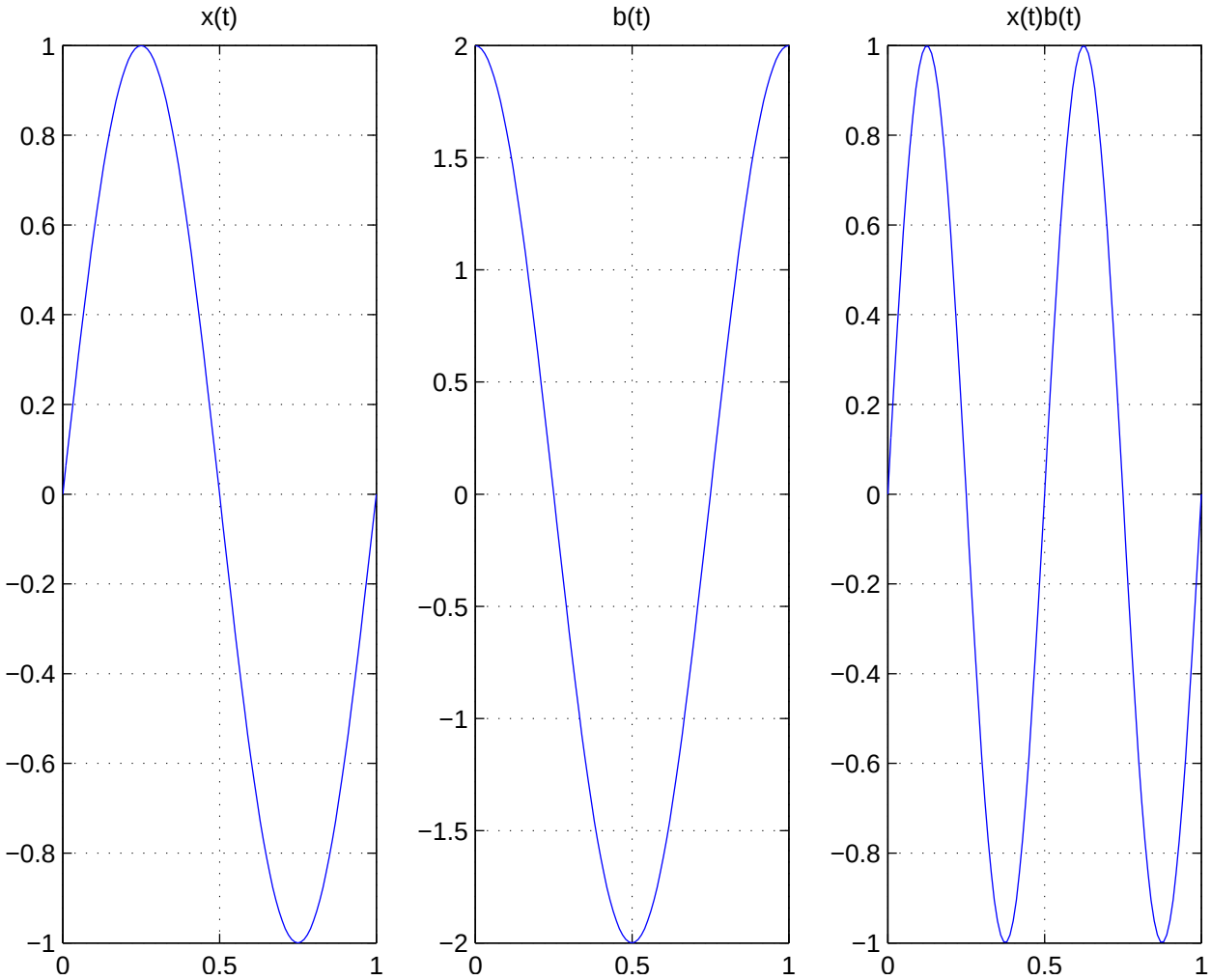
2x faster cosine ?

$b(t) = 2\cos(4\pi t)$ $\int_0^1 s(t)b(t)dt = 0$ ... is not similar



Now lets play with sine function $s(t) = \sin(2\pi t), \quad b(t) = 2\cos(2\pi t) \quad \int_0^1 s(t)b(t)dt = 0$

... no



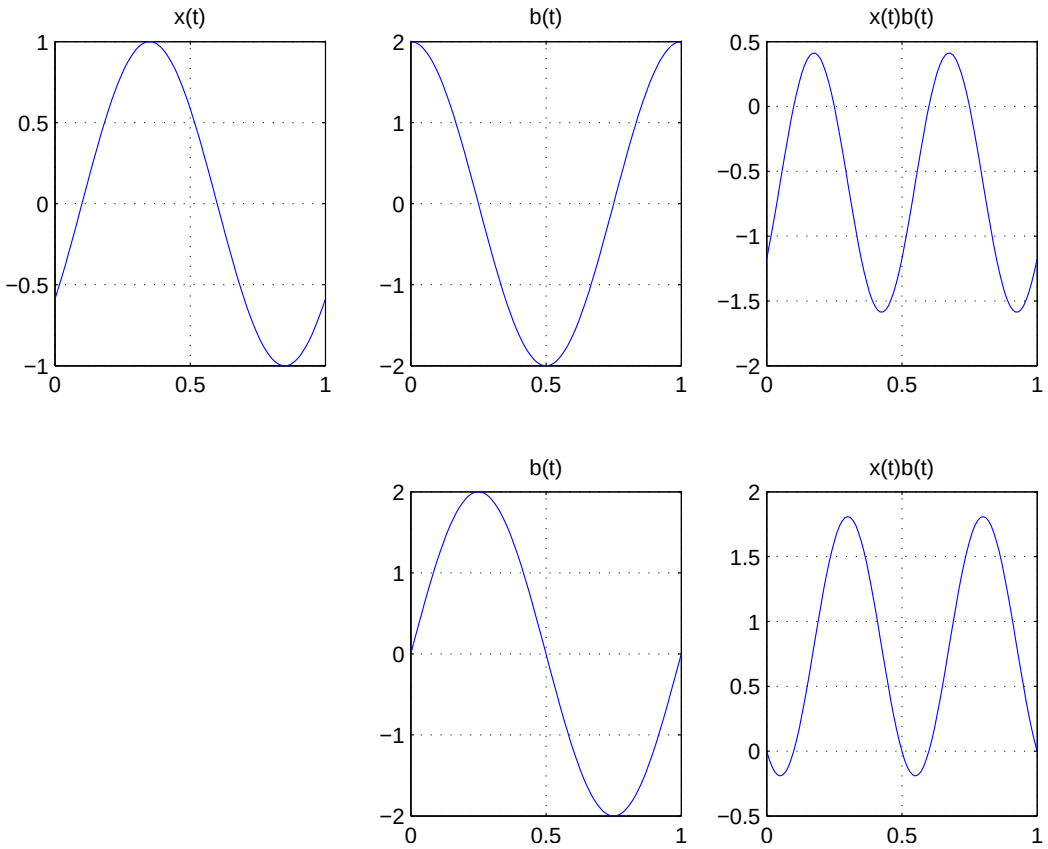
$\Rightarrow$ we will need cosine and sine !!!

Shifted sine ...

$$s(t) = \sin(2\pi t - \pi/5)$$

$$b_1(t) = 2\cos(2\pi t) \qquad \int_0^1 s(t)b_1(t)dt = -0.59$$

$$b_2(t) = 2\sin(2\pi t) \qquad \int_0^1 s(t)b_1(t)dt = 0.81$$



We will be working with complex exponentials !

that conjugate cosine and sine in themselves !

$b_1(t) = e^{j\omega t} = \cos \omega t + j \sin \omega t$

coefficient calculation: $c_1 = \int x(t) b_1^\star(t) dt$

product of such coefficient and the base is complex... so we need to add a complex cojugate base:

$b_{-1}(t) = e^{-j\omega t} = \cos \omega t - j \sin \omega t$

coefficient computation: $c_{-1} = \int x(t) b_{-1}^\star(t) dt$

Derivation of coefficients of FS

We said that a periodic function can be decomposed to the series:

$$\boxed{x(t) = \sum_{k=-\infty}^{+\infty} c_k e^{jk\omega_1 t}}$$

The coefficients c_k are computed using the formula:

$$\boxed{c_k = \frac{1}{T_1} \int_{T_1} x(t) e^{-jk\omega_1 t} dt,}$$

where $\int_{T_1}$ denotes integral over one period (typically, from 0 to T_1 or from $-\frac{T_1}{2}$ to $\frac{T_1}{2}$, but also can be from $25T_1$ to $26T_1$)

Plotting FS coefficients

The coefficients are indexed from $k = -\infty$ to ∞. Remember, they are tied with angular frequencies: c_1 is associated with ω_1, c_2 with $2\omega_1$ Coefficients with negative indeces are associated with corresponding negative frequencies: c_{-1} is associated with $-\omega_1$, c_{-2} with $-2\omega_1$ Negative frequency means that the exponential $c_{-k}e^{-jk\omega_1 t}$ is rotating the opposite direction that $c_k e^{jk\omega_1 t}$. Their composition will give us a normal cosine $C_k \cos(k\omega_1 t + \phi_k)$.

As the coefficients are complex numbers, we will plot their modules and aguments in separate graphs along frequency axes. Thus we are expressing a signal in frequency domain. Coefficients' position and value will form a **SPECTRUM** of a signal. For a periodic signal the spectrum is descrete (composed of vertical lines).

Example 1.

