Chapter 5

Restrictions and Extensions

This chapter introduces several restricted and extended versions of scattered context grammars. Two types of these modifications are discussed—modifications that only change the definition of the derivation step, and modifications of the whole concept of scattered context grammars. Most importantly, we investigate the generative power of scattered context grammars modified in this way.

5.1 *n*-Limited Derivations

As formal language theory has always studied various left restrictions placed on grammatical derivations, we investigate this classical topic in terms of propagating scattered context grammars in this section as well. More specifically, we discuss the language families generated by propagating scattered context grammars and propagating unordered scattered context grammars whose derivations are n-limited, where n is a positive integer. In these derivations, a scattered context production is always applied within the first n occurrences of nonterminals in the current sentential form. We demonstrate that this restriction gives rise to an infinite hierarchy of language families. In addition, we prove that every family of this hierarchy is properly included in the family of context-sensitive languages. Based upon this proper inclusion, we obtain several conclusions and formulate new open problems. Perhaps most importantly, we point out that the language family generated by propagating scattered context grammars that make derivations in the above n-limited way is properly contained in the context-sensitive language family, so in this sense, we partially solve Open Problem 3.28. First, we define n-limited derivations formally.

Definition 5.1. Let $G = (V, T, P, S)$ be a propagating scattered context grammar. If

$$(A_1, \ldots, A_k) \rightarrow (x_1, \ldots, x_k) \in P,$$

$u = u_1 A_1 \ldots u_k A_k u_{k+1}$, and $v = u_1 x_1 \ldots u_k x_k u_{k+1}$, where $u_i \in V^*$, for all $1 \leq i \leq k+1$, and $u \Rightarrow_G v$ satisfies

$$|u_1 A_1 \ldots u_k A_k|_{V-T} \leq n,$$

then the derivation step is *n-limited*, symbolically written as

$$u \ {}_{\text{lim}}\!\overset{n}{\Rightarrow}_G\ v.$$

An *n-limited derivation*, denoted by $x\ {}_{lim}{\overset{n}{\Rightarrow}}{}^{*}_{G}\ y$, is a derivation in which every derivation step $u\ {}_{lim}{\overset{j}{\Rightarrow}}_{G}\ v$ satisfies $j \leq n$. Define the *language of degree n* generated by G as

$$L(G, lim, n) = \{x : x \in T^*, S\ {}_{lim}{\overset{n}{\Rightarrow}}{}^{*}_{G}\ x\}.$$

The family of languages of degree n generated by propagating scattered context grammars is denoted by $\mathscr{L}(PSC, lim, n)$, and

$$\mathscr{L}(PSC, lim, \infty) = \bigcup_{i=1}^{\infty} \mathscr{L}(PSC, lim, i).$$

Analogously, we define n-limited derivations for propagating unordered scattered context grammars. The family of languages of degree n generated by propagating unordered scattered context grammars is denoted by $\mathscr{L}(PSC, un, lim, n)$, and

$$\mathscr{L}(PSC, un, lim, \infty) = \bigcup_{i=1}^{\infty} \mathscr{L}(PSC, un, lim, i).$$

We prove the main result, $\mathscr{L}(PSC, lim, n) = \mathscr{L}(ST, n)$, for all $n \geq 1$, by demonstrating that $\mathscr{L}(ST, n) \subseteq \mathscr{L}(PSC, lim, n)$ and $\mathscr{L}(PSC, lim, n) \subseteq \mathscr{L}(ST, n)$ in Lemmas 5.2 and 5.3, respectively.

Lemma 5.2. $\mathscr{L}(ST, n) \subseteq \mathscr{L}(PSC, lim, n)$*, for all* $n \geq 1$.

Proof. Let $G = (V, T, K, P, S, p_0)$ be a state grammar of degree n. Set

$$\begin{aligned} N_1 &= \{\langle A, p, k\rangle : A \in V - T, p \in K, 1 \leq k \leq n\}, \\ N_2 &= \{\langle \hat{A}, p, k\rangle : A \in V - T, p \in K, 1 \leq k \leq n\}, \\ N_3 &= \{\langle A', p, n-1\rangle : A \in V - T, p \in K\}, \\ N_4 &= \{\hat{A} : A \in V - T\}. \end{aligned}$$

Set $\alpha(p) = \{A : (A, p) \rightarrow (x, q) \in P\}$, for each $p \in K$. Define the propagating scattered context grammar

$$\bar{G} = \big(V \cup N_1 \cup N_2 \cup N_3 \cup N_4 \cup \{\bar{S}\}, T, \bar{P}, \bar{S}\big)$$

with $\bar{P}$ constructed as follows (throughout the construction, we add intuitive explanation of the purpose of the constructed productions):

1. Add $(\bar{S}) \rightarrow (\langle \hat{S}, p_0, 1\rangle)$ to $\bar{P}$.
2. For each $A_1, \ldots, A_k \in V - T$, where $1 \leq k \leq n$, each

$$(A_r, p) \rightarrow (x_1 B_1 \ldots x_t B_t x_{t+1}, q) \in P,$$

where $1 \leq r \leq k$, $B_1, \ldots, B_t \in V - T$, $x_1, \ldots, x_{t+1} \in T^*$, for some $t \geq 0$, $A_i \notin \alpha(p)$, for every $1 \leq i < r$ ($A_1, \ldots, A_k$ denote the first k nonterminals in the sentential form; k is the number of nonterminals present in the sentential form if it contains fewer than n nonterminals, otherwise $k = n$; A_r is the nonterminal whose rewriting is simulated; t is the number of nonterminals appearing on the right-hand side of the simulated production),

(a) and $r + t - 1 > n$, add

i. (used when the sentential form contains more than n nonterminals)

$$\begin{aligned}&\big(\langle A_1, p, n\rangle, \ldots, \langle A_{r-1}, p, n\rangle, \langle A_r, p, n\rangle,\\ &\quad\langle A_{r+1}, p, n\rangle, \ldots, \langle A_n, p, n\rangle\big)\\ \to &\big(\langle A_1, q, n\rangle, \ldots, \langle A_{r-1}, q, n\rangle,\\ &\quad x_1 \langle B_1, q, n\rangle \ldots x_{n-r+1} \langle B_{n-r+1}, q, n\rangle\\ &\quad x_{n-r+2} B_{n-r+2} \ldots x_t B_t x_{t+1}, A_{r+1}, \ldots, A_n\big)\end{aligned}$$

to $\bar{P}$;

ii. (used when the sentential form contains at most n nonterminals and A_r is not the last nonterminal)
if $r < k$, add

$$\begin{aligned}&\big(\langle A_1, p, k\rangle, \ldots, \langle A_{r-1}, p, k\rangle, \langle A_r, p, k\rangle,\\ &\quad\langle A_{r+1}, p, k\rangle, \ldots, \langle A_{k-1}, p, k\rangle, \langle \hat{A}_k, p, k\rangle\big)\\ \to &\big(\langle A_1, q, n\rangle, \ldots, \langle A_{r-1}, q, n\rangle,\\ &\quad x_1 \langle B_1, q, n\rangle \ldots x_{n-r+1} \langle B_{n-r+1}, q, n\rangle\\ &\quad x_{n-r+2} B_{n-r+2} \ldots x_t B_t x_{t+1}, A_{r+1}, \ldots, A_{k-1}, \hat{A}_k\big)\end{aligned}$$

to $\bar{P}$;

iii. (used when the sentential form contains at most n nonterminals and A_r is the last nonterminal)
if $r = k$, add

$$\begin{aligned}&\big(\langle A_1, p, k\rangle, \ldots, \langle A_{k-1}, p, k\rangle, \langle \hat{A}_k, p, k\rangle\big)\\ \to &\big(\langle A_1, q, n\rangle, \ldots, \langle A_{k-1}, q, n\rangle,\\ &\quad x_1 \langle B_1, q, n\rangle \ldots x_{n-k+1} \langle B_{n-k+1}, q, n\rangle\\ &\quad x_{n-k+2} B_{n-k+2} \ldots x_{t-1} B_{t-1} x_t \hat{B}_t x_{t+1}\big)\end{aligned}$$

to $\bar{P}$;

(b) and $r + t - 1 \leq n$, $k + t - 1 > n$, add

i. (used when the sentential form contains more than n nonterminals)

$$\begin{aligned}&\big(\langle A_1,p,n\rangle,\ldots,\langle A_{r-1},p,n\rangle,\langle A_r,p,n\rangle,\\&\langle A_{r+1},p,n\rangle,\ldots,\langle A_{n-t+1},p,n\rangle,\\&\langle A_{n-t+2},p,n\rangle,\ldots,\langle A_n,p,n\rangle\big)\\\rightarrow&\big(\langle A_1,q,n\rangle,\ldots,\langle A_{r-1},q,n\rangle,x_1\langle B_1,q,n\rangle\ldots x_t\langle B_t,q,n\rangle x_{t+1},\\&\langle A_{r+1},q,n\rangle,\ldots,\langle A_{n-t+1},q,n\rangle,A_{n-t+2},\ldots,A_n\big)\end{aligned}$$

to $\bar{P}$;

ii. (used when the sentential form contains at most n nonterminals and A_r is not the last nonterminal)
if $r<k$, add

$$\begin{aligned}&\big(\langle A_1,p,k\rangle,\ldots,\langle A_{r-1},p,k\rangle,\langle A_r,p,k\rangle,\\&\langle A_{r+1},p,k\rangle,\ldots,\langle A_{n-t+1},p,k\rangle,\\&\langle A_{n-t+2},p,k\rangle,\ldots,\langle A_{k-1},p,k\rangle,\langle \hat{A}_k,p,k\rangle\big)\\\rightarrow&\big(\langle A_1,q,n\rangle,\ldots,\langle A_{r-1},q,n\rangle,x_1\langle B_1,q,n\rangle\ldots x_t\langle B_t,q,n\rangle x_{t+1},\\&\langle A_{r+1},q,n\rangle,\ldots,\langle A_{n-t+1},q,n\rangle,A_{n-t+2},\ldots,A_{k-1},\hat{A}_k\big)\end{aligned}$$

to $\bar{P}$;

