Periodic signals - Fourier series

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• Why do we like exponentials.
• Fourier series.
• Basis.
• FS of square signals.
• Example from real world.
• Hints.
• Mean power and Parseval theorem.
By means of FS, we can decompose a periodic signal into a series of complex exponentials. ⇒ Any cosine function with an arbitrary phase can be expressed as sum of exponentials $\cos(x) = \frac{e^{jx} + e^{-jx}}{2}$:

$$C_1 \cos(\omega_1 t + \phi_1) =$$

$$= \frac{C_1}{2} e^{j(\omega_1 t + \phi)} + \frac{C_1}{2} e^{-j(\omega_1 t + \phi)} =$$

$$= \frac{C_1}{2} e^{j\phi_1} e^{j\omega_1 t} + \frac{C_1}{2} e^{-j\phi_1} e^{-j\omega_1 t},$$

where $c_1 = \frac{C_1}{2} e^{j\phi_1}$, $c_{-1} = \frac{C_1}{2} e^{-j\phi_1}$ are complex constants.
Transfer of an exponential through an LTI system results in the original exponential scaled (multiplied) by a constant: the input signal is defined as $x(t) = e^{st}$, where $s$ is a complex number. Mainly, we will be interested in the case when $s$ is purely imaginary, $s = j\omega$. However, we will demonstrate the computation on a general complex $s$. 
LTI system output is given by convolution of the input signal and the impulse response:

\[ y(t) = h(t) \ast x(t) = \int_{-\infty}^{+\infty} h(\tau)x(t-\tau)d\tau = \int_{-\infty}^{+\infty} h(\tau)e^{s(t-\tau)}d\tau \]

The expression \( e^{s(t-\tau)} \) can be decomposed (using equation \( e^{a+b} = e^a e^b \)) and the component independent of \( \tau \) is moved in front of the integral

\[ y(t) = \int_{-\infty}^{+\infty} h(\tau)e^{st} e^{-s\tau}d\tau = e^{st} \int_{-\infty}^{+\infty} h(\tau)e^{-s\tau}d\tau. \]

The integral becomes now a function of the complex number \( s \) and the system’s impulse response, but NOT a function of the input! Let us denote the integral as a complex function \( H(s) \). Factor \( e^{st} \) is the input, thus we can write:

\[ y(t) = e^{st} H(s), \]

Here we see that the output is the same exponential multiplied by the constant defined as:

\[ H(s) = \int_{-\infty}^{+\infty} h(\tau)e^{-s\tau}d\tau. \]
... what happens when we multiply complex exp. $c_1 e^{j\omega_1 t}$ by complex $H(s)$?

Short recap:

- Every complex number can be decomposed into a module and an argument (phase): 
  \[ H(s) = |H(s)|e^{j \arg H(s)}. \]
- Multiplication rule: 
  \[ z_1 z_2 = r_1 r_2 e^{j(\phi_1 + \phi_2)}. \]

How will it be? We start off with:

\[ H(s) c_1 e^{j\omega_1 t} = \ldots \]

We do nothing with $e^{j\omega_1 t}$ and define a new complex constant where 
\[ c'_1 = H(s) c_1. \]

As both $H(s)$ and $c_1$ are complex, during multiplication we multiply the modules and add up the arguments. The output of LTI system is thus a complex exponential that is scaled (wider or thinner) and has a different phase. The "speed" of the exponential does not change relative the original input signal.
Example: complex exponential $c_1 e^{j\omega_1 t} = 2.5 e^{-j \frac{\pi}{4}} e^{j100\pi t}$ is multiplied by $H(s) = 2e^{j \frac{\pi}{4}}$.

$$c'_1 = c_1 H(s) = 2.5 e^{-j \frac{\pi}{4}} 2e^{j \frac{\pi}{4}} = 2.5 \times 2e^{-j \frac{\pi}{4} + j \frac{\pi}{4}} = 5$$
A periodic signal $x(t) = x(t + T_1)$, with fundamental period $T_1$, is decomposed into a series of exponentials:

$$x(t) = \sum_{k=-\infty}^{+\infty} c_k e^{jk\omega_1 t}$$

$\omega_1$ is the fundamental period of the signal, $\omega_1 = \frac{2\pi}{T_1}$. Function $e^{jk\omega t}$ for $k = 0, \pm 1, \pm 2 \ldots$ is called harmonic complex exponential.
Basic properties of the coefficients:

- For real signals $x(t)$: $c_k$ and $c_{-k}$ are always complex conjugate:

$$c_k = c_{-k}^*$$

- $c_0$ is complex conjugate with itself, thus is real: $c_0 \in \mathbb{R}$. As $e^{jk\omega_1 t}$ equals to 1 for $k = 0$, $c_0$ must be direct component.