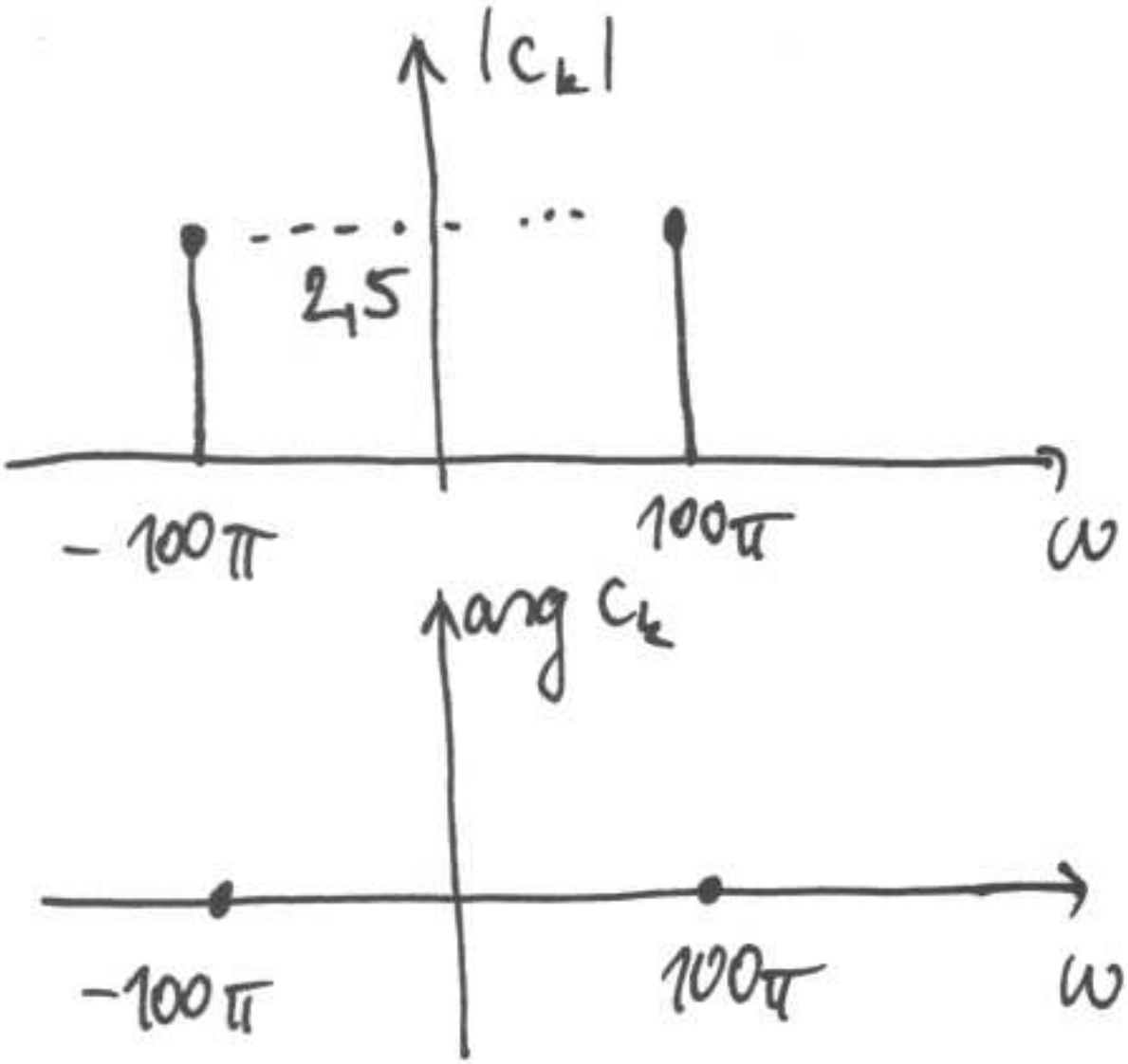
Derive FS coefficients for signal: $x(t) = 5\cos(100\pi t)$. The basic angular frequency is $\omega_1 = 100\pi$. In this example we don't integrate because we can rewrite cosine function as:

$$x(t) = \frac{5}{2}e^{j100\pi t} + \frac{5}{2}e^{-j100\pi t},$$

thus, the coefficients' values are :

$$c_1 = \frac{5}{2}, \qquad c_{-1} = \frac{5}{2}$$

The coefficients are real.


|c_k|
2,5
-100π
100π
ω
arg c_k
-100π
100π
ω

Example 2.

Derive FS coefficients of the signal: $x(t) = 5\cos(100\pi t - \frac{\pi}{4})$ Again, we will manage with no integration. Either we can rewrite cosine into the form:

$$2.5e^{-j\frac{\pi}{4}}e^{j100\pi t} + 2.5e^{+j\frac{\pi}{4}}e^{-j100\pi t},$$

thus, for c_1 and c_{-1} we get:

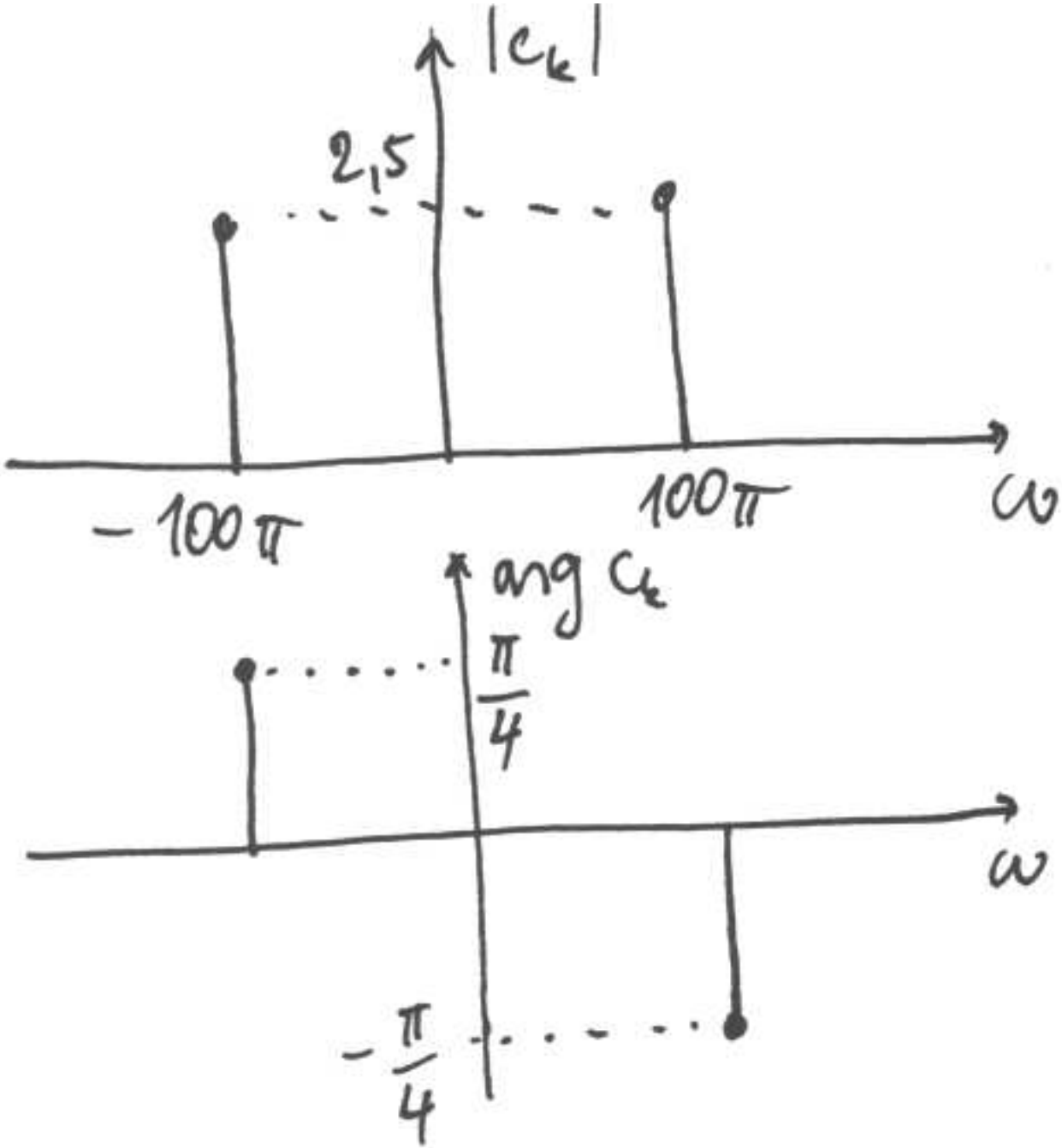
$$c_1 = 2.5e^{-j\frac{\pi}{4}}, \qquad c_{-1} = 2.5e^{+j\frac{\pi}{4}}$$

OR we look at how the coefficients c_k are related to the magnitude and initial phase of the harmonics (here, only one): $C_1 = 5$ and $\phi_1 = -\frac{\pi}{4}$.

$$|c_1| = \frac{C_1}{2} = 2.5, \quad \arg c_1 = \phi_1 = -\frac{\pi}{4}, \quad \text{so} \quad c_1 = 2.5e^{-j\frac{\pi}{4}}.$$

Coefficient c_{-1} is a complex conjugate to c_1 (same module, reverse angle):