(c) and $k+t-1\leq n$, and

i. if $t=0$, add

A. (used when the sentential form contains more than n nonterminals and A_r is rewritten to $x_1\in T^*$)

$$\begin{aligned}&\big(\langle A_1,p,n\rangle,\ldots,\langle A_{r-1},p,n\rangle,\langle A_r,p,n\rangle,\\&\langle A_{r+1},p,n\rangle,\ldots,\langle A_n,p,n\rangle\big)\\\rightarrow&\big(\langle A'_1,q,n-1\rangle,\ldots,\langle A'_{r-1},q,n-1\rangle,x_1,\\&\langle A'_{r+1},q,n-1\rangle,\ldots,\langle A'_n,q,n-1\rangle\big),\end{aligned}$$

B. (used immediately after (2.c.i.A))

$$\begin{aligned}&\big(\langle A'_1,q,n-1\rangle,\ldots,\langle A'_{r-1},q,n-1\rangle,\\&\langle A'_{r+1},q,n-1\rangle,\ldots,\langle A'_n,q,n-1\rangle,A_{n+1}\big)\\\rightarrow&\big(\langle A_1,q,n\rangle,\ldots,\langle A_{r-1},q,n\rangle,\\&\langle A_{r+1},q,n\rangle,\ldots,\langle A_n,q,n\rangle,\langle A_{n+1},q,n\rangle\big),\end{aligned}$$

where $A_{n+1}\in(V-T)\cup N_4$, to $\bar{P}$;

ii. (used when the sentential form contains at most n nonterminals and A_r is not the last nonterminal)
if $r < k$, add

$$\begin{aligned}&\big(\langle A_1, p, k\rangle, \ldots, \langle A_{r-1}, p, k\rangle,\\ &\langle A_r, p, k\rangle, \langle A_{r+1}, p, k\rangle, \ldots, \langle A_{k-1}, p, k\rangle, \langle \hat{A}_k, p, k\rangle\big)\\ \rightarrow &\big(\langle A_1, q, k+t-1\rangle, \ldots, \langle A_{r-1}, q, k+t-1\rangle,\\ &x_1\langle B_1, q, k+t-1\rangle \ldots x_t\langle B_t, q, k+t-1\rangle x_{t+1},\\ &\langle A_{r+1}, q, k+t-1\rangle, \ldots, \langle A_{k-1}, q, k+t-1\rangle,\\ &\langle \hat{A}_k, q, k+t-1\rangle\big)\end{aligned}$$

to $\bar{P}$;

iii. (used when the sentential form contains at most n nonterminals and A_r is the last nonterminal)
if $r = k$

A. and $k > 1$ or $t \neq 0$, add

$$\begin{aligned}&\big(\langle A_1, p, k\rangle, \ldots, \langle A_{k-1}, p, k\rangle, \langle \hat{A}_k, p, k\rangle\big)\\ \rightarrow &\big(\langle A_1, q, k+t-1\rangle, \ldots, \langle A_{k-1}, q, k+t-1\rangle,\\ &x_1\langle B_1, q, k+t-1\rangle \ldots x_{t-1}\langle B_{t-1}, q, k+t-1\rangle\\ &x_t\langle \hat{B}_t, q, k+t-1\rangle x_{t+1}\big)\end{aligned}$$

to $\bar{P}$;

B. (simulates the last derivation step of G)
and $k = 1$, $t = 0$, add
$\big(\langle \hat{A}_1, p, 1\rangle\big) \rightarrow (x_1)$ to $\bar{P}$.

Basic Idea.
For every sentential form (x, p) of G, strings u and v can be found so that $x = uv$ and either $|u|_{V-T} = n$ and $|v|_{V-T} \geq 1$ or $|u|_{V-T} = k$, where $k \leq n$, and $|v|_{V-T} = 0$. As a result, only nonterminals occurring in u can be rewritten by a production of G. As n is a finite number, it is possible to construct a propagating scattered context grammar $\bar{G}$ that rewrites all nonterminals occurring in u in every derivation step. In this way, $\bar{G}$ simulates the rewriting of the leftmost nonterminal for a given state by considering all possible forms of u and constructing productions of $\bar{G}$ accordingly. The constructed grammar simulates every sentential form of G by its division into two parts. The first part contains only nonterminals from $N_1 \cup N_2$, which can be rewritten by the constructed productions. The other part contains nonterminals from $(V-T) \cup N_4$, which no production rewrites (with the exception of the productions from (2.c.i.B) whose application is explained next).

By rewriting a nonterminal in the first part, the number of nonterminals appearing in the first part might exceed n. To prevent this, the constructed productions move the extra nonterminals from the end of the first part to the beginning of the second part (see all productions from (2.a) and (2.b)), so the number of nonterminals appearing in the first part is no more than n.

Apart from adding nonterminals, a nonterminal can be removed from the first part as well. This happens when a nonterminal is rewritten to a string over T. In this case, a special action is in order when the second part contains some nonterminals. For this purpose, the grammar records the last nonterminal of the sentential form. If the last nonterminal appears within the first part, it is represented by a symbol from N_2, and if it occurs in the second part, it is represented by a symbol from N_4. Now, consider the case when a nonterminal from the first part is rewritten to a string over T and the second part contains some nonterminals; in other words, a symbol from N_2 does not appear at the end of the first part. In this case, the first nonterminal of the second part is removed, converted to a symbol from N_1 (or N_2 if it is the last nonterminal), and added to the end of the first part (see the productions introduced in (2.c.i.A) and (2.c.i.B)). Therefore, the number of nonterminals that appear in the first part remains n. If a symbol from N_2 appears at the end of the first part, the second part can be ignored because it does not contain any nonterminal. In this way, the grammar guarantees that if the first part of the sentential form contains fewer than n nonterminals, the second part does not contain any nonterminal at all. Productions from (2.c.ii) and (2.c.iii) are used when the second part remains empty after the sentential form is rewritten.

Every production changes the current state p incorporated into every nonterminal of the first part to the new state q. In addition, each of these nonterminals records the number of nonterminals, k, occurring in the first part of the sentential form and this number is updated after every single derivation step. Therefore, productions that simulate the rewriting in a different state and productions that rewrite a different number of nonterminals are not applicable.

Formal Proof.
By examining the constructed productions, we see that the derivations of G and $\bar{G}$ resemble each other very much. In most cases, one production of $\bar{G}$ simulates one production of G. However, when a production from (2.c.i.A) is applied, this application is followed by (2.c.i.B), so in this case, one derivation step in G corresponds to two derivation steps in $\bar{G}$. Formally, we define the term *sf-correspondence* between the sentential forms of G and $\bar{G}$ by the following recursive definition, and use this term in the formulation of Claim 1:

1. The sentential form (S, p_0) of G sf-corresponds to the sentential form $\langle \hat{S}, p_0, 1 \rangle$ in $\bar{G}$.
2. Let $(x, p) \Rightarrow_G (y, q)$ $[\alpha]$, where (x, p) sf-corresponds to some $\bar{x}$ in $\bar{G}$.
   - If $(x, p) \Rightarrow_G (y, q)$ $[\alpha]$ satisfies $|x|_{V-T} > n$, $k + t - 1 \leq n$, and $t = 0$, then (y, q) sf-corresponds to $\bar{z}$ in $\bar{G}$, where $\bar{x} \Rightarrow^2_{\bar{G}} \bar{z}$ $[\bar{\alpha}_1 \bar{\alpha}_2]$, $\bar{\alpha}_1$ and $\bar{\alpha}_2$ are productions from (2.c.i.A) and (2.c.i.B), respectively, whose construction is based on α.
   - Otherwise, (y, q) sf-corresponds to $\bar{y}$ in $\bar{G}$, where $\bar{x} \Rightarrow_{\bar{G}} \bar{y}$ $[\bar{\alpha}]$ and the construction of $\bar{\alpha}$ is based on α.

Claim 1. Every sentential form $(y_1A_1 \dots y_mA_my_{m+1}, p)$ of G, where $y_1, \dots, y_{m+1} \in T^*$, $p \in K$, $A_1, \dots, A_m \in V - T$, for some $m \geq 0$, sf-corresponds to one of the following sentential forms in $\bar{G}$:

1. $y_1\langle A_1, p, m\rangle \dots y_{m-1}\langle A_{m-1}, p, m\rangle y_m\langle \hat{A}_m, p, m\rangle y_{m+1}$, for $m \leq n$;
2. $y_1\langle A_1, p, n\rangle \dots y_n\langle A_n, p, n\rangle y_{n+1}A_{n+1} \dots y_{m-1}A_{m-1}y_m\hat{A}_my_{m+1}$, for $m > n$.

Proof. Every derivation in $\bar{G}$ starts by the production from (1), and this production is not used during the rest of the derivation process, so

$$S \Rightarrow_{\bar{G}} \langle \hat{S}, p_0, 1\rangle.$$

The rest of the claim is proved by induction on length h of derivations, for $h \geq 0$.

Basis. Let $h = 0$. Then, $(S, p_0) \Rightarrow^0_G (S, p_0)$ corresponds to $\langle \hat{S}, p_0, 1\rangle \Rightarrow^0_{\bar{G}} \langle \hat{S}, p_0, 1\rangle$.