- We can rewrite the series into the form:

$$s(t) = c_0 + \sum_{k=1}^{\infty} \left[ c_k e^{jk\omega_1 t} + c_{-k} e^{-jk\omega_1 t} \right] = C_k \cos(k\omega_1 t + \phi_k)$$

- for $k > 0$
  - magnitude of the $k$-th harmonic component is $C_k = 2|c_k|$.
  - initial phase of the $k$-th harmonic is $\phi_k = \text{arg} \ c_k$. 
Recapitulation - basis and projection into basis

Basis

- we express one 'thing' by means of some other 'thing'.
- how similar is some input (vector, signal, function) to some reference.

\[ x = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \]

\[ b_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad b_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \]

\[ x_1 = [1 \ 0] \begin{bmatrix} 2 \\ 3 \end{bmatrix} = 2 \quad x_2 = [0 \ 1] \begin{bmatrix} 2 \\ 3 \end{bmatrix} = 3 \]

...scalar product
\[ x = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \]

\[ b_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \quad b_2 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \]

\[ y_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = 3.53 \]

\[ y_2 = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = 0.707 \]
How can we plot a higher dimensional space?

\[ \mathbf{x} = [3 \ 2 \ 1 \ 0 \ 1 \ 2 \ 3 \ 4], \quad \mathbf{b}_1 = \sqrt{\frac{1}{8}}[1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1], \quad b_{2n} = \frac{1}{2} \cos(2\pi/8n) \]

\[ y_1 = 5.65, \quad \text{similarity} \quad y_2 = 2.41 \quad \text{similarity}. \]
vector representation $y_n = b_n^T x,$

matrix representation $y = Bx$
Properties of basis

• One coefficient should not be dependent on the others - orthogonality:

\[ \mathbf{b}_i^T \mathbf{b}_j = 0 \]

• All basis are of the same 'importance' - basis magnitude equal to 1

\[ |\mathbf{b}_i| = 1 \]

• Orthogonality and unit magnitude of basis gives us an orthonormal system
Signal and basis can be functions!

- one function can be represented by means of some other function
- how two functions are similar.

Analogiously to the scalar product:

- multiplication with the basis
- adding up - integral \( \int \) for continuous signals or sum \( \sum \) for discrete

Example - what is cosine similar to?
To a direct signal?

\[ b(t) = 1 \quad \int_{0}^{1} s(t)b(t) dt = 0 \quad \text{... is not similar} \]
$b(t) = 2 \cos(2\pi t)$  \quad \int_0^1 s(t)b(t)dt = 1 \quad ... \text{is similar}
$b(t) = 2 \cos(4\pi t)$  \[ \int_0^1 s(t)b(t)dt = 0 \quad \text{... is not similar} \]
Now let's play with sine function $s(t) = \sin(2\pi t)$, $b(t) = 2\cos(2\pi t)$, $\int_0^1 s(t)b(t)dt = 0$

... no

⇒ we will need cosine and sine !!!
\[ s(t) = \sin(2\pi t - \pi/5) \]
\[ b_1(t) = 2 \cos(2\pi t) \quad \int_0^1 s(t) b_1(t) \, dt = -0.59 \]
\[ b_2(t) = 2 \sin(2\pi t) \quad \int_0^1 s(t) b_1(t) \, dt = 0.81 \]
We will be working with complex exponentials!

that conjugate cosine and sine in themselves!

\[ b_1(t) = e^{j\omega t} = \cos \omega t + j \sin \omega t \]

coefficient calculation: \[ c_1 = \int x(t)b_1^*(t)dt \]

product of such coefficient and the base is complex... so we need to add a complex conjugate base:

\[ b_{-1}(t) = e^{-j\omega t} = \cos \omega t - j \sin \omega t \]

coefficient computation: \[ c_{-1} = \int x(t)b_{-1}^*(t)dt \]
Derivation of coefficients of FS

We said that a periodic function can be decomposed to the series:

\[ x(t) = \sum_{k=-\infty}^{+\infty} c_k e^{jk\omega_1 t} \]