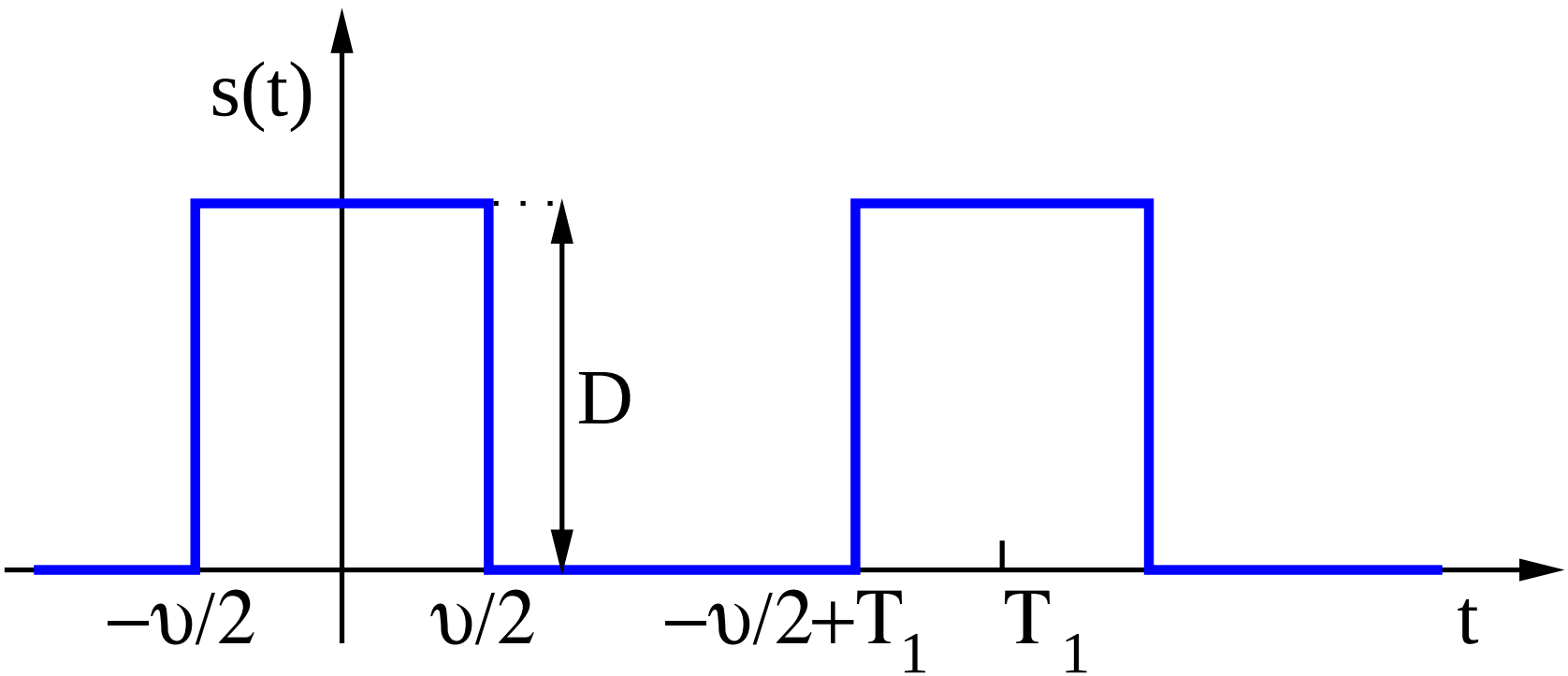
$$|c_{-1}| = \frac{C_1}{2} = 2.5, \quad \arg c_{-1} = -\phi_1 = \frac{\pi}{4}, \quad \text{so} \quad c_{-1} = 2.5e^{j\frac{\pi}{4}}.$$


$|c_k|$
2,5
-100π
100π
ω
$\arg c_k$
$\frac{\pi}{4}$
ω
$-\frac{\pi}{4}$

Example 3. . . . let's integrate!! We have a periodic signal of square impulses:

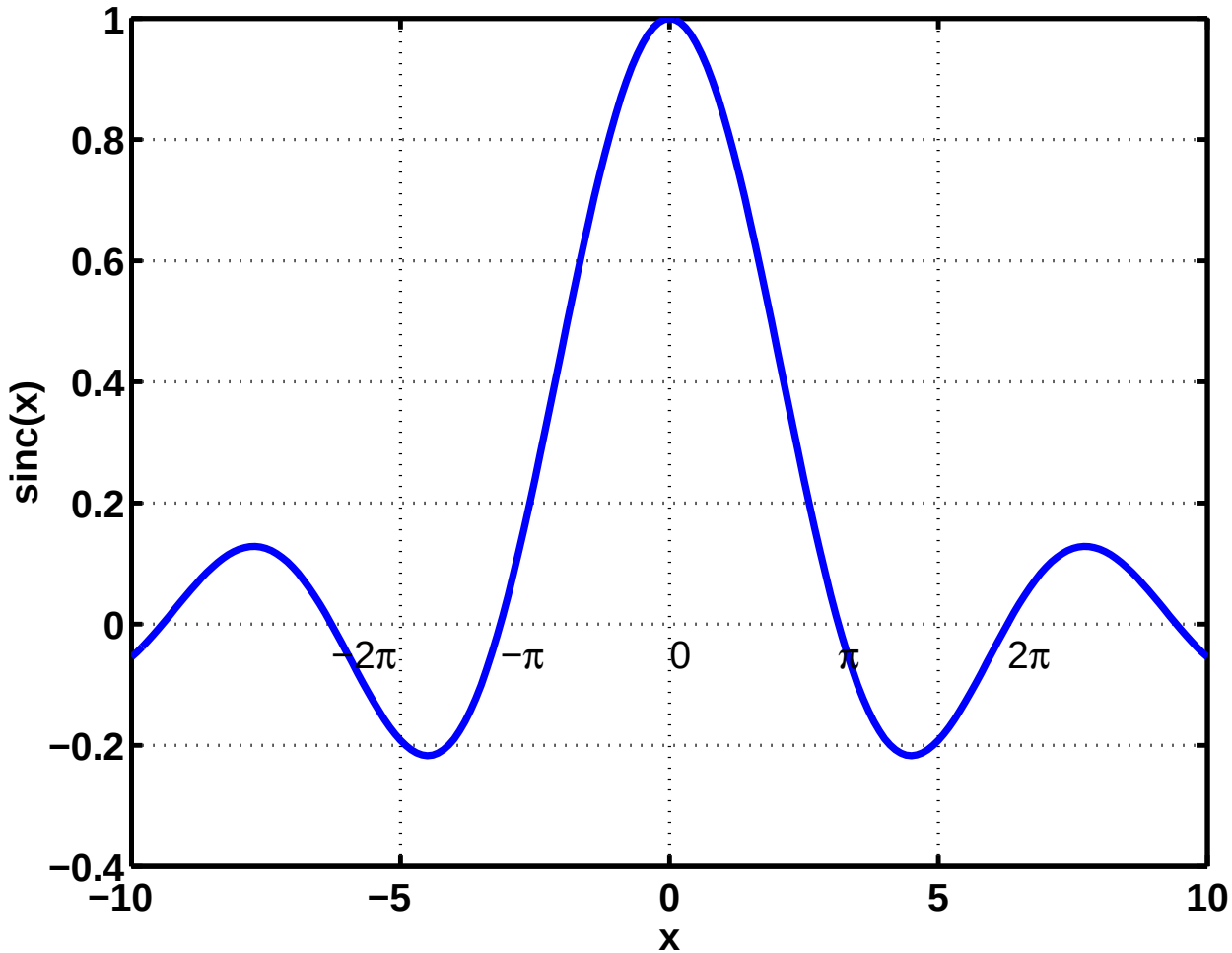
$$x(t) = \begin{cases} D & \text{for} \quad -\frac{\vartheta}{2} \leq t \leq \frac{\vartheta}{2} \\ 0 & \text{for} \quad -\frac{T_1}{2} \leq t < -\frac{\vartheta}{2} \quad \text{a} \quad \frac{\vartheta}{2} < t \leq \frac{T_1}{2} \end{cases}$$

with period T_1



Preparation 1. — Function sinc$(\cdot)$

$$\text{sinc}(x) = \begin{cases} \dfrac{\sin(x)}{x} & \text{for} \quad x \neq 0 \\ 1 & \text{for} \quad x = 0 \end{cases}$$



... Matlab! sinc is defined as $\frac{sin(\pi x)}{\pi x}$!!!

Preparation 2. — Computation of integral e^{jxy}

$$I(x) = \int_{-b}^{b} e^{\pm jxy} dy$$

a) for $x = 0$:

$$I(0) = 2b$$

b) for $x \neq 0$:

$$I(x) \quad = \left[\frac{e^{\pm jxy}}{\pm jx} \right]_{-b}^{b} = \frac{e^{jxb} - e^{-jxb}}{jx} =$$

$$= \frac{2}{x} \frac{e^{jxb} - e^{-jxb}}{2j} = 2b \frac{\sin bx}{bx}.$$

$$\boxed{\int_{-b}^{b} e^{\pm jxy} dy = 2b \ \mathsf{sinc}(bx).}$$

Spectrum of a periodic signal of square impulses

The signal is defined by ϑ, D, T_1.

$$c_k = \frac{1}{T_1}\int_{-\frac{T_1}{2}}^{+\frac{T_1}{2}} x(t)e^{-jk\omega_1 t}dt$$

$$= \frac{1}{T_1}\int_{-\frac{\vartheta}{2}}^{+\frac{\vartheta}{2}} De^{-jk\omega_1 t}dt = \frac{D}{T_1}\int_{-\frac{\vartheta}{2}}^{+\frac{\vartheta}{2}} e^{-jk\omega_1 t}dt.$$