Induction Hypothesis. Suppose that the claim holds for all derivations of length h or less, for some $h \geq 0$.

Induction Step. First, consider a sentential form $(y_1A_1 \dots y_mA_my_{m+1}, p)$ of G, where $m \leq n$, and a production

$$(A_r, p) \to (x_1B_1 \dots x_tB_tx_{t+1}, q) \in P,$$

where $1 \leq r \leq m$, $B_1, \dots, B_t \in V - T$, $x_1, \dots, x_{t+1} \in T^*$, for some $t \geq 0$, that is applicable to the above sentential form (that is, $A_i \notin \alpha(p)$, for every $1 \leq i < r$). Then,

$$\begin{aligned}&(y_1A_1 \dots y_{r-1}A_{r-1}y_rA_ry_{r+1}A_{r+1} \dots y_mA_my_{m+1}, p)\\ \Rightarrow_G\ &(y_1A_1 \dots y_{r-1}A_{r-1}y_rx_1B_1 \dots x_tB_tx_{t+1}y_{r+1}A_{r+1} \dots y_mA_my_{m+1}, q).\end{aligned}$$

By the induction hypothesis, for $m \leq n$, the sentential form of $\bar{G}$ sf-corresponding to

$$(y_1A_1 \dots y_{r-1}A_{r-1}y_rA_ry_{r+1}A_{r+1} \dots y_mA_my_{m+1}, p)$$

is of the form

$$y_1\langle A_1, p, m\rangle \dots y_r\langle A_r, p, m\rangle \dots y_{m-1}\langle A_{m-1}, p, m\rangle y_m\langle \hat{A}_m, p, m\rangle y_{m+1}.$$

Now, one of the productions from (2.a.ii), (2.a.iii), (2.b.ii), (2.c.ii), or (2.c.iii) is applicable depending on the simulated production, m, and n:

- If $r + t - 1 > n$ and $r < m$, then a production introduced in (2.a.ii) is applied, so

$$\begin{aligned}&y_1\langle A_1, p, m\rangle \dots y_{r-1}\langle A_{r-1}, p, m\rangle y_r\langle A_r, p, m\rangle\\ &y_{r+1}\langle A_{r+1}, p, m\rangle \dots y_{m-1}\langle A_{m-1}, p, m\rangle y_m\langle \hat{A}_m, p, m\rangle y_{m+1}\\ {}_{lim}^{\ n}\!\Rightarrow_{\bar{G}}\ &y_1\langle A_1, q, n\rangle \dots y_{r-1}\langle A_{r-1}, q, n\rangle y_r\\ &x_1\langle B_1, q, n\rangle \dots x_{n-r+1}\langle B_{n-r+1}, q, n\rangle x_{n-r+2}B_{n-r+2} \dots x_tB_tx_{t+1}\\ &y_{r+1}A_{r+1} \dots y_{m-1}A_{m-1}y_m\hat{A}_my_{m+1}.\end{aligned}$$

- If $r+t-1>n$ and $r=m$, then a production introduced in (2.a.iii) is applied, so

$$\begin{aligned}&y_1\langle A_1,p,m\rangle\dots y_{m-1}\langle A_{m-1},p,m\rangle y_m\langle \hat{A}_m,p,m\rangle y_{m+1}\\ {}_{lim}\!\overset{n}{\Rightarrow}_{\bar{G}}\ &y_1\langle A_1,q,n\rangle\dots y_{m-1}\langle A_{m-1},q,n\rangle y_m\\ &x_1\langle B_1,q,n\rangle\dots x_{n-m+1}\langle B_{n-m+1},q,n\rangle\\ &x_{n-m+2}B_{n-m+2}\dots x_{t-1}B_{t-1}x_t\hat{B}_t x_{t+1}.\end{aligned}$$

- If $r+t-1\le n$, $m+t-1>n$, and $r<m$, then a production introduced in (2.b.ii) is applied, so

$$\begin{aligned}&y_1\langle A_1,p,m\rangle\dots y_{r-1}\langle A_{r-1},p,m\rangle y_r\langle A_r,p,m\rangle\\ &y_{r+1}\langle A_{r+1},p,m\rangle\dots y_{m-1}\langle A_{m-1},p,m\rangle y_m\langle \hat{A}_m,p,m\rangle y_{m+1}\\ {}_{lim}\!\overset{n}{\Rightarrow}_{\bar{G}}\ &y_1\langle A_1,q,n\rangle\dots y_{r-1}\langle A_{r-1},q,n\rangle y_r x_1\langle B_1,q,n\rangle\dots x_t\langle B_t,q,n\rangle x_{t+1}\\ &y_{r+1}\langle A_{r+1},q,n\rangle\dots y_{n-t+1}\langle A_{n-t+1},q,n\rangle\\ &y_{n-t+2}A_{n-t+2}\dots y_{m-1}A_{m-1}y_m\hat{A}_m y_{m+1}.\end{aligned}$$

- If $m+t-1\le n$ and $r<m$, then a production introduced in (2.c.ii) is applied, so

$$\begin{aligned}&y_1\langle A_1,p,m\rangle\dots y_{r-1}\langle A_{r-1},p,m\rangle y_r\langle A_r,p,m\rangle\\ &y_{r+1}\langle A_{r+1},p,m\rangle\dots y_{m-1}\langle A_{m-1},p,m\rangle y_m\langle \hat{A}_m,p,m\rangle y_{m+1}\\ {}_{lim}\!\overset{n}{\Rightarrow}_{\bar{G}}\ &y_1\langle A_1,q,m+t-1\rangle\dots y_{r-1}\langle A_{r-1},q,m+t-1\rangle y_r\\ &x_1\langle B_1,q,m+t-1\rangle\dots x_t\langle B_t,q,m+t-1\rangle x_{t+1}\\ &y_{r+1}\langle A_{r+1},q,m+t-1\rangle\dots y_{m-1}\langle A_{m-1},q,m+t-1\rangle\\ &y_m\langle \hat{A}_m,q,m+t-1\rangle y_{m+1}.\end{aligned}$$

- If $m+t-1\le n$ and $r=m$, and $m>1$ or $t\ne 0$, then a production introduced in (2.c.iii.A) is applied, so

$$\begin{aligned}&y_1\langle A_1,p,m\rangle\dots y_{m-1}\langle A_{m-1},p,m\rangle y_m\langle \hat{A}_m,p,m\rangle y_{m+1}\\ {}_{lim}\!\overset{n}{\Rightarrow}_{\bar{G}}\ &y_1\langle A_1,q,m+t-1\rangle\dots y_{m-1}\langle A_{m-1},q,m+t-1\rangle y_m\\ &x_1\langle B_1,q,m+t-1\rangle\dots x_{t-1}\langle B_{t-1},q,m+t-1\rangle\\ &x_t\langle \hat{B}_t,q,m+t-1\rangle x_{t+1}y_{m+1}.\end{aligned}$$

- If $m=1$, $t=0$, then a production introduced in (2.c.iii.B) is applied, so

$$y_1\langle \hat{A}_1,p,1\rangle y_2\ {}_{lim}\!\overset{n}{\Rightarrow}_{\bar{G}}\ y_1x_1y_2.$$

  Observe that this production removes the last symbol from $N_1\cup N_2$ from the sentential form and that this symbol appears on the left-hand side of every production introduced in (2). As a result, this production can be used only during the very last derivation step.

Second, consider a sentential form $(y_1A_1 \dots y_mA_my_{m+1}, p)$ of G, where $m > n$, and a production

$$(A_r, p) \to (x_1B_1 \dots x_tB_tx_{t+1}, q) \in P,$$

where $1 \le r \le m$, $B_1, \dots, B_t \in V - T$, $x_1, \dots, x_{t+1} \in T^*$, for some $t \ge 0$, that is applicable to the above sentential form. Then,

$$\begin{aligned} &(y_1A_1 \dots y_{r-1}A_{r-1}y_rA_ry_{r+1}A_{r+1} \dots y_nA_n \dots y_mA_my_{m+1}, p) \\ \Rightarrow_G\ &(y_1A_1 \dots y_{r-1}A_{r-1}y_rx_1B_1 \dots x_tB_tx_{t+1} \\ &\ y_{r+1}A_{r+1} \dots y_nA_n \dots y_mA_my_{m+1}, q). \end{aligned}$$

By the induction hypothesis, for $m > n$, the sentential form of $\bar{G}$ sf-corresponding to

$$(y_1A_1 \dots y_{r-1}A_{r-1}y_rA_ry_{r+1}A_{r+1} \dots y_nA_n \dots y_mA_my_{m+1}, p)$$

is of the form

$$y_1\langle A_1, p, n\rangle \dots y_r\langle A_r, p, n\rangle \dots y_n\langle A_n, p, n\rangle y_{n+1}A_{n+1} \dots y_{m-1}A_{m-1}y_m\hat{A}_my_{m+1}.$$

Now, one of the productions from (2.a.i), (2.b.i), and (2.c.i.A) is applicable depending on the simulated production, m, and n:

- If $r + t - 1 > n$, then a production introduced in (2.a.i) is applied, so

$$\begin{aligned} &y_1\langle A_1, p, n\rangle \dots y_{r-1}\langle A_{r-1}, p, n\rangle y_r\langle A_r, p, n\rangle \\ &y_{r+1}\langle A_{r+1}, p, n\rangle \dots y_n\langle A_n, p, n\rangle \\ &y_{n+1}A_{n+1} \dots y_{m-1}A_{m-1}y_m\hat{A}_my_{m+1} \\ {}_{lim}\!\overset{n}{\Rightarrow}_{\bar{G}}\ &y_1\langle A_1, q, n\rangle \dots y_{r-1}\langle A_{r-1}, q, n\rangle y_r \\ &x_1\langle B_1, q, n\rangle \dots x_{n-r+1}\langle B_{n-r+1}, q, n\rangle x_{n-r+2}B_{n-r+2} \dots x_tB_tx_{t+1} \\ &y_{r+1}A_{r+1} \dots y_nA_ny_{n+1}A_{n+1} \dots y_{m-1}A_{m-1}y_m\hat{A}_my_{m+1}. \end{aligned}$$

- If $r + t - 1 \le n$ and $m + t - 1 > n$, then a production introduced in (2.b.i) is applied, so

$$\begin{aligned} &y_1\langle A_1, p, n\rangle \dots y_{r-1}\langle A_{r-1}, p, n\rangle y_r\langle A_r, p, n\rangle \\ &y_{r+1}\langle A_{r+1}, p, n\rangle \dots y_{n-t+1}\langle A_{n-t+1}, p, n\rangle \\ &y_{n-t+2}\langle A_{n-t+2}, p, n\rangle \dots y_n\langle A_n, p, n\rangle \\ &y_{n+1}A_{n+1} \dots y_{m-1}A_{m-1}y_m\hat{A}_my_{m+1} \\ {}_{lim}\!\overset{n}{\Rightarrow}_{\bar{G}}\ &y_1\langle A_1, q, n\rangle \dots y_{r-1}\langle A_{r-1}, q, n\rangle y_r \\ &x_1\langle B_1, q, n\rangle \dots x_t\langle B_t, q, n\rangle x_{t+1} \\ &y_{r+1}\langle A_{r+1}, q, n\rangle \dots y_{n-t+1}\langle A_{n-t+1}, q, n\rangle \\ &y_{n-t+2}A_{n-t+2} \dots y_nA_ny_{n+1}A_{n+1} \dots y_{m-1}A_{m-1}y_m\hat{A}_my_{m+1}. \end{aligned}$$

- If $m+t-1 \leq n$ and $t = 0$, then a production introduced in (2.c.i.A) is applied, so

$$\begin{aligned}
& y_1 \langle A_1, p, n\rangle \dots y_{r-1} \langle A_{r-1}, p, n\rangle y_r \langle A_r, p, n\rangle \\
& y_{r+1} \langle A_{r+1}, p, n\rangle \dots y_n \langle A_n, p, n\rangle \\
& y_{n+1} A_{n+1} \dots y_{m-1} A_{m-1} y_m \hat{A}_m y_{m+1} \\
{}_{lim}\!\overset{n}{\Rightarrow}_{\bar{G}}\; & y_1 \langle A'_1, q, n-1\rangle \dots y_{r-1} \langle A'_{r-1}, q, n-1\rangle y_r x_1 \\
& y_{r+1} \langle A'_{r+1}, q, n-1\rangle \dots y_n \langle A'_n, q, n-1\rangle \\
& y_{n+1} A_{n+1} \dots y_{m-1} A_{m-1} y_m \hat{A}_m y_{m+1}.
\end{aligned}$$

  Recall that the last nonterminal in every sentential form of $\bar{G}$ is from $N_2 \cup N_4$. As $\langle A_n, p, n\rangle \notin N_2 \cup N_4$, there is at least one nonterminal in the sentential form following $\langle A_n, p, n\rangle$. Therefore, a production from (2.c.i.B) can be used. This production rewrites a nonterminal $A \in (V-T) \cup N_4$ in its last component. Because we generate the language of degree n, $A = A_{n+1}$, so

  - either

$$\begin{aligned}
& y_1 \langle A'_1, q, n-1\rangle \dots y_{r-1} \langle A'_{r-1}, q, n-1\rangle y_r x_1 \\
& y_{r+1} \langle A'_{r+1}, q, n-1\rangle \dots y_n \langle A'_n, q, n-1\rangle \\
& y_{n+1} A_{n+1} y_{n+2} A_{n+2} \dots y_{m-1} A_{m-1} y_m \hat{A}_m y_{m+1} \\
{}_{lim}\!\overset{n}{\Rightarrow}_{\bar{G}}\; & y_1 \langle A_1, q, n\rangle \dots y_{r-1} \langle A_{r-1}, q, n\rangle y_r x_1 \\
& y_{r+1} \langle A_{r+1}, q, n\rangle \dots y_n \langle A_n, q, n\rangle \\
& y_{n+1} \langle A_{n+1}, q, n\rangle y_{n+2} A_{n+2} \dots y_{m-1} A_{m-1} y_m \hat{A}_m y_{m+1},
\end{aligned}$$

    for $A_{n+1} \in V-T$,

  - or

$$\begin{aligned}
& y_1 \langle A'_1, q, n-1\rangle \dots y_{r-1} \langle A'_{r-1}, q, n-1\rangle y_r x_1 \\
& y_{r+1} \langle A'_{r+1}, q, n-1\rangle \dots y_n \langle A'_n, q, n-1\rangle y_{n+1} \hat{A}_{n+1} y_{n+2} \\
{}_{lim}\!\overset{n}{\Rightarrow}_{\bar{G}}\; & y_1 \langle A_1, q, n\rangle \dots y_{r-1} \langle A_{r-1}, q, n\rangle y_r x_1 \\
& y_{r+1} \langle A_{r+1}, q, n\rangle \dots y_n \langle A_n, q, n\rangle y_{n+1} \langle \hat{A}_{n+1}, q, n\rangle y_{n+2},
\end{aligned}$$

    for $\hat{A}_{n+1} \in N_4$.

To see that for a given sentential form and a production of G, there exists only one of the above derivations in $\bar{G}$, let us make the following observations:

- Notice that every sentential form of G determines the number of nonterminals m, first $k \leq n$ nonterminals, and the state p. Furthermore, a production of G determines the new state q, B_1, …, B_t, and the constants r and t. Observe that for any sentential form and a production like these, there exists only one production in $\bar{G}$ that simulates this production.

- The simulating production rewrites all nonterminals from $N_1 \cup N_2$ appearing in the sentential form of $\bar{G}$. Indeed, as k is included in each of these nonterminals, no production rewriting fewer than k nonterminals can be used.

- The last nonterminal of the sentential form in $\bar{G}$ is always from $N_2 \cup N_4$.
- A production of G is properly simulated by the corresponding production of $\bar{G}$—that is, all the constructed productions satisfy $A_i \notin \alpha(p)$, for every $1 \le i < r$, and p is changed to q in all nonterminals from $N_1 \cup N_2$. In addition, in every nonterminal from $N_1 \cup N_2$, k is properly updated.

Finally, notice that if a sentential form of $\bar{G}$ is of the form (1) or (2) as described in Claim 1, the sentential form obtained after performing a derivation step is of one of these forms as well. As the right-hand side of the production introduced in (1) of the construction is of the form (1), every sentential form obtained during the derivation process satisfies the properties given in Claim 1. □

From Claim 1 and the derivations described in its proof, it is easy to see that $\bar{G}$ rewrites at most n first nonterminals in a sentential form and that $L(G,n) = L(\bar{G}, lim, n)$. ■

Lemma 5.3. *$\mathscr{L}(PSC, lim, n) \subseteq \mathscr{L}(ST, n)$, for all $n \ge 1$.*

Proof. Let $L(G, lim, n)$ be a language of degree n generated by a propagating scattered context grammar $G = (V, T, P, S)$. Set

$$N = \big\{\langle A, i\rangle : A \in V - T, 1 \le i \le n\big\}.$$

Furthermore, set $K_1 = \big\{\langle p, i\rangle : p \in P, 0 \le i < n\big\}$ and

$$K_2 = \big\{\langle p, i, j\rangle : p = (A_1, \dots, A_k) \to (x_1, \dots, x_k) \in P, 0 \le i \le n, 0 \le j \le k\big\}.$$

Define the state grammar

$$\bar{G} = \big(V \cup N \cup \{\bar{S}\}, T, K_1 \cup K_2 \cup \{p_0\}, \bar{P}, \bar{S}, p_0\big)$$

with $\bar{P}$ constructed as follows:

1. For each $p = (S) \to (x) \in P$, add
   $(\bar{S}, p_0) \to \big(S, \langle p, 0\rangle\big)$ to $\bar{P}$.
2. For each $A \in V - T$, $p = (A_1, \dots, A_k) \to (x_1, \dots, x_k) \in P$, $0 \le i < n$, add
   (a) $\big(A, \langle p, i\rangle\big) \to \big(\langle A, i+1\rangle, \langle p, i+1\rangle\big)$,
   (b) $\big(A, \langle p, i\rangle\big) \to \big(\langle A, i+1\rangle, \langle p, i+1, k\rangle\big)$ to $\bar{P}$.
3. For each $p = (A_1, \dots, A_j, \dots, A_k) \to (x_1, \dots, x_j, \dots, x_k) \in P$, $q \in P$, $A \in V - T$, $1 \le i \le n$, $0 \le j \le k$, add
   (a) $\big(\langle A, i\rangle, \langle p, i, j\rangle\big) \to \big(A, \langle p, i-1, j\rangle\big)$,

(b) if $j \geq 1$, add
$\big(\langle A_j, i\rangle, \langle p, i, j\rangle\big) \to \big(x_j, \langle p, i-1, j-1\rangle\big)$ to $\bar{P}$.

(c) Add $\big(A, \langle p, 0, 0\rangle\big) \to \big(A, \langle q, 0\rangle\big)$ to $\bar{P}$.