The coefficients \( c_k \) are computed using the formula:

\[ c_k = \frac{1}{T_1} \int_{T_1} x(t) e^{-jk\omega_1 t} dt, \]

where \( \int_{T_1} \) denotes integral over one period (typically, from 0 to \( T_1 \) or from \( -\frac{T_1}{2} \) to \( \frac{T_1}{2} \), but also can be from \( 25T_1 \) to \( 26T_1 \) )
The coefficients are indexed from $k = -\infty$ to $\infty$. Remember, they are tied with angular frequencies: $c_1$ is associated with $\omega_1$, $c_2$ with $2\omega_1$. Coefficients with negative indices are associated with corresponding negative frequencies: $c_{-1}$ is associated with $-\omega_1$, $c_{-2}$ with $-2\omega_1$. Negative frequency means that the exponential $c_{-k}e^{-j\omega_1 t}$ is rotating the opposite direction than $c_{k}e^{jk\omega_1 t}$. Their composition will give us a normal cosine $C_k \cos(k\omega_1 t + \phi_k)$. 
As the coefficients are complex numbers, we will plot their modules and arguments in separate graphs along frequency axes. Thus we are expressing a signal in frequency domain. Coefficients’ position and value will form a **SPECTRUM** of a signal. For a periodic signal the spectrum is discrete (composed of vertical lines).

**Example 1.**
Derive FS coefficients for signal: \( x(t) = 5 \cos(100\pi t) \). The basic angular frequency is \( \omega_1 = 100\pi \). In this example we don’t integrate because we can rewrite cosine function as:

\[
x(t) = \frac{5}{2} e^{j100\pi t} + \frac{5}{2} e^{-j100\pi t},
\]

thus, the coefficients’ values are:

\[
c_1 = \frac{5}{2}, \quad c_{-1} = \frac{5}{2}
\]

The coefficients are real.
Example 2.
Derive FS coefficients of the signal: $x(t) = 5 \cos(100\pi t - \frac{\pi}{4})$ Again, we will manage with no integration. Either we can rewrite cosine into the form:

$$2.5e^{-j \frac{\pi}{4}} e^{j100\pi t} + 2.5e^{+j \frac{\pi}{4}} e^{-j100\pi t},$$

thus, for $c_1$ and $c_{-1}$ we get:

$$c_1 = 2.5e^{-j \frac{\pi}{4}}, \quad c_{-1} = 2.5e^{+j \frac{\pi}{4}}$$

OR we look at how the coefficients $c_k$ are related to the magnitude and initial phase of the harmonics (here, only one): $C_1 = 5$ and $\phi_1 = -\frac{\pi}{4}$.

$$|c_1| = \frac{C_1}{2} = 2.5, \quad \arg c_1 = \phi_1 = -\frac{\pi}{4}, \quad \text{so} \quad c_1 = 2.5e^{-j \frac{\pi}{4}}.$$  

Coefficient $c_{-1}$ is a complex conjugate to $c_1$ (same module, reverse angle):

$$|c_{-1}| = \frac{C_1}{2} = 2.5, \quad \arg c_{-1} = -\phi_1 = \frac{\pi}{4}, \quad \text{so} \quad c_{-1} = 2.5e^{j \frac{\pi}{4}}.$$
Example 3. . . let’s integrate!! We have a periodic signal of square impulses:

\[ x(t) = \begin{cases} 
D & \text{for } -\frac{\vartheta}{2} \leq t \leq \frac{\vartheta}{2} \\
0 & \text{for } -\frac{T_1}{2} \leq t < -\frac{\vartheta}{2} \text{ and } \frac{\vartheta}{2} < t \leq \frac{T_1}{2} 
\end{cases} \]

with period \( T_1 \)
Preparation 1. — Function sinc(·)

\[
sinc(x) = \begin{cases} 
\frac{\sin(x)}{x} & \text{for } x \neq 0 \\
1 & \text{for } x = 0 
\end{cases}
\]

... Matlab! sinc is defined as \( \frac{\sin(\pi x)}{\pi x} \) !!!
Preparation 2. — Computation of integral $e^{jxy}$

\[ I(x) = \int_{-b}^{b} e^{\pm jxy} \, dy \]

a) for $x = 0$:

\[ I(0) = 2b \]

b) for $x \neq 0$:

\[ I(x) = \left[ \frac{e^{\pm jxy}}{\pm jx} \right]_{-b}^{b} = \frac{e^{jxb} - e^{-jxb}}{jx} = \]

\[ = \frac{2}{x} \frac{e^{jxb} - e^{-jxb}}{2j} = 2b \frac{\sin bx}{bx} . \]

\[ \int_{-b}^{b} e^{\pm jxy} \, dy = 2b \ \text{sinc}(bx). \]
Spectrum of a periodic signal of square impulses

The signal is defined by \( \psi \), \( D \), \( T_1 \).