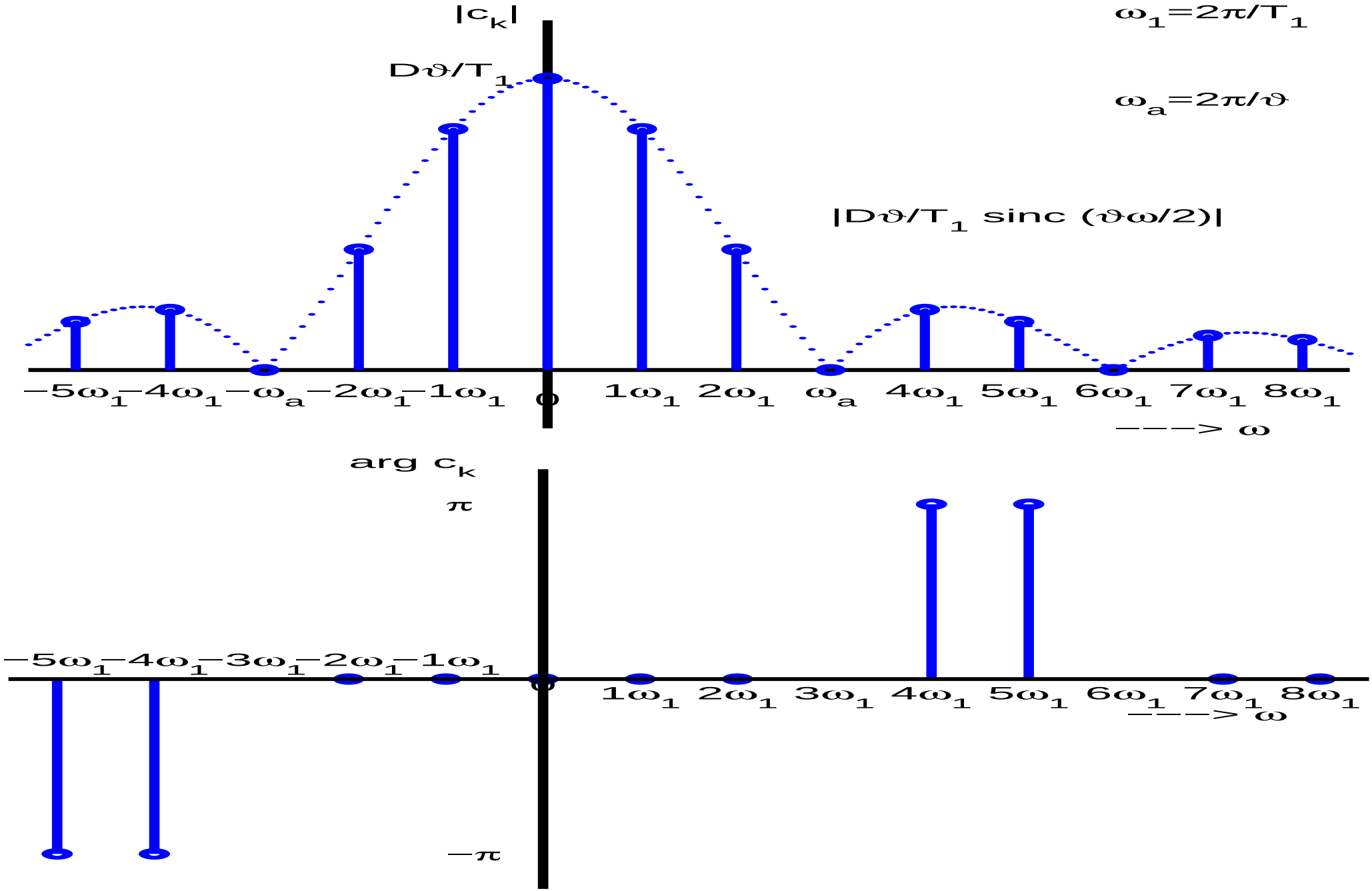
When we define: $t = y$, $b = \vartheta/2$ and $x = k\omega_1$, we obtain:

$$c_k = \frac{D}{T_1}2\frac{\vartheta}{2}\text{sinc}\left(\frac{\vartheta}{2}k\omega_1\right) = D\frac{\vartheta}{T_1}\text{sinc}\left(\frac{\vartheta}{2}k\omega_1\right).$$

Plotting results is not trivial :-(

- Prepare graphs ω-$|c_k|$ and ω-$\arg(c_k)$.
- Realize that the resulting coefficients are real numbers- either positive or negative depending on the sign of the funtion sinc.

- Prepare an auxiliary function (dotted) for a general ω (we will be placing the coefficients under the aux func): $\left| D\frac{\vartheta}{T_1}\text{sinc}\left(\frac{\vartheta}{2}\omega\right)\right|$. Note, that the principal (central) fold of the function is twise as wide as the side folds. Very important point is ω_a, where the function touchs the axis for the first time. To estimate ω_a, set the argument of the sinc equal to π:
$$\frac{\vartheta}{2}\omega_a = \pi \quad \rightarrow \quad \omega_a = \frac{2\pi}{\vartheta}.$$
- indicate the coefficients on the aux function at every fold of ω_1.
- argument of a positive coefficients (sinc$(\cdot) > 0$) is 0. Argument of a negative coefficient (sinc$(\cdot) < 0$) must be $\pm\pi$. Convention: for positive ω it will be $+\pi$ and for negative ω it will be $-\pi$. (It will not be incorrect when its the opposite thought).


|c_k|
ω_1=2π/T_1
Dϑ/T_1
ω_a=2π/ϑ
|Dϑ/T_1 sinc (ϑω/2)|
−5ω_1 −4ω_1 −ω_a −2ω_1 −1ω_1 0 1ω_1 2ω_1 ω_a 4ω_1 5ω_1 6ω_1 7ω_1 8ω_1
---> ω
arg c_k
π
−5ω_1 −4ω_1 −3ω_1 −2ω_1 −1ω_1 0 1ω_1 2ω_1 3ω_1 4ω_1 5ω_1 6ω_1 7ω_1 8ω_1
---> ω
−π

c_0 and mean of a signal

The coefficient c_0 lies in the maximum of the auxiliary function $\left| D\frac{\vartheta}{T_1}\text{sinc}\left(\frac{\vartheta}{2}\omega\right)\right|$ for $\omega = 0$. Its value is thus:

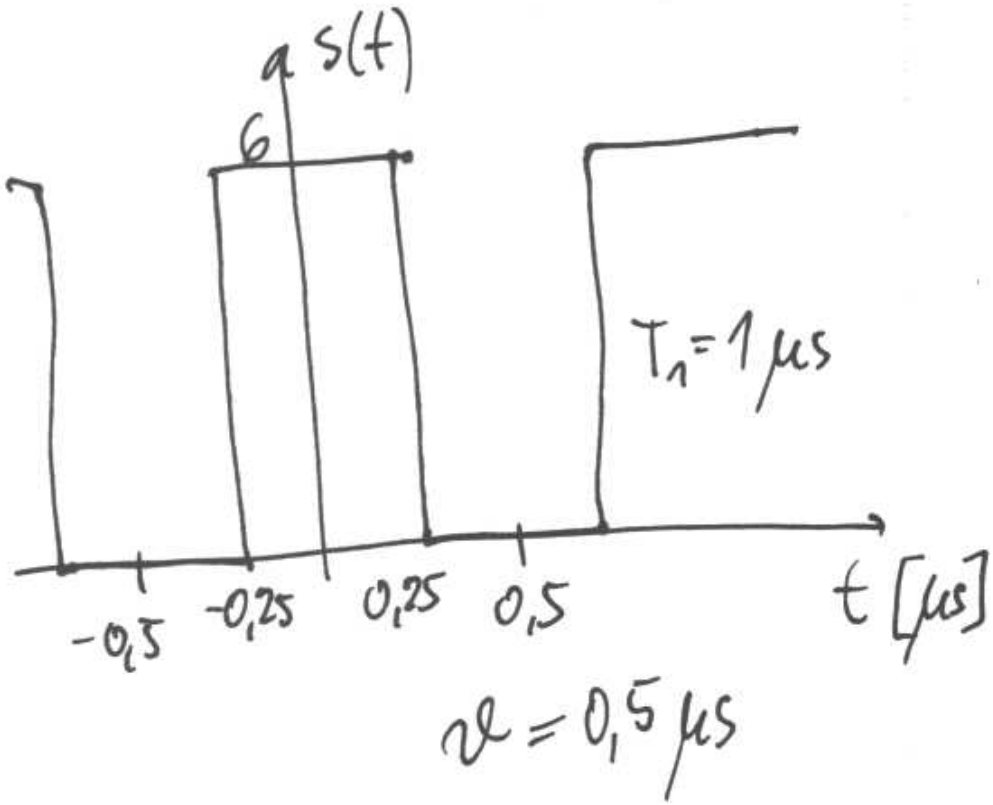
$$c_0 = D\frac{\vartheta}{T_1}.$$

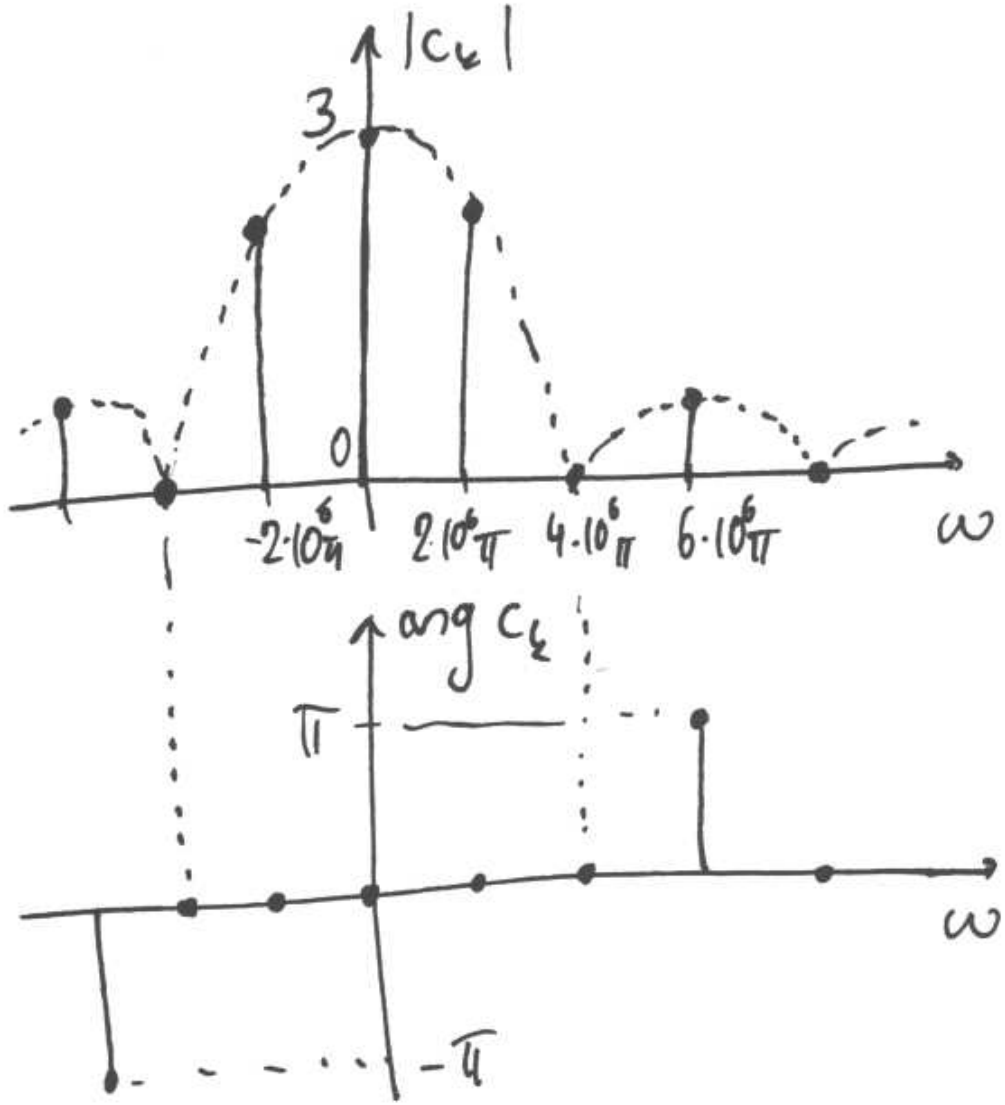
We also know that the value of c_0 must be equal to the direct component of the signal, which is its mean