Basic Idea.
Every derivation step of G is simulated in two phases in $\bar{G}$. In the first phase, performed by productions from (2), $\bar{G}$ assigns a sequence number to the first m nonterminals in the sentential form, where $m \leq n$ is selected non-deterministically. The form of the constructed productions guarantees that no nonterminal is skipped during this phase. $\bar{G}$ enters the second phase by a production from (2b). In the second phase, $\bar{G}$ simulates the scattered context production backwards; it starts by simulating the last context-free component of the scattered context production and ends by simulating its first context-free component. The previously numbered nonterminals are processed backwards from this point on; as before, none of them can be skipped. The state of $\bar{G}$ consists of three components. First, it contains the scattered context production that is being simulated; second, it contains the position of the nonterminal within the sentential form that is being rewritten; finally, it contains the position of the context-free component within the scattered context production whose rewriting is being simulated. If the current nonterminal coincides with the left-hand side of the currently simulated context-free component, the simulation can be performed by a production from (3b). Every nonterminal can be skipped by a production from (3a), which only removes the sequence number assigned during the first phase. Finally, when the simulation of the whole scattered context production is completed (the last component of the state equals 0) and the sequence numbers are removed from all nonterminals (the second component of the state equals 0), the simulation of the following scattered context production can be initiated by a production from (3c). Otherwise, the simulation is unsuccessful and the derivation is blocked.

Formal Proof.
The derivation starts in $\bar{G}$ by a production introduced in (1), and because no production contains $\bar{S}$ on its right-hand side, none of the productions from (1) is used during the rest of the derivation.

Consider a sentential form $u_1A_1 \dots u_kA_ku_{k+1}$ of G, where $u_1, \dots, u_{k+1} \in V^*$, and a production

$$p = (A_1, \dots, A_k) \to (x_1, \dots, x_k) \in P.$$

Obviously, for a sentential form satisfying $|u_1A_1 \dots u_kA_k|_{V-T} \leq n$,

$$u_1A_1 \dots u_kA_ku_{k+1} \ {}_{lim}\!\overset{n}{\Rightarrow}_G \ u_1x_1 \dots u_kx_ku_{k+1} \ [p].$$

Consider now a sentential form

$$\big(u_1A_1 \dots u_kA_ku_{k+1}, \langle p, 0\rangle\big)$$

of $\bar{G}$ corresponding to the above sentential form of G. Notice that $\big(S, \langle q, 0\rangle\big)$, where $q = (S) \to (x) \in P$, obtained by the application of a production from (1) is a sentential form of this kind as well. To describe the derivation of G, we express

$$\big(u_1A_1 \dots u_kA_ku_{k+1}, \langle p,0\rangle\big)$$

as

$$\big(w_1B_1 \dots w_mB_m \dots w_nB_n \dots w_tB_tw_{t+1}, \langle p,0\rangle\big),$$

where $w_1, \dots, w_{t+1} \in T^*$, $B_1, \dots, B_t \in V-T$, $t = |u_1A_1 \dots u_kA_ku_{k+1}|_{V-T}$, $B_{l_i} = A_i$, for some $1 \le l_i \le m$, for all $1 \le i \le k$, and $l_j < l_{j+1}$, for all $1 \le j \le k-1$. (Discussion of other possible kinds of this sentential form, for example when $t < n$, is left to the reader.) Then, the derivation performed by productions from (2a) can be expressed as

$$\begin{aligned}
&\big(w_1B_1 \dots w_mB_m \dots w_tB_tw_{t+1}, \langle p,0\rangle\big)\\
\Rightarrow_{\bar{G}}\ &\big(w_1\langle B_1,1\rangle w_2B_2 \dots w_mB_m \dots w_tB_tw_{t+1}, \langle p,1\rangle\big)\\
&\vdots\\
\Rightarrow_{\bar{G}}\ &\big(w_1\langle B_1,1\rangle \dots w_{m-1}\langle B_{m-1},m-1\rangle w_mB_m \dots w_tB_tw_{t+1}, \langle p,m-1\rangle\big).
\end{aligned}$$

Finally, a production from (2b) is used, so

$$\begin{aligned}
&\big(w_1\langle B_1,1\rangle \dots w_{m-1}\langle B_{m-1},m-1\rangle w_mB_m \dots w_tB_tw_{t+1}, \langle p,m-1\rangle\big)\\
\Rightarrow_{\bar{G}}\ &\big(w_1\langle B_1,1\rangle \dots w_m\langle B_m,m\rangle w_{m+1}B_{m+1} \dots w_tB_tw_{t+1}, \langle p,m,k\rangle\big).
\end{aligned}$$

Next, productions from (3) simulate all context-free components of p in reverse order. The simulation of $A_k \rightarrow x_k$ is performed as follows:

$$\begin{aligned}
&\big(w_1\langle B_1,1\rangle \dots w_m\langle B_m,m\rangle w_{m+1}B_{m+1} \dots w_tB_tw_{t+1}, \langle p,m,k\rangle\big)\\
\Rightarrow_{\bar{G}}\ &\big(w_1\langle B_1,1\rangle \dots w_{m-1}\langle B_{m-1},m-1\rangle w_mB_m \dots w_tB_tw_{t+1}, \langle p,m-1,k\rangle\big)\\
&\vdots\\
\Rightarrow_{\bar{G}}\ &\big(w_1\langle B_1,1\rangle \dots w_{l_k}\langle B_{l_k},l_k\rangle w_{l_k+1}B_{l_k+1} \dots w_tB_tw_{t+1}, \langle p,l_k,k\rangle\big)\\
\Rightarrow_{\bar{G}}\ &\big(w_1\langle B_1,1\rangle \dots w_{l_k-1}\langle B_{l_k-1},l_k-1\rangle\\
&\quad w_{l_k}x_kw_{l_k+1}B_{l_k+1} \dots w_tB_tw_{t+1}, \langle p,l_k-1,k-1\rangle\big).
\end{aligned}$$

The context-free components $A_{k-1} \rightarrow x_{k-1}, \dots, A_1 \rightarrow x_1$ are simulated analogously until a sentential form

$$\big(u_1x_1 \dots u_kx_ku_{k+1}, \langle p,0,0\rangle\big)$$

is obtained. Notice that when a state $\langle p,0,i\rangle$, where $i \ge 1$, is reached, the derivation is blocked. This means that either in the non-deterministic part of the derivation the value of m was chosen too low so the whole scattered context production cannot be simulated or more than n first nonterminals need to be rewritten to simulate the scattered context production. Finally, a production from (3c) finishes the simulation of p and starts the simulation of some $q \in P$:

$$\begin{aligned} &\big(u_1x_1 \dots u_kx_ku_{k+1}, \langle p,0,0\rangle\big) \\ \Rightarrow_{\bar{G}} &\big(u_1x_1 \dots u_kx_ku_{k+1}, \langle q,0\rangle\big). \end{aligned}$$

This simulation continues until we obtain the sentential form $\big(w, \langle r,0,0\rangle\big)$, where $w \in T^*$ and $r \in P$. ■

As $\mathscr{L}(ST,n) \subseteq \mathscr{L}(PSC, lim,n)$ and $\mathscr{L}(PSC, lim,n) \subseteq \mathscr{L}(ST,n)$, for all $n \geq 1$, we obtain the the following result.

Theorem 5.4. *$\mathscr{L}(PSC, lim,n) = \mathscr{L}(ST,n)$, for all $n \geq 1$.* ■

Next, we reformulate Theorem 5.4 in terms of $\mathscr{L}(PSC, un, lim,n)$.

Theorem 5.5. *$\mathscr{L}(PSC, un, lim,n) = \mathscr{L}(ST,n)$, for all $n \geq 1$.*

Proof. To prove this theorem, we demonstrate that $\mathscr{L}(ST,n) \subseteq \mathscr{L}(PSC, un, lim,n)$ and $\mathscr{L}(PSC, un, lim,n) \subseteq \mathscr{L}(ST,n)$, for all $n \geq 1$.

The first inclusion can be proved similarly to the proof of Lemma 5.2. However, in this case, the constructed grammar must also record the sequence number of each of the first n nonterminals in the sentential form in order to guarantee that the right component of the unordered scattered context production is used; without this record, any permutation of the unordered scattered context production could be applied. Therefore, the elements of the sets N_1, N_2, and N_3 have to be changed to contain one more component that determines the order of the nonterminal in the sentential form. In addition, each of the constructed productions has to be modified so that it preserves the correct sequence numbers of these nonterminals in every sentential form. This can be easily accomplished because each of these productions rewrites all the nonterminals from $N_1 \cup N_2 \cup N_3$ in the sentential form. The rest of the proof is analogous to the proof of Lemma 5.2 and, therefore, left to the reader.

The second inclusion, $\mathscr{L}(PSC, un, lim,n) \subseteq \mathscr{L}(ST,n)$, can be proved as follows. For any propagating unordered scattered context grammar G of degree n, we can construct a propagating scattered context grammar $\bar{G}$ of the same degree such that $L(G, lim,n) = L(\bar{G}, lim,n)$. Indeed, if we construct $\bar{G}$ so that it contains all possible permutations of every production of G, we obtain a grammar satisfying these properties. Therefore, $L(G, lim,n) = L(\bar{G}, lim,n)$ and $\mathscr{L}(PSC, un, lim,n) \subseteq \mathscr{L}(PSC, lim,n)$. As $\mathscr{L}(PSC, lim,n) = \mathscr{L}(ST,n)$, we obtain $\mathscr{L}(PSC, un, lim,n) \subseteq \mathscr{L}(ST,n)$. ■

Recall that

$$\mathscr{L}(CF) = \mathscr{L}(ST,1) \subset \mathscr{L}(ST,2) \subset \cdots \subset \mathscr{L}(ST,\infty) \subset \mathscr{L}(ST) = \mathscr{L}(CS),$$

where every $\mathscr{L}(ST,n)$, for some $n \geq 1$, is an abstract family of languages (see Theorems 2.32 and 2.33). These properties together with Theorems 5.4 and 5.5 imply the following two corollaries.