\[
    c_k = \frac{1}{T_1} \int_{-\frac{T_1}{2}}^{+\frac{T_1}{2}} x(t)e^{-jk\omega_1 t} dt
    = \frac{1}{T_1} \int_{-\frac{\psi}{2}}^{+\frac{\psi}{2}} De^{-jk\omega_1 t} dt = \frac{D}{T_1} \int_{-\frac{\psi}{2}}^{+\frac{\psi}{2}} e^{-jk\omega_1 t} dt.
\]

When we define: \( t = y \), \( b = \frac{\psi}{2} \) and \( x = k\omega_1 \), we obtain:

\[
    c_k = \frac{D}{T_1} 2\frac{\psi}{2} \text{sinc} \left( \frac{\psi}{2} k\omega_1 \right) = D \frac{\psi}{T_1} \text{sinc} \left( \frac{\psi}{2} k\omega_1 \right).
\]

Plotting results is not trivial :-(

- Prepare graphs \( \omega-|c_k| \) and \( \omega-\text{arg}(c_k) \).
- Realize that the resulting coefficients are real numbers- either positive or negative depending on the sign of the function sinc.
• Prepare an auxiliary function (dotted) for a general $\omega$ (we will be placing the coefficients under the aux func): $|D_{\vartheta} \frac{\vartheta}{T_1} \text{sinc} \left( \frac{\vartheta}{2 \omega} \right)|$. Note, that the principal (central) fold of the function is twice as wide as the side folds. Very important point is $\omega_a$, where the function touches the axis for the first time. To estimate $\omega_a$, set the argument of the sinc equal to $\pi$:

$$\frac{\vartheta}{2} \omega_a = \pi \quad \rightarrow \quad \omega_a = \frac{2\pi}{\vartheta}.$$  

• indicate the coefficients on the aux function at every fold of $\omega_1$.

• argument of a positive coefficients ($\text{sinc}(\cdot) > 0$) is 0. Argument of a negative coefficient ($\text{sinc}(\cdot) < 0$) must be $\pm \pi$. Convention: for positive $\omega$ it will be $+\pi$ and for negative $\omega$ it will be $-\pi$. (It will not be incorrect when its the opposite thought).
\[ |c_k| \]

\[ D\vartheta/T_1 \]

\[ |D\vartheta/T_1 \text{sinc} (\vartheta\omega/2)| \]

\[ \omega_1 = 2\pi/T_1 \]

\[ \omega_a = 2\pi/\vartheta \]

\[ \arg c_k \]

\[ \pi \]

\[ -\pi \]
The coefficient $c_0$ lies in the maximum of the auxiliary function $\left| D \frac{\vartheta}{T_1} \text{sinc} \left( \frac{\vartheta}{2 \omega} \right) \right|$ for $\omega = 0$. Its value is thus:

$$c_0 = D \frac{\vartheta}{T_1}.$$ 

We also know that the value of $c_0$ must be equal to the direct component of the signal, which is its mean

$$\bar{x} = \frac{1}{T_1} \int_{-\frac{T_1}{2}}^{+\frac{T_1}{2}} x(t) dt = \frac{1}{T_1} \int_{-\frac{\vartheta}{2}}^{+\frac{\vartheta}{2}} D dt = \frac{1}{T_1} \left[ D t \right]_{-\frac{\vartheta}{2}}^{+\frac{\vartheta}{2}} = D \frac{\vartheta}{T_1}.$$ 

Proved!
Example 1. $D = 6, T_1 = 1 \, \mu s, \vartheta = 0.5 \, \mu s$. **Solution:** $f_1 = 1 \, \text{MHz}, \omega_1 = 2 \times 10^6 \pi \, \text{rad/s}$, auxiliary function: $|D\frac{\vartheta}{T_1} \text{sinc} \left( \frac{\vartheta}{2} \omega \right)| = |3 \text{sinc}(0.25 \times 10^{-6} \omega)|$. It touches zero when $0.25 \times 10^{-6} \omega = k\pi$, so $\omega = k4 \times 10^6 \pi$. 
Example 2. $D = 6$, $T_1 = 1 \, \mu s$, $\vartheta = 0.25 \, \mu s$. Solution: $f_1 = 1 \, \text{MHz}$, $\omega_1 = 2 \times 10^6 \pi \, \text{rad/s}$, auxiliary function: $\left| D \frac{\vartheta}{T_1} \text{sinc} \left( \frac{\vartheta}{2} \omega \right) \right| = |1.5 \text{sinc}(0.125 \times 10^{-6} \omega)|$. It touches zeros for $0.125 \times 10^{-6} \omega = k\pi$, so for $\omega = k8 \times 10^6 \pi$. 
Example 3. ... same as example 1 but the signal is normalized (mean is 0)  
Same result with the only difference: $c_0$ is equal to 0.
Again example 1. – how it is composed: \( k = 0, \pm 1, \pm 3, \pm 5 \)
$k = \pm 7, \pm 9, \pm 11, \pm 13$
What else can we decompose?