$$\bar{x} = \frac{1}{T_1}\int_{-\frac{T_1}{2}}^{+\frac{T_1}{2}} x(t)dt = \frac{1}{T_1}\int_{-\frac{\vartheta}{2}}^{+\frac{\vartheta}{2}} Ddt = \frac{1}{T_1}\left[Dt\right]_{-\frac{\vartheta}{2}}^{+\frac{\vartheta}{2}} = \frac{D\vartheta}{T_1}.$$

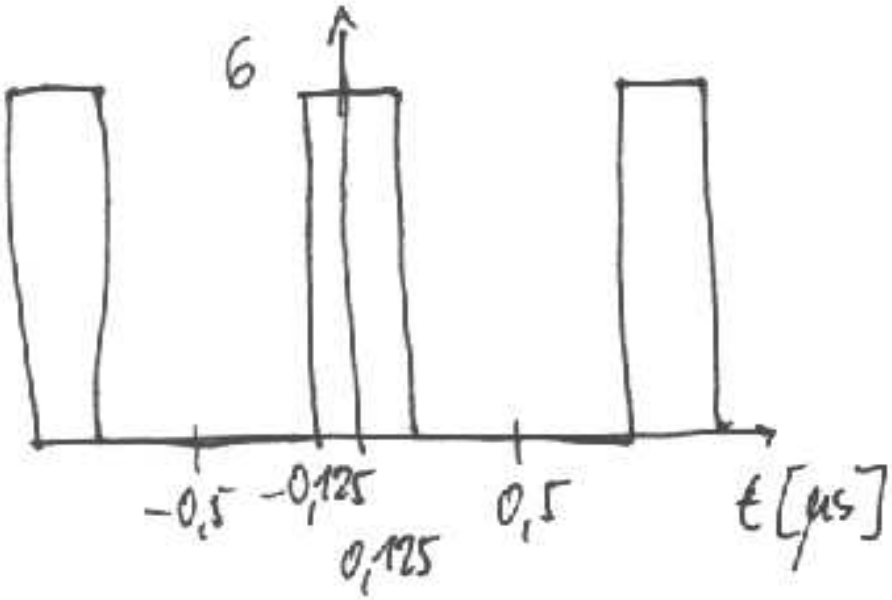
Proved!

Example 1. $D = 6$, $T_1 = 1$ μs, $\vartheta = 0.5$ μs. **Solution:** $f_1 = 1$ MHz, $\omega_1 = 2 \times 10^6 \pi$ rad/s, auxiliary function: $\left| D \frac{\vartheta}{T_1} \text{sinc}\left(\frac{\vartheta}{2}\omega\right) \right| = |3\text{sinc}(0.25 \times 10^{-6}\omega)|$. It touches zero when $0.25 \times 10^{-6}\omega = k\pi$, so $\omega = k4 \times 10^6 \pi$.



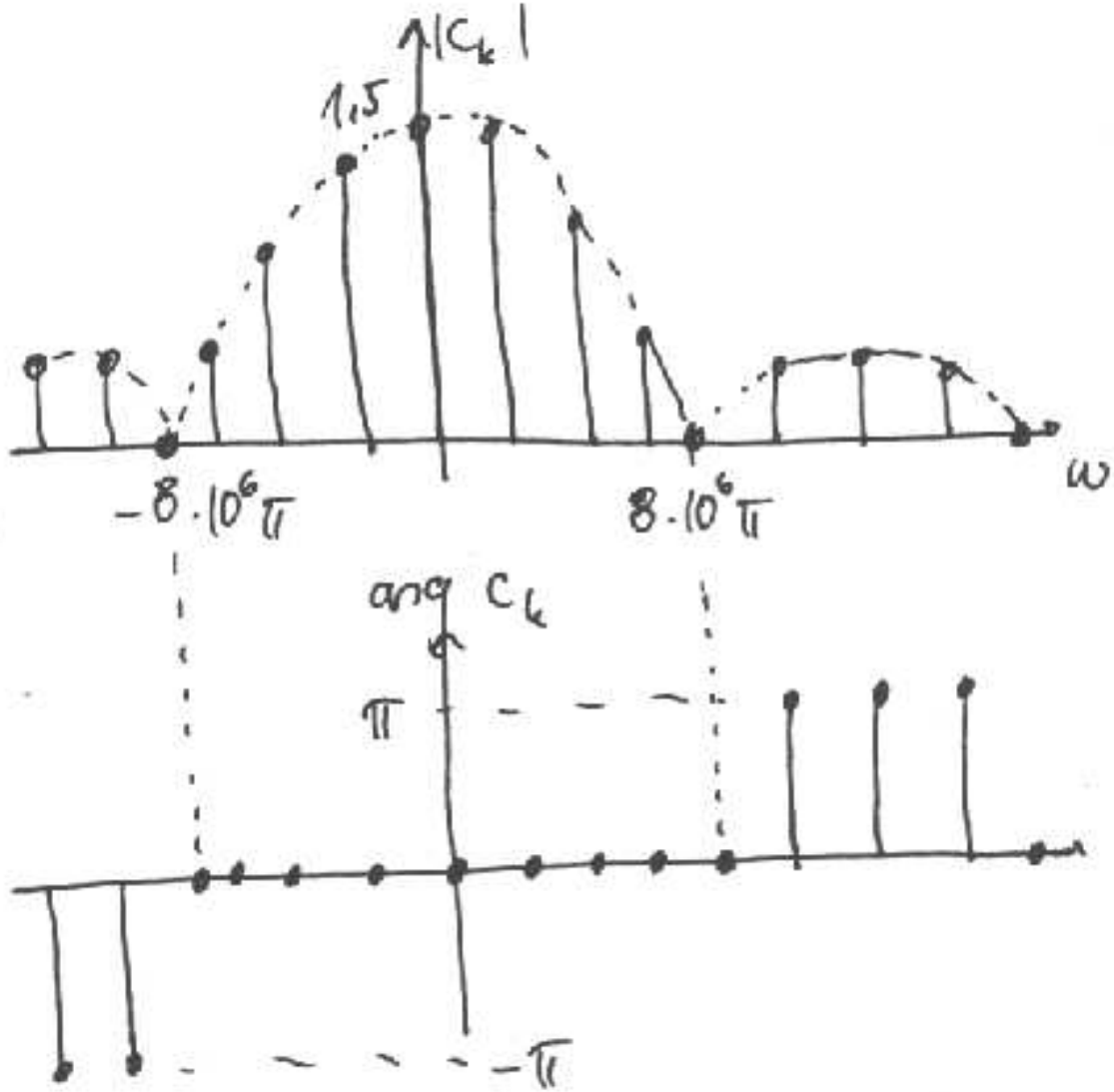


Example 2. $D = 6$, $T_1 = 1$ μs, $\vartheta = 0.25$ μs. **Solution:** $f_1 = 1$ MHz, $\omega_1 = 2 \times 10^6 \pi$ rad/s, auxiliary function: $\left| D \frac{\vartheta}{T_1} \text{sinc}\left(\frac{\vartheta}{2}\omega\right) \right| = |1.5\text{sinc}(0.125 \times 10^{-6}\omega)|$. It touches zeros for $0.125 \times 10^{-6}\omega = k\pi$, so for $\omega = k8 \times 10^6 \pi$.