Corollary 5.6.

$$
\begin{aligned}
&\mathscr{L}(PSC, un, lim, 1) &= \mathscr{L}(PSC, lim, 1) &= \mathscr{L}(ST, 1) = \mathscr{L}(CF) \\
\subset\ &\mathscr{L}(PSC, un, lim, 2) &= \mathscr{L}(PSC, lim, 2) &= \mathscr{L}(ST, 2) \\
&\vdots \\
\subset\ &\mathscr{L}(PSC, un, lim, \infty) &= \mathscr{L}(PSC, lim, \infty) &= \mathscr{L}(ST, \infty) \\
\subset\ &\mathscr{L}(CS).
\end{aligned}
$$

■

Corollary 5.7. *Every $\mathscr{L}(PSC, lim, n)$ and $\mathscr{L}(PSC, un, lim, n)$, where $n \geq 1$, is an abstract family of languages.* ■

We have demonstrated that limiting derivations performed by propagating scattered context grammars and propagating unordered scattered context grammars to the first *n* nonterminals gives rise to an infinite hierarchy of languages. This result is of some practical interest in terms of compilers. When constructing a compiler based on a grammatical model, we usually need to restrict this model in order to make the compiler more effective. The presented result shows that if the model is based on propagating scattered context grammars, by limiting the width of the window in which the context dependency is checked, we also limit the power of the resulting compiler. In certain situations, such as parsing of streamed data, limiting the context dependency checks to a finite window is necessary because the exact length of the input is unknown.

From a theoretical point of view, the achieved results are interesting too. It is known that $\mathscr{L}(PSC, un) \subset \mathscr{L}(PSC)$ (see Corollary 5.20). However, when using *n*-limited derivations, $\mathscr{L}(PSC, un, lim, n) = \mathscr{L}(PSC, lim, n)$. On the other hand, the definition of *n*-limited derivations induces the following problem.

Open Problem 5.8. Can we construct ordinary propagating scattered context grammars that automatically perform only *n*-limited derivations—that is, they make these derivations without any explicitly required *n*-limited derivation restriction placed on them? If so, can we establish the results of this section for them? How would this modification influence the generative power of propagating unordered scattered context grammars?

5.2 Leftmost Derivations

While the previous section discussed a left restriction placed on sentential forms, the present section studies a left restriction placed on the use of productions in propagating scattered context grammars. In essence, this restriction requires that in the current sentential form, every production *p* is applied so that *p* rewrites the leftmost possible occurrences of the nonterminals corresponding to its left-hand side. We prove that propagating scattered context grammars restricted in this way are equivalent to context-sensitive grammars. This result is of some interest for two reasons. First, in terms of ordinary propagating scattered context grammars, this equivalence represents a famous long-standing open problem (see Open Problem 3.28). Second, regarding all four grammars underlying the Chomsky hierarchy, analogical restriction does not affect their generative power at all.

From a historical perspective, the equivalence mentioned above was originally established in [15] by a rather complicated and difficult-to-follow proof. Later on, the same result was proved by [25] in a simpler way. Naturally, we base this section on the latter.

Definition 5.9. A *propagating scattered context grammar that uses leftmost* or *rightmost derivations* is a propagating scattered context grammar $G = (V, T, P, S)$ whose language is defined as

$$L(G, lm) = \{x : x \in T^*, S \;{}_{lm}\!\Rightarrow_G^* x\}, \text{ or}$$
$$L(G, rm) = \{x : x \in T^*, S \;{}_{rm}\!\Rightarrow_G^* x\},$$

respectively. The family of languages generated by propagating scattered context grammars that use leftmost or rightmost derivations is denoted by $\mathscr{L}(PSC, lm)$ or $\mathscr{L}(PSC, rm)$, respectively.

The following theorem demonstrates that propagating scattered context grammars that perform their derivations in a leftmost way are equivalent to context-sensitive grammars.

Theorem 5.10. $\mathscr{L}(PSC, lm) = \mathscr{L}(CS)$.

Proof. As propagating scattered context grammars do not contain erasing productions, their derivations can be simulated by context-sensitive grammars. As a result, $\mathscr{L}(PSC, lm) \subseteq \mathscr{L}(CS)$. In what follows, we demonstrate that also $\mathscr{L}(CS) \subseteq \mathscr{L}(PSC, lm)$ holds true by demonstrating that for every context-sensitive grammar in Kuroda normal form, there exists an equivalent propagating scattered context grammar that uses leftmost derivations.

Let $G = (V, T, P, S)$ be a context-sensitive grammar in Kuroda normal form (see Definition 2.17). Set

$$N_1 = (V - T) \cup \{\bar{a} : a \in T\}$$

(suppose that $(V - T) \cap \{\bar{a} : a \in T\} = \emptyset$), and $\hat{N}_1 = \{\hat{A} : A \in N_1\}$. Let $n = |N_1|$; then, we denote the elements of N_1 as $\{A_1, \ldots, A_n\}$. Define the homomorphism α from V^* to N_1^* as $\alpha(A) = A$, for each $A \in V - T$, and $\alpha(a) = \bar{a}$, for each $a \in T$. Set

$$\begin{aligned}
N_2' &= \{A' : A \in V - T\},\\
N_3 &= \{\langle ab \rangle : a, b \in V\},\\
N_4' &= \{\langle Aa \rangle' : A \in V - T, a \in V\},\\
N_5 &= \{\langle a, 0 \rangle, \langle ab, 0 \rangle : a, b \in V\}\\
&\quad \cup \{\langle a, i, j \rangle : a \in V - T, 1 \le i \le 3, 1 \le j \le n\}\\
&\quad \cup \{\langle ab, 4 \rangle : a, b \in T\}.
\end{aligned}$$

Without loss of generality, assume that the sets N_1, $\hat{N}_1$, N_2', N_3, N_4', N_5, $\{\bar{S}, X\}$, and T are pairwise disjoint. Define the propagating scattered context grammar

$$\bar{G} = \big(N_1 \cup \hat{N}_1 \cup N_2' \cup N_3 \cup N_4' \cup N_5 \cup \{\bar{S}, X\} \cup T, T, \bar{P}, \bar{S}\big),$$

where $\bar{P}$ is constructed as follows:

1. (a) For each $a \in L(G)$, where $a \in T$, add $(\bar{S}) \to (a)$ to $\bar{P}$.
   (b) For each $S \Rightarrow_G ab$, where $a, b \in V$, add $(\bar{S}) \to \big(\langle ab, 0\rangle X\big)$ to $\bar{P}$.
2. For each $a, b, c \in V$, add
   (a) $\big(\langle a, 0\rangle, \alpha(b)\big) \to \big(\alpha(a), \langle b, 0\rangle\big)$,
   (b) $\big(\alpha(a), \langle b, 0\rangle\big) \to \big(\langle a, 0\rangle, \alpha(b)\big)$,
   (c) $\big(\langle a, 0\rangle, \langle bc\rangle\big) \to \big(\alpha(a), \langle bc, 0\rangle\big)$,
   (d) $\big(\alpha(a), \langle bc, 0\rangle\big) \to \big(\langle a, 0\rangle, \langle bc\rangle\big)$ to $\bar{P}$.
3. For each $A \to a \in P$ and $b \in V$, add
   (a) $\big(\langle A, 0\rangle\big) \to \big(\langle a, 0\rangle\big)$,
   (b) $\big(\langle Ab, 0\rangle\big) \to \big(\langle ab, 0\rangle\big)$,
   (c) $\big(\langle bA, 0\rangle\big) \to \big(\langle ba, 0\rangle\big)$ to $\bar{P}$.
4. For each $A \to BC \in P$ and $a \in V$, add
   (a) $\big(\langle A, 0\rangle\big) \to \big(B\langle C, 0\rangle\big)$,
   (b) $\big(\langle Aa, 0\rangle\big) \to \big(B\langle Ca, 0\rangle\big)$,
   (c) $\big(\langle aA, 0\rangle\big) \to \big(\alpha(a)\langle BC, 0\rangle\big)$ to $\bar{P}$.
5. For each $AB \to CD \in P$, $a \in V$, $E \in N_3 \cup N_4'$, $F' \in \{B', \langle Ba\rangle'\}$, $1 \le i \le n$, and $1 \le j \le n-1$, add
   (a) $\big(\langle AB, 0\rangle\big) \to \big(\langle CD, 0\rangle\big)$,
   (b) i. $\big(\langle A, 0\rangle, B, X\big) \to \big(\langle A, 1, 1\rangle, B', A_1\big)$,
       ii. $\big(\langle A, 0\rangle, \langle Ba\rangle, X\big) \to \big(\langle A, 1, 1\rangle, \langle Ba\rangle', A_1\big)$,
   (c) i. $\big(\langle A, 1, i\rangle, A_i\big) \to \big(\langle A, 2, i\rangle, \hat{A}_i\big)$,
       ii. $\big(\langle A, 2, i\rangle, F', \hat{A}_i\big) \to \big(\langle A, 3, i\rangle, F', A_i\big)$,
       iii. $\big(\langle A, 3, j\rangle, E, A_j\big) \to \big(\langle A, 1, j+1\rangle, E, A_{j+1}\big)$,
   (d) i. $\big(\langle A, 3, n\rangle, B', E, A_n\big) \to \big(\langle C, 0\rangle, D, E, X\big)$,
       ii. $\big(\langle A, 3, n\rangle, \langle Ba\rangle', A_n\big) \to \big(\langle C, 0\rangle, \langle Da\rangle, X\big)$ to $\bar{P}$.
6. For each $a, b, c \in T$, add

(a) $\big(\langle ab,0\rangle\big) \to \big(\langle ab,4\rangle\big)$,
(b) $\big(\bar{c},\langle ab,4\rangle\big) \to \big(c,\langle ab,4\rangle\big)$,
(c) $\big(\langle ab,4\rangle,X\big) \to (a,b)$ to $\bar{P}$.