Wovel 'a', $F_s = 16$ kHz,
selected 1 period, repeted $50 \times \Rightarrow xx.wav$.

Coefficients of FS:

$yytill14.wav - k = 0, \pm 1, \pm 2, \pm 3, \pm 4$.

$yytill19.wav - k = 0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \pm 6, \pm 7, \pm 8, \pm 9$. etc...
### Properties of spectra of periodic signals

<table>
<thead>
<tr>
<th>Property</th>
<th>Original Function $x(t)$</th>
<th>Spectrum $c_k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Linearity</td>
<td>$a x_a(t) + b x_b(t)$</td>
<td>$a c_{a,k} + b c_{b,k}$</td>
</tr>
<tr>
<td>Time shift</td>
<td>$x(t - \tau)$</td>
<td>$c_k e^{-j k \omega_1 \tau}$</td>
</tr>
<tr>
<td>Scale of time axis</td>
<td>$x(mt)$</td>
<td>$c_k$</td>
</tr>
</tbody>
</table>
FS coefficients are computed as $c_k = \frac{1}{T_1} \int_{T_1} x(t)e^{-jk\omega_1 t} dt$. FS coefficients of a shifted signal $x(t - \tau)$ are then:

$$c_{k,\tau} = \frac{1}{T_1} \int_{T_1} x(t - \tau)e^{-jk\omega_1 t} dt = \left| \begin{array}{l} r = t - \tau \\ t = r + \tau \\ dr = dt \end{array} \right| = \frac{1}{T_1} \int_{T_1} x(r)e^{-jk\omega_1 (r+\tau)} dr =
$$

$$= \frac{1}{T_1} \int_{T_1} x(r)e^{-jk\omega_1 r}e^{-jk\omega_1 \tau} dr = e^{-jk\omega_1 \tau}c_k$$

How have the coefficients changed? (Recall, complex numbers multiplication)

- magnitude - NO, as $|e^{-jk\omega_1 \tau}| = 1$: $|c_{k,\tau}| = c_k$.
- phase - YES - shift by factor $-k\omega_1 \tau$: $\arg c_{k,\tau} = \arg c_k - k\omega_1 \tau$. 
signal $x(t)$ from example 1.: $D = 6$, $T_1 = 1 \mu s$, $\vartheta = 0.5 \mu s$. Shifted signal $x(t - 0.25 \times 10^{-6})$: coefficients are multiplied by $|e^{-jk\omega_1 \tau}| = 1$

$-k\omega_1 \tau = -k \frac{2\pi}{T_1} \tau = -k \frac{2\pi}{1 \times 10^{-6}} (0.25 \times 10^{-6}) = -k \frac{\pi}{2}$. 
Mean power of periodic signals – Parseval theorem

\[ P_s = \frac{1}{T_1} \int_{T_1} |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |c_k|^2. \]

The signal power thus can be calculated not only through integration of \(|\cdot|^2\) in time, but also by summing up squares of all FS coefficients. Summation is usually easier realized than integration.

Why? What is the power of \(k\)-th complex exponential:

\[ \frac{1}{T_1} \int_{T_1} |c_k e^{j\omega_1 t}|^2 dt = \frac{1}{T_1} \int_{T_1} |c_k|^2 dt = |c_k|^2 \]

The power is then \(|c_k|^2\), when powers of all exponentials are summed, we get overall mean power \(P_s\) of the signal.
Convergency of FS

3 Dirichlet properties:

1. $x(t)$ must be absolutely integrable:

   $\int_{T_1} |x(t)| < \infty$

   not fulfilled for example for: $x(t) = 1/t$ for $0 < t \leq 1$, periodic with $T_1 = 1$.

2. $x(t)$ must have finitely variable: a finite interval must contain a finite number of minima and maxima.

   not fulfilled for example for: $x(t) = \sin \frac{2\pi}{t}$ for $0 < t \leq 1$, periodic with $T_1 = 1$.

3. On finite interval the signal must have finite number of discontinuities.

We are going to work with signals that fully satisfy these restrictions.
Conclusions

- Spectrum of periodic signal is discrete.
- When signal is narrowed — spectrum is spread.
- When signal is delayed phase is dropped downward, when signal is advanced phase is tipped upward