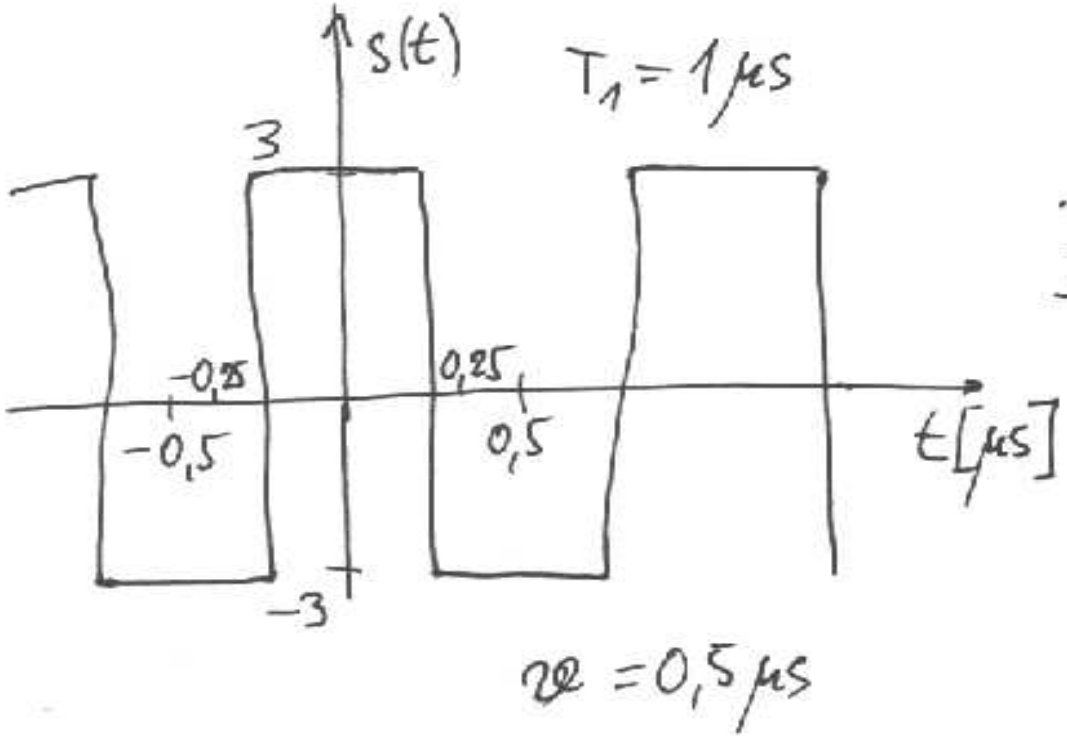


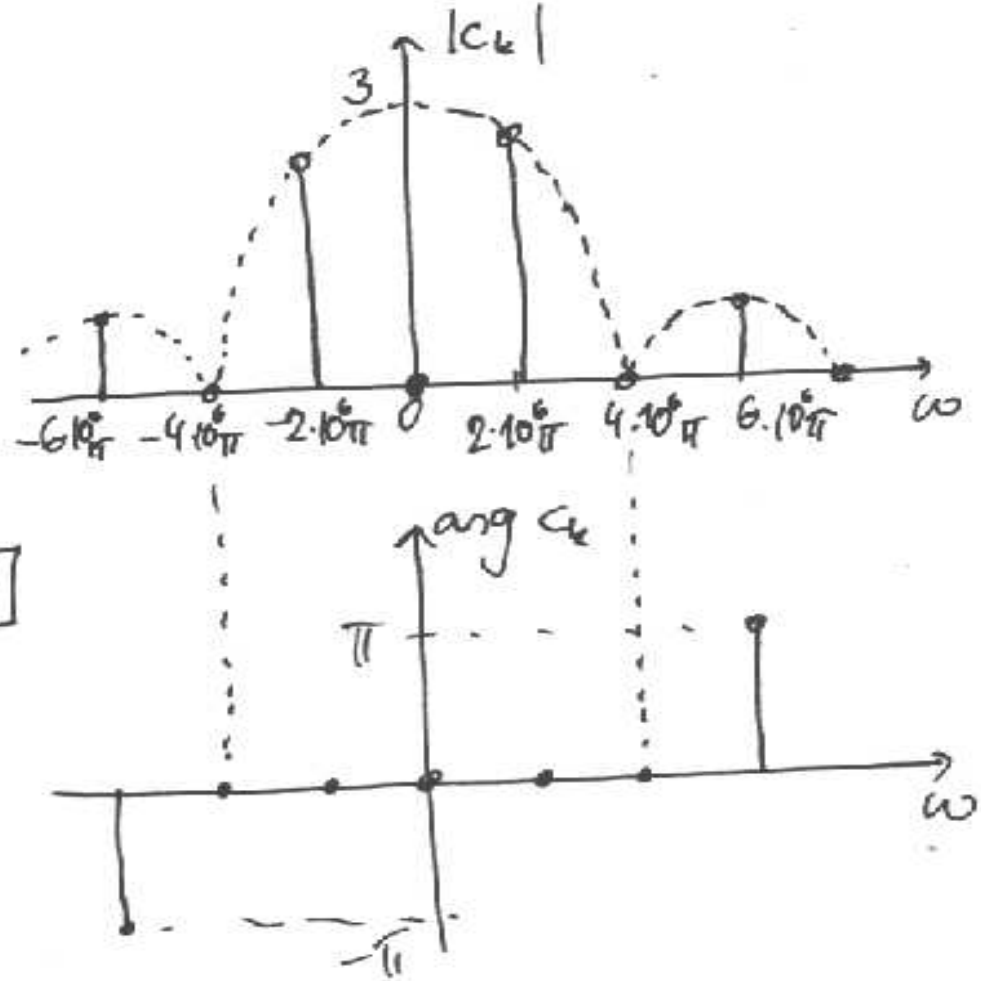
$T_1 = 1\,\mu s$

$\vartheta = 0{,}25\,\mu s$

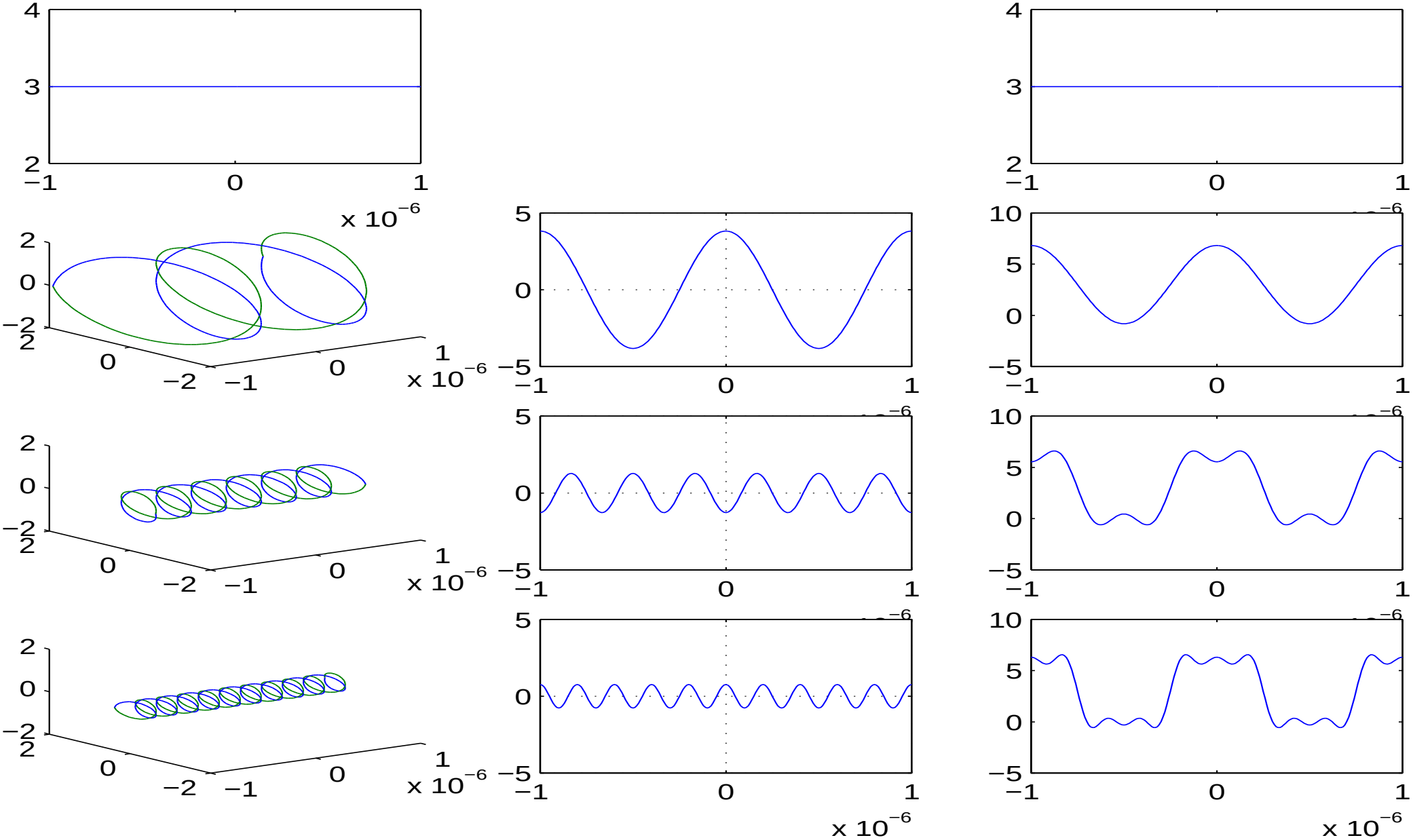


Example 3. ... same as example 1 but the signal is normalized (mean is 0)
Same result with the only difference: c_0 is equal to 0.