Basic Idea.
Productions introduced in (1) initiate the derivation. Productions from (2) select the nonterminal to be rewritten. Productions from (3), (4), and (5) simulate productions of G of the form $A \to a$, $A \to BC$, and $AB \to CD$, respectively. Finally, productions from (6) finish the derivation.

Formal Proof.
Every derivation starts either by a production introduced in (1a) to generate sentences $a \in L(G)$, where $a \in T$, or by a production introduced in (1b) to generate sentences $x \in L(G)$, where $|x| \geq 2$. As $\bar{S}$ does not occur on the right-hand side of any production, productions from (1) are not used during the rest of the derivation.

Consider a sentential form $a_1 \dots a_k$ of G, where $a_1, \dots, a_k \in V$, for some $k \geq 2$. In $\bar{G}$, this sentential form corresponds

- either to
$$b_1 \dots b_{r-1}\langle a_r,0\rangle b_{r+1}b_{r+2}\dots b_{k-2}\langle a_{k-1}a_k\rangle X,$$
where $b_i = \alpha(a_i)$, for all
$$i \in \{1,2,\dots,r-1,r+1,r+2,\dots,k-2\},$$
for some $1 \leq r \leq k-2$,
- or to
$$b_1 \dots b_{k-2}\langle a_{k-1}a_k,0\rangle X,$$
where $b_i = \alpha(a_i)$, for all $1 \leq i \leq k-2$ (observe that every right-hand side of a production from (1b) represents a sentential form of this kind).

To simulate a production of G, the leftmost nonterminal from its left-hand side has to be selected in the sentential form of $\bar{G}$. This is made by appending 0 to the selected symbol by productions from (2). Specifically, for a symbol $a \in V$, (2a) selects the leftmost symbol a immediately following the currently selected symbol, and (2b) selects the leftmost symbol a preceding the currently selected symbol. Productions from (2c) and (2d) select and deselect the penultimate nonterminal in a sentential form of $\bar{G}$, which consists of two symbols from V. Observe that in this way, any symbol (except for the final X) in every sentential form of $\bar{G}$ can be selected. Furthermore, observe that during a derivation, always one symbol is selected.

After the nonterminal is selected, the application of the production of G can be simulated. Productions of the form $A \to a$ are simulated by (3a) for every selected nonterminal $a_1, \dots, a_{k-2}$, and by (3b), (3c) if the penultimate nonterminal (which contains a_{k-1}, a_k) of the sentential form of $\bar{G}$ is selected. Analogously, productions of the form $A \to BC$ are simulated by productions from (4).

Productions from (5a) simulate applications of productions of the form $AB \to CD$ within the penultimate nonterminal of a sentential form of $\bar{G}$. In what follows, we demonstrate how productions from (5b), (5c), and (5d) are used if this production is simulated within $a_1 \ldots a_{k-2}$. Suppose that the sentential form in $\bar{G}$ has the form

$$b_1 \ldots b_{r-1} \langle a_r, 0 \rangle b_{r+1} b_{r+2} \ldots b_{k-2} \langle a_{k-1} a_k \rangle X,$$

and we simulate the application of $a_r a_{r+1} \to c_r c_{r+1} \in P$. Recall that $N_1 = \{A_1, \ldots, A_n\}$ denotes the set of all symbols that may occur in $b_{r+1} b_{r+2} \ldots b_{k-2}$. First, to select $b_{r+1} = \alpha(a_{r+1})$, the production

$$\big(\langle a_r, 0\rangle, b_{r+1}, X\big) \to \big(\langle a_r, 1, 1\rangle, b'_{r+1}, A_1\big)$$

from (5bi) is applied in a successful derivation, so

$$\begin{aligned} & b_1 \ldots b_{r-1} \langle a_r, 0 \rangle b_{r+1} b_{r+2} \ldots b_{k-2} \langle a_{k-1} a_k \rangle X \\ {}_{lm}\!\Rightarrow_{\bar{G}}\ & b_1 \ldots b_{r-1} \langle a_r, 1, 1 \rangle b'_{r+1} b_{r+2} b_{r+3} \ldots b_{k-2} \langle a_{k-1} a_k \rangle A_1. \end{aligned}$$

Observe that if b_{r+1} does not immediately follow $\langle a_r, 0\rangle$, the leftmost

$$b \in \mathrm{alph}(b_{r+2} b_{r+3} \ldots b_{k-2})$$

satisfying $b = b_{r+1}$ is selected by the production from (5bi). The purpose of productions from (5c) is to verify that the nonterminal immediately following $\langle a_r, 0\rangle$ has been selected. First, the production

$$\big(\langle a_r, 1, 1\rangle, A_1\big) \to \big(\langle a_r, 2, 1\rangle, \hat{A}_1\big)$$

from (5ci) is applied to tag the first A_1 following $\langle a_r, 1, 1\rangle$, so

$$\begin{aligned} & b_1 \ldots b_{r-1} \langle a_r, 1, 1 \rangle b'_{r+1} b_{r+2} b_{r+3} \ldots b_{k-2} \langle a_{k-1} a_k \rangle A_1 \\ {}_{lm}\!\Rightarrow_{\bar{G}}\ & b_1 \ldots b_{r-1} \langle a_r, 2, 1 \rangle b'_{r+1} y_1 \langle a_{k-1} a_k \rangle d_1, \end{aligned}$$

where

- either $d_1 = A_1$,

  $$y_1 = b_{r+2} b_{r+3} \ldots b_{m-1} \hat{A}_1 b_{m+1} b_{m+2} \ldots b_{k-2},$$

  satisfying $A_1 \notin \mathrm{alph}(b_{r+2} b_{r+3} \ldots b_{m-1})$, for some $1 \leq m \leq k-2$,
- or $d_1 = \hat{A}_1$,

  $$y_1 = b_{r+2} b_{r+3} \ldots b_{k-2},$$

  satisfying $A_1 \notin \mathrm{alph}(y_1)$.

Then, the production

$$\big(\langle a_r,2,1\rangle,b'_{r+1},\hat{A}_1\big)\rightarrow\big(\langle a_r,3,1\rangle,b'_{r+1},A_1\big)$$

from (5cii) is applied to untag the first symbol $\hat{A}_1$ following b'_{r+1}, so

$$\begin{aligned}&b_1\ldots b_{r-1}\langle a_r,2,1\rangle b'_{r+1}y_1\langle a_{k-1}a_k\rangle d_1\\ {}_{lm}\Rightarrow_{\bar{G}}\ &b_1\ldots b_{r-1}\langle a_r,3,1\rangle b'_{r+1}b_{r+2}b_{r+3}\ldots b_{k-2}\langle a_{k-1}a_k\rangle A_1.\end{aligned}$$

This means that if A_1 occurs between $\langle a_r,2,1\rangle$ and b'_{r+1}, it is tagged by the production from (5ci), but it cannot be untagged by any production from (5cii), so the derivation is blocked. Finally, the production

$$\big(\langle a_r,3,1\rangle,\langle a_{k-1}a_k\rangle,A_1\big)\rightarrow\big(\langle a_r,1,2\rangle,\langle a_{k-1}a_k\rangle,A_2\big)$$

from (5ciii) is applied, so

$$\begin{aligned}&b_1\ldots b_{r-1}\langle a_r,3,1\rangle b'_{r+1}b_{r+2}b_{r+3}\ldots b_{k-2}\langle a_{k-1}a_k\rangle A_1\\ {}_{lm}\Rightarrow_{\bar{G}}\ &b_1\ldots b_{r-1}\langle a_r,1,2\rangle b'_{r+1}b_{r+2}b_{r+3}\ldots b_{k-2}\langle a_{k-1}a_k\rangle A_2,\end{aligned}$$

and the same verification can be performed for A_2. This verification proceeds for all symbols from N_1, so this part of the derivation can be expressed as

$$\begin{array}{lll} u_1 & {}_{lm}\Rightarrow_{\bar{G}}\ v_1\ [p_{11}] & {}_{lm}\Rightarrow_{\bar{G}}\ w_1\ [p_{12}]\\ {}_{lm}\Rightarrow_{\bar{G}}\ u_2\ [p_{13}] & {}_{lm}\Rightarrow_{\bar{G}}\ v_2\ [p_{21}] & {}_{lm}\Rightarrow_{\bar{G}}\ w_2\ [p_{22}]\\ \vdots & & \\ {}_{lm}\Rightarrow_{\bar{G}}\ u_n\ [p_{(n-1)3}] & {}_{lm}\Rightarrow_{\bar{G}}\ v_n\ [p_{n1}] & {}_{lm}\Rightarrow_{\bar{G}}\ w_n\ [p_{n2}]\end{array}$$

with

$$\begin{aligned}u_i&=b_1\ldots b_{r-1}\langle a_r,1,i\rangle b'_{r+1}b_{r+2}b_{r+3}\ldots b_{k-2}\langle a_{k-1}a_k\rangle A_i,\\ v_i&=b_1\ldots b_{r-1}\langle a_r,2,i\rangle b'_{r+1}y_i\langle a_{k-1}a_k\rangle d_i,\\ w_i&=b_1\ldots b_{r-1}\langle a_r,3,i\rangle b'_{r+1}b_{r+2}b_{r+3}\ldots b_{k-2}\langle a_{k-1}a_k\rangle A_i,\end{aligned}$$

where p_{i1}, p_{i2}, and p_{j3} are productions from (5ci), (5cii), and (5ciii), respectively, for all $1\leq i\leq n$, $1\leq j\leq n-1$, and

- either $d_i=A_i$,

  $$y_i=b_{r+2}b_{r+3}\ldots b_{i_{m-1}}\hat{A}_i b_{i_{m+1}}b_{i_{m+2}}\ldots b_{k-2},$$

  satisfying $A_i\notin\text{alph}(b_{r+2}b_{r+3}\ldots b_{i_{m-1}})$, for some $1\leq i_m\leq k-2$,

- or $d_i=\hat{A}_i$,

  $$y_i=b_{r+2}b_{r+3}\ldots b_{k-2},$$

  satisfying $A_i\notin\text{alph}(y_i)$.

