Converting Finite Automata to Regular Expressions

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Outline

- **Introduction**
  - Basic terms
  - Why?

- **Methods**
  - Transitive Closure Method
  - State Removal Method
  - Brzozowski Algebraic Method

- **Comparison**
Basic Terms

- Finite automata (NFAs, DFAs)
- Regular expressions (REGEXPs)
- ...

Converting Finite Automata to Regular Expressions
Two possible transformations:

- Regular expression $\rightarrow$ Finite automaton
  - ✓
- Finite automaton $\rightarrow$ Regular expression
  - Uhm... Why?
• Rather theoretical approach.

\[ q_1 \xrightarrow{a} q_2 \xrightarrow{b} q_3 \]

• Sketch of the method:
  1. Let \( Q = \{q_1, q_2, \ldots, q_m\} \) be the set of all automatons states.
  2. Suppose that regular expression \( R_{ij} \) represents the set of all strings that transition the automaton from \( q_i \) to \( q_j \).
  3. Wanted regular expression will be the union of all \( R_{sf} \), where \( q_s \) is the starting state and \( q_f \) is one the final states.

• The main problem is how to construct \( R_{ij} \) for all states \( q_i, q_j \).
How to construct $R_{ij}$?

- Suppose $R_{ij}^k$ represents the set of all strings that transition the automaton from $q_i$ to $q_j$ without passing through any state higher than $q_k$. We can construct $R_{ij}$ by successively constructing $R_{ij}^1, R_{ij}^2, \ldots, R_{ij}^{|Q|} = R_{ij}$.

- $R_{ij}^k$ is recursively defined as:

$$R_{ij}^k = R_{ij}^{k-1} + R_{ik}^{k-1} (R_{kk}^{k-1})^* R_{kj}^{k-1}$$

- Assuming we have initialized $R_{ij}^0$ to be:

$$R_{ij}^0 = \begin{cases} 
  r & \text{if } i \neq j \text{ and } r \text{ transitions NFA from } q_i \text{ to } q_j \\
  r + \varepsilon & \text{if } i = j \text{ and } r \text{ transitions NFA from } q_i \text{ to } q_j \\
  \emptyset & \text{otherwise}
\end{cases}$$
Transform the following NFA to the corresponding REGEXP using Transitive Closure Method:
1) Initialize $R_{ij}^0$:

\[
R_{11}^0 = \varepsilon + 1 \\
R_{12}^0 = 0 \\
R_{21}^0 = \emptyset \\
R_{22}^0 = \varepsilon + 0 + 1
\]
2) Compute $R_{ij}^1$:

![Diagram of a finite automaton with states $q_1$ and $q_2$, transitions labeled 0 and 1 to $q_2$, and a loop at $q_1$ labeled 1.]

<table>
<thead>
<tr>
<th></th>
<th>By direct substitution</th>
<th>Simplified</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_{11}^1$</td>
<td>$\varepsilon + 1 + (\varepsilon + 1)(\varepsilon + 1)^*(\varepsilon + 1)$</td>
<td>$1^*$</td>
</tr>
<tr>
<td>$R_{12}^1$</td>
<td>$0 + (\varepsilon + 1)(\varepsilon + 1)^*0$</td>
<td>$1^*0$</td>
</tr>
<tr>
<td>$R_{21}^1$</td>
<td>$\emptyset + \emptyset(\varepsilon + 1)^*(\varepsilon + 1)$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>$R_{22}^1$</td>
<td>$\varepsilon + 0 + 1 + \emptyset(\varepsilon + 1)^*0$</td>
<td>$\varepsilon + 0 + 1$</td>
</tr>
</tbody>
</table>
3) Compute $R_{ij}^2$:

By direct substitution

<table>
<thead>
<tr>
<th>$R_{ij}^2$</th>
<th>Simplified</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_{11}^2$</td>
<td>$1^* + 1^<em>0(\varepsilon + 0 + 1)^</em>\emptyset$</td>
</tr>
<tr>
<td>$R_{12}^2$</td>
<td>$1^*0 + 1^<em>0(\varepsilon + 0 + 1)^</em>(\varepsilon + 0 + 1)$</td>
</tr>
<tr>
<td>$R_{21}^2$</td>
<td>$\emptyset + (\varepsilon + 0 + 1)(\varepsilon + 0 + 1)^*\emptyset$</td>
</tr>
<tr>
<td>$R_{22}^2$</td>
<td>$\varepsilon + 0 + 1 + (\varepsilon + 0 + 1)(\varepsilon + 0 + 1)^*(\varepsilon + 0 + 1)$</td>
</tr>
</tbody>
</table>
4) Get the resulting regular expression:

\[ R_{12}^2 = R_{12} = 1^*0(0 + 1)^* \] is the REGEXP corresponding to the NFA.
State Removal Method

- Based on a transformation from NFA to GNFA (generalized nondeterministic finite automaton).

- Identifies patterns within the graph and removes states, building up regular expressions along each transition.

- Sketch of the method:
  1. Unify all final states into a single final state using $\varepsilon$-trans.
  2. Unify all multi-transitions into a single transition that contains union of inputs.
  3. Remove states (and change transitions accordingly) until there is only the starting a the final state.
  4. Get the resulting regular expression by direct calculation.

- The main problem is how to remove states correctly so the accepted language won’t be changed.
Transform the following NFA to the corresponding REGEXP using State Removal Method:
1) Remove the "middle" state:
2) Get the resulting regular expression \( r \):

\[
r = (ae^*d)^* ae^*b (ce^*b + ce^*d (ae^*d)^* ae^*b)^* .
\]
Brzozowski Algebraic Method

- Janusz Brzozowski, 1964

- Utilizes equations over regular expressions.

- Sketch of the method:
  1. Create a system of regular equations with one regular expression unknown for each state in the NFA.
  2. Solve the system.
  3. The regular expression corresponding to the NFA is the regular expression associated with the starting state.

- The main problem is how to create the system and how to solve it.
Transform the following NFA to the corresponding REGEXP using Brzozowski Method:
1) Create a characteristic regular equation for state 1:

\[ X_1 = aX_1 + bX_2 \]
2) Create a characteristic regular equation for state 2:

\[ X_2 = \varepsilon + bX_1 + cX_2 \]
4) Solve the arisen system of regular expressions:

\[
\begin{align*}
X_1 &= aX_1 + bX_2 \\
X_2 &= \varepsilon + bX_1 + cX_2
\end{align*}
\]
Solution:

\[ X_1 = (a + bc^* b)^* bc^* \]
\[ X_2 = c^* [\varepsilon + b(a + bc^* b)^* bc^*] \]

\[ \Rightarrow \] \( X_1 \) is the REGEXP corresponding to the NFA.
Comparison of presented methods

- Transitive Closure Method
  + clear and simple implementation
  - tedious for manual use
  - tends to create very long regular expressions

- State Removal Method
  + intuitive, useful for manual inspection
  - not as straightforward to implement as other methods

- Brzozowski Algebraic Method
  + elegant
  + generates reasonably compact regular expressions
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Discussion