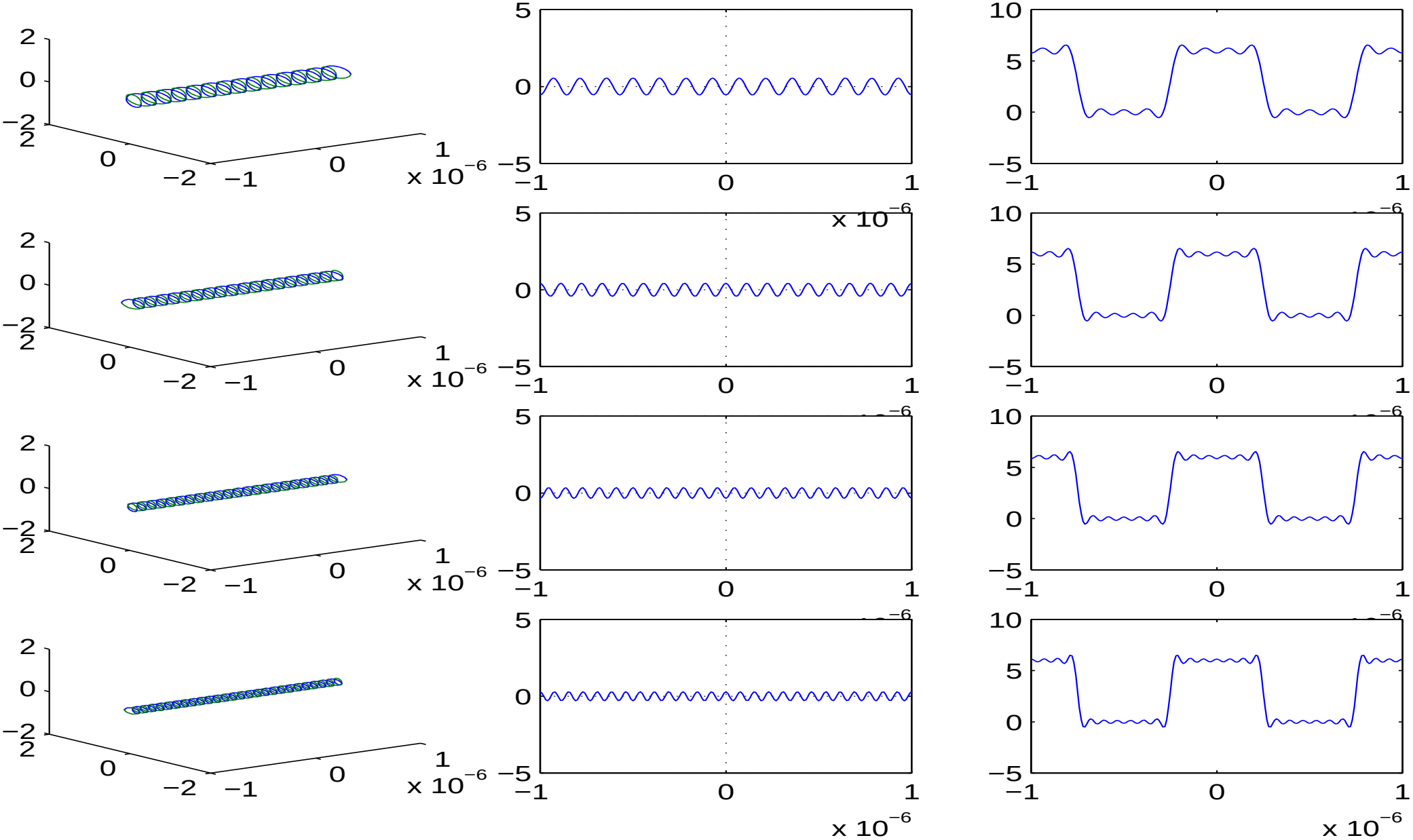




Again example 1. – how it is composed: $k = 0, \pm 1, \pm 3, \pm 5$



$k = \pm 7, \pm 9, \pm 11, \pm 13$



What else can we decompose ?

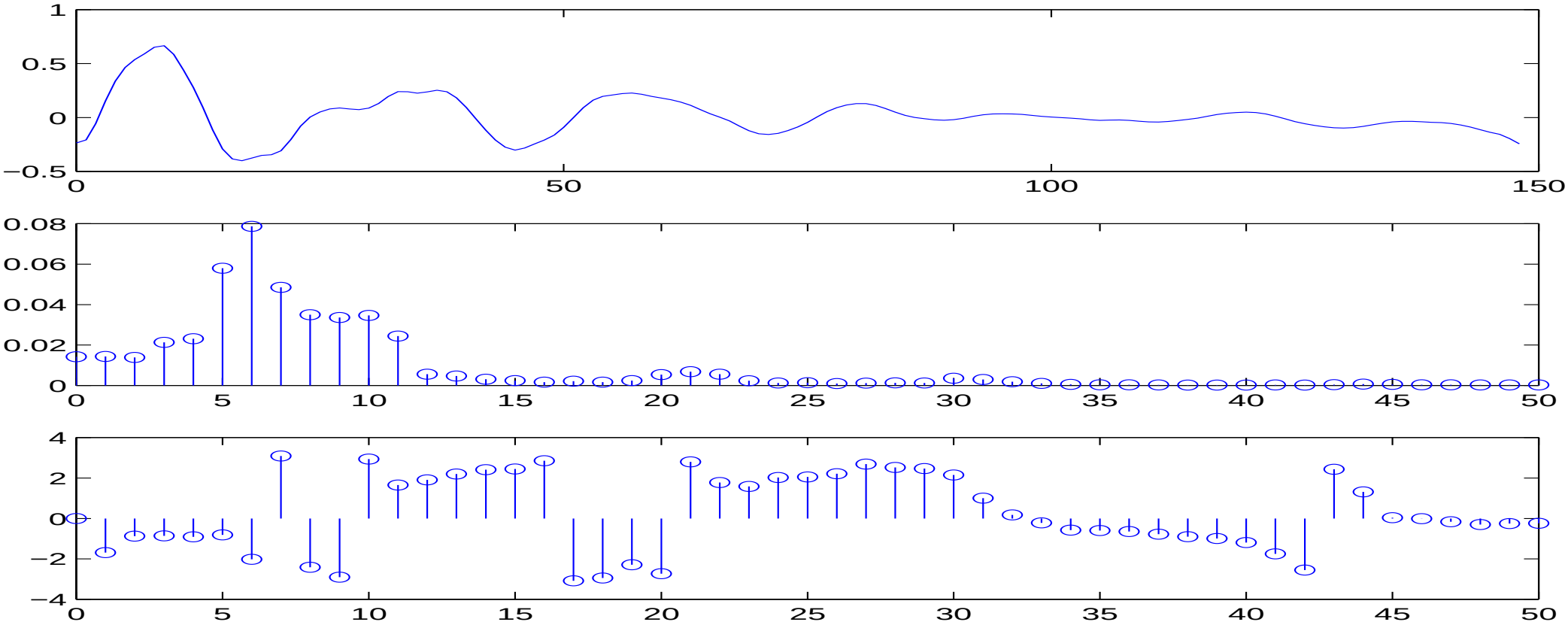
Wovel 'a', $F_s = 16$ kHz,

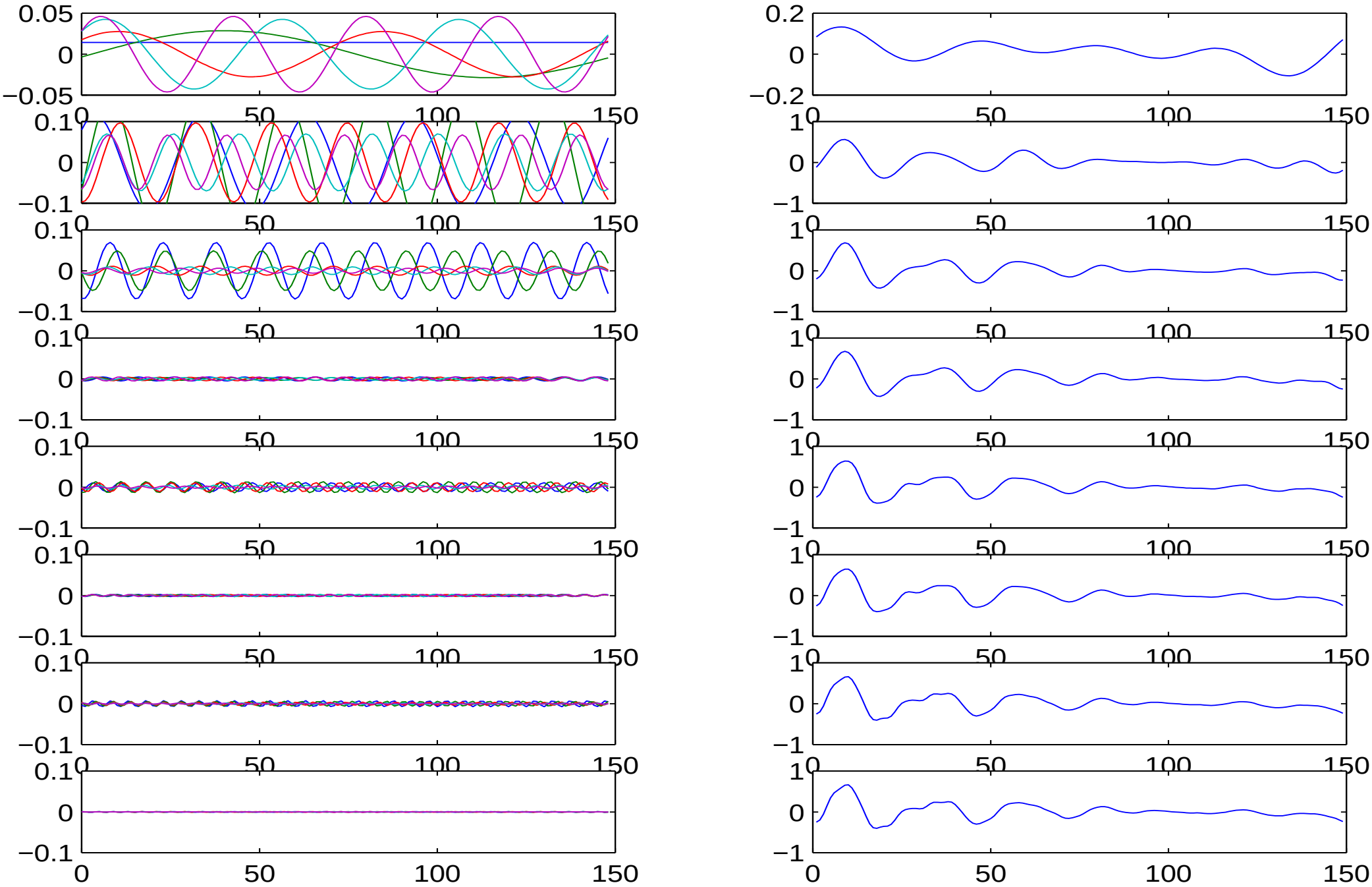
selected 1 period, repeted 50$\times$ $\Rightarrow$ `xx.wav`.

Coefficients of FS:

`yytill14.wav` – $k = 0, \pm1, \pm2, \pm3, \pm4$.

`yytill19.wav` – $k = 0, \pm1, \pm2, \pm3, \pm4, \pm5, \pm6, \pm7, \pm8, \pm9$. etc. . .