After the verification is finished, the application of $a_r a_{r+1} \to c_r c_{r+1} \in P$ is simulated by

$$\big(\langle a_r, 3, n\rangle, b'_{r+1}, \langle a_{k-1}a_k\rangle, A_n\big) \to \big(\langle c_r, 0\rangle, c_{r+1}, \langle a_{k-1}a_k\rangle, X\big)$$

from (5di), so

$$\begin{aligned} & b_1 \ldots b_{r-1}\langle a_r, 3, n\rangle b'_{r+1} b_{r+2} b_{r+3} \ldots b_{k-2}\langle a_{k-1}a_k\rangle A_n \\ {}_{lm}\Rightarrow_{\bar{G}}\ & b_1 \ldots b_{r-1}\langle c_r, 0\rangle c_{r+1} b_{r+2} b_{r+3} \ldots b_{k-2}\langle a_{k-1}a_k\rangle X. \end{aligned}$$

Observe that in order to simulate a production of the form $AB \to CD$ within $a_{k-2}a_{k-1}$, productions from (5bii) and (5dii) have to be used instead of productions from (5bi) and (5di) in the simulation described above. The details of this simulation are left to the reader.

Finally, consider a sentence $a_1 \ldots a_k \in T^+$ of G. After the simulation is completed, this sentence corresponds in $\bar{G}$

- either to
$$\bar{a}_1 \ldots \bar{a}_{r-1}\langle a_r, 0\rangle \bar{a}_{r+1}\bar{a}_{r+2} \ldots \bar{a}_{k-2}\langle a_{k-1}a_k\rangle X$$
- or to
$$\bar{a}_1 \ldots \bar{a}_{k-2}\langle a_{k-1}a_k, 0\rangle X.$$

To enter the final phase in $\bar{G}$, we need the sentential form to be in the second above described form. This can be achieved by applying a production from (2c) to the first sentential form. The rest of the derivation can be expressed as

$$\begin{aligned} & \bar{a}_1 \ldots \bar{a}_{k-2}\langle a_{k-1}a_k, 0\rangle X \\ {}_{lm}\Rightarrow_{\bar{G}}\ & \bar{a}_1 \ldots \bar{a}_{k-2}\langle a_{k-1}a_k, 4\rangle X\ [p_{6a}] \\ {}_{lm}\Rightarrow_{\bar{G}}^{k-2}\ & a_1 \ldots a_{k-2}\langle a_{k-1}a_k, 4\rangle X\ [\Xi_{6b}] \\ {}_{lm}\Rightarrow_{\bar{G}}\ & a_1 \ldots a_{k-2}a_{k-1}a_k \qquad [p_{6c}], \end{aligned}$$

where p_{6a} and p_{6c} are productions introduced in (6a) and (6c), respectively, and Ξ_{6b} is a sequence of $k-2$ productions from (6b). As a result, $x \in L(\bar{G}, lm)$ if and only if $x \in L(G)$. Therefore, $\mathscr{L}(CS) \subseteq \mathscr{L}(PSC, lm)$.

As $\mathscr{L}(PSC, lm) \subseteq \mathscr{L}(CS)$ and $\mathscr{L}(CS) \subseteq \mathscr{L}(PSC, lm)$, we get $\mathscr{L}(PSC, lm) = \mathscr{L}(CS)$, so the theorem holds. ■

The following corollary reformulates Theorem 5.10 in terms of right derivations.

Corollary 5.11. $\mathscr{L}(PSC, rm) = \mathscr{L}(CS)$.

Proof. This corollary can be proved by a straightforward modification of the proof of Theorem 5.10 and its proof is, therefore, left to the reader. ■

5.3 Maximal and Minimal Derivations

In this section, we introduce two natural and simple modifications of propagating scattered context grammars. As a matter of fact, both modifications only change the way propagating scattered context grammars perform their derivations while keeping their grammatical concept unchanged. More specifically, during every derivation step, these modified versions select an applicable production containing the maximal or, in contrast, the minimal number of nonterminals on its left-hand side. We prove that both of these versions generate the family of context-sensitive languages.

Definition 5.12. Let $G = (V, T, P, S)$ be a scattered context grammar. Define the *maximal derivation step* as

$$u \;_{max}{\Rightarrow}_G\; v\;[p]$$

if and only if $u \Rightarrow_G v\;[p]$ and there is no $r \in P$ satisfying $\text{len}(r) > \text{len}(p)$ such that $u \Rightarrow_G w\;[r]$. Similarly, define the *minimal derivation step* as

$$u \;_{min}{\Rightarrow}_G\; v\;[p]$$

if and only if $u \Rightarrow_G v\;[p]$ and there is no $r \in P$ satisfying $\text{len}(r) < \text{len}(p)$ such that $u \Rightarrow_G w\;[r]$. Define the transitive closure and the reflexive and transitive closure of $_{max}{\Rightarrow}_G$ and $_{min}{\Rightarrow}_G$ in the standard way. The *language of a scattered context grammar G that uses maximal* or *minimal derivations* is denoted by $L(G, max)$ or $L(G, min)$, and defined as

$$L(G, max) = \{x : x \in T^*, S \;_{max}{\Rightarrow}_G^*\; x\}, \text{ or}$$
$$L(G, min) = \{x : x \in T^*, S \;_{min}{\Rightarrow}_G^*\; x\},$$

respectively. The corresponding language families are denoted by $\mathscr{L}(PSC, max)$ and $\mathscr{L}(PSC, min)$.

Next, we demonstrate that propagating scattered context grammars that use either maximal or minimal derivations characterize the family of context-sensitive languages.

Theorem 5.13. $\mathscr{L}(CS) = \mathscr{L}(PSC, max)$.

Proof. Let L be a context-sensitive language. As state grammars characterize the family of context-sensitive languages (see Theorem 2.32), we suppose that L is described by a state grammar $\bar{G} = (\bar{V}, T, K, \bar{P}, \bar{S}, p_0)$. Set

$$Y = \big\{\langle A, q\rangle : A \in \bar{V} - T, q \in K\big\}$$

and $Z = \{\bar{a} : a \in T\}$. Define the homomorphism α form $\bar{V}^*$ to $\big((\bar{V} - T) \cup Z\big)^*$ as $\alpha(A) = A$, for all $A \in \bar{V} - T$, and $\alpha(a) = \bar{a}$, for all $a \in T$. Set $V = \bar{V} \cup Y \cup Z \cup \{S, X\}$. Define the propagating scattered context grammar $G = (V, T, P, S)$, where P is constructed as follows:

1. For each $x \in L(\bar{G})$, where $|x| \leq 2$, add
   $(S) \to (x)$ to P.
2. For each

   $$(x,q) \in \Big\{(x,q) : (\bar{S}, p_0) \Rightarrow^{+}_{\bar{G}} (x,q),\ \text{for some } q \in K, \\ \text{and } 3 \leq |x| \leq \max\big(\{3\} \cup \{|y| : (B,p) \to (y,p') \in \bar{P}\}\big)\Big\},$$

   where

   (a) $x \in T^*$, add
   $(S) \to (x)$ to P;

   (b) $x = x_1 A x_2$, $A \in \bar{V} - T$, $x_1, x_2 \in \bar{V}^*$, add
   $(S) \to \big(\alpha(x_1)\langle A,q\rangle\alpha(x_2)\big)$ to P.
3. For each $(A,p) \to (x,q)$, $(B,p) \to (y,r) \in \bar{P}$, $C \in \bar{V}$, $\Gamma_{21} \in \text{perm}(2,1)$, and

   $$z = \text{reorder}\Big(\big(B, \langle A,p\rangle, \alpha(C)\big), \Gamma_{21}\Big),$$

   add $z \to (X,X,X)$ to P.
4. For each $(A,p) \to (x,q) \in \bar{P}$, $B \in \bar{V} - T$, $C \in \bar{V}$, $\Gamma_{11} \in \text{perm}(1,1)$, and

   $$y = \text{reorder}\Big(\big(\langle A,p\rangle, \alpha(C)\big), \Gamma_{11}\Big),$$

   add

   (a) $\big(B, \langle A,p\rangle\big) \to \big(\langle B,q\rangle, \alpha(x)\big)$,

   (b) $\big(\langle A,p\rangle, B\big) \to \big(\alpha(x), \langle B,q\rangle\big)$ to P;

   (c) if $x = vBw$, v, $w \in \bar{V}^*$, for each

   $$z = \text{reorder}\Big(\big(\alpha(v)\langle B,q\rangle\alpha(w), \alpha(