0.05
0
−0.05
0.1
−0.1
0.2
−0.2
1
−1
0
50
100
150

Properties of spectra of periodic signals

	$x(t)$	c_k
linearity	$a\,x_a(t) + b\,x_b(t)$	$a\,c_{a,k} + b\,c_{b,k}$
time shift	$x(t-\tau)$	$c_k e^{-jk\omega_1\tau}$
scale of time axis	$x(mt)$	c_k

A little bit more on time shif

FS coefficients are computed as $c_k = \frac{1}{T_1}\int_{T_1} x(t)e^{-jk\omega_1 t}dt$. FS coefficients of a shifted signal $x(t-\tau)$ are then:

$$c_{k,\tau} = \frac{1}{T_1}\int_{T_1} x(t-\tau)e^{-jk\omega_1 t}dt = \left| \begin{array}{c} r = t-\tau \\ t = r+\tau \\ dr = dt \end{array} \right| = \frac{1}{T_1}\int_{T_1} x(r)e^{-jk\omega_1 (r+\tau)}dr =$$

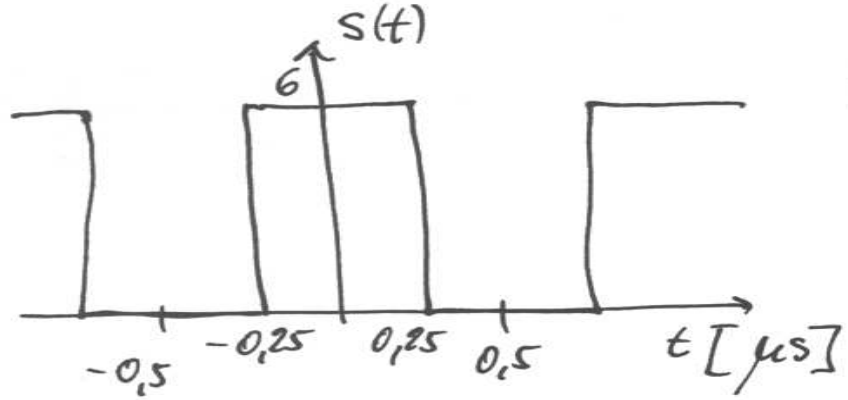
$$= \frac{1}{T_1}\int_{T_1} x(r)e^{-jk\omega_1 r}e^{-jk\omega_1 \tau}dr = e^{-jk\omega_1 \tau}c_k$$

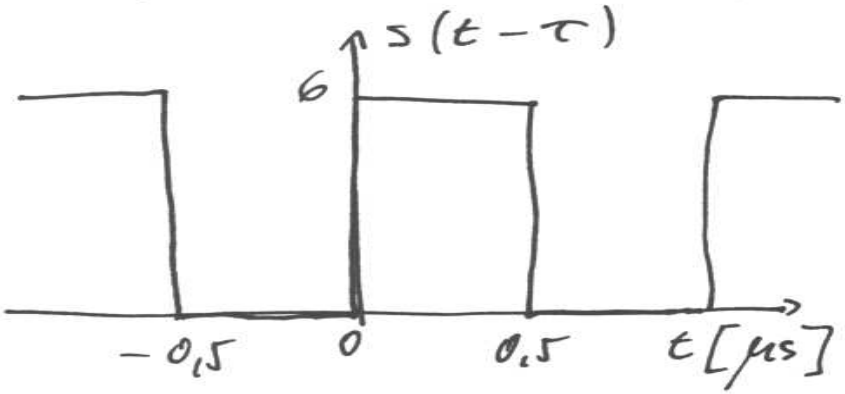
How have the coefficients changed? (Recall, complex numbers multiplication)

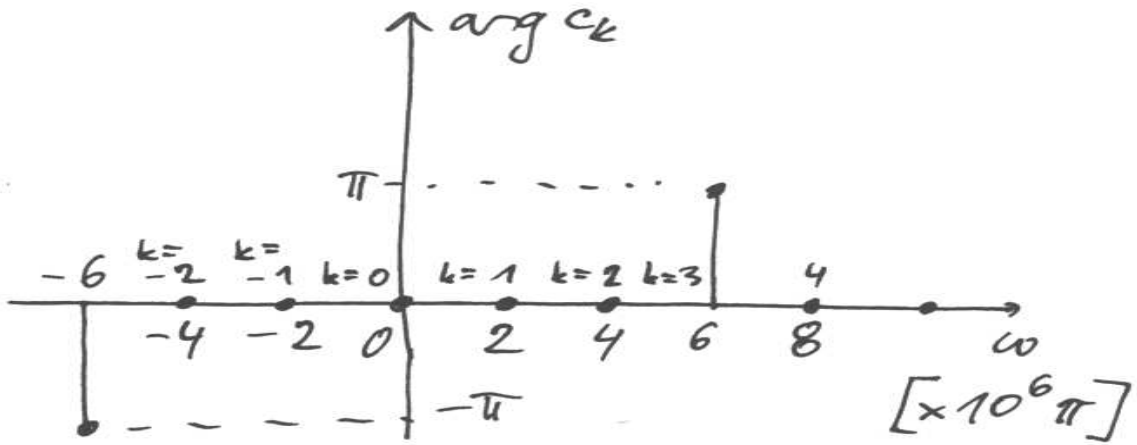
- magnitude - NO, as $|e^{-jk\omega_1\tau}| = 1$: $|c_{k,\tau}| = c_k$.
- phase - YES - shift by factor $-k\omega_1\tau$: $\arg c_{k,\tau} = \arg c_k - k\omega_1\tau$.

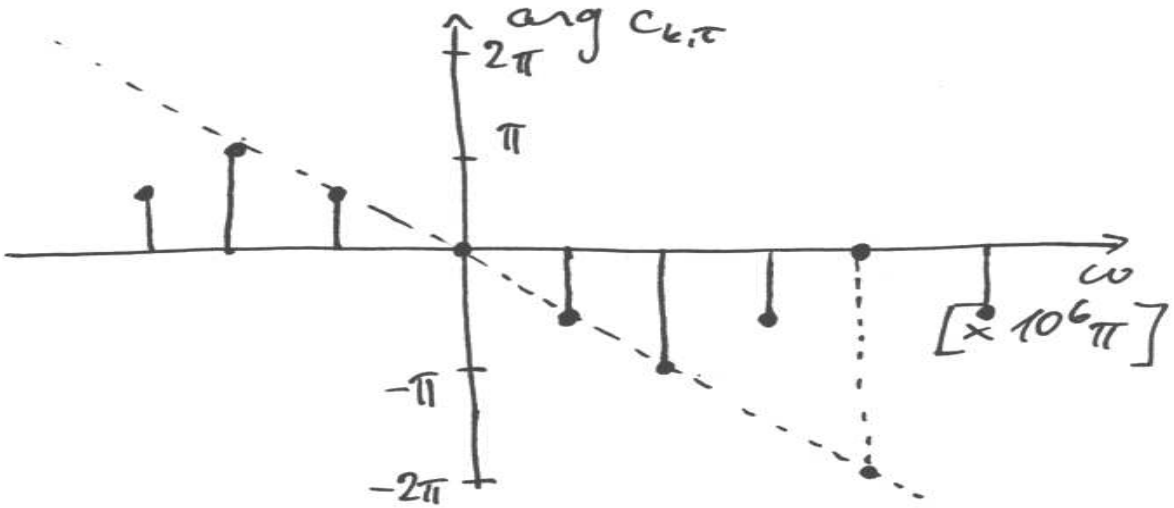
Time shift – example

signal $x(t)$ from example 1.: $D = 6$, $T_1 = 1$ μs, $\vartheta = 0.5$ μs. Shifted signal $x(t - 0.25 \times 10^{-6})$: coefficients are multiplied by $|e^{-jk\omega_1\tau}| = 1$ $:-k\omega_1\tau = -k\frac{2\pi}{T_1}\tau = -k\frac{2\pi}{1\times 10^{-6}}(0.25 \times 10^{-6}) = -k\frac{\pi}{2}$.


s(t)
6
-0,5
-0,25
0,25
0,5
t [μs]


s(t − τ)
6
-0,5
0
0,5
t [μs]


arg c_k
π
-π
-6
k=-2
k=-1
k=0
k=1
k=2
k=3
4
-4
-2
0
2
4
6
8
ω
[×10^6 π]


arg c_k,τ
2π
π
-π
-2π
ω
[×10^6 π]

Mean power of periodic signals – Parseval theorem

$$P_s = \frac{1}{T_1}\int_{T_1} |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |c_k|^2.$$

The signal power thus can be calculated not only through integration of $|\cdot|^2$ in time, but also by summing up squares of all FS coefficients. Summation is usually easier realized than integration.

Why? What is the power of k-th complex exponential:

$$\frac{1}{T_1}\int_{T_1} |c_k e^{jk\omega_1 t}|^2 dt = \frac{1}{T_1}\int_{T_1} |c_k|^2 dt = |c_k|^2$$

The power is then $|c_k|^2$, when powers of all exponentials are summed, we get overall mean power P_s of the signal.

Convergency of FS

3 Dirichlet properties:

1. $x(t)$ must be absolutelly integrable:

$$\int_{T_1} |x(t)| < \infty$$

   not fullfilled for example for: $x(t) = 1/t$ for $0 < t \leq 1$, periodic with $T_1 = 1$.

2. $x(t)$ must have finitly variable: a finit interval must contain a finit number of minima and maxima.
   not fullfilled for example for: $x(t) = \sin \frac{2\pi}{t}$ for $0 < t \leq 1$, periodic with $T_1 = 1$.

3. on finite interval the signal must have finite number of discontinuities.

We are going to work with signals that fully satisfy these restrictions.

Conclusions

- **Spectrum of periodic signal is discrete.**
- **When signal is narrowed — spectrum is spread.**
- **When signal is delayed phase is dropped downward, when signal is advanced phase is tipped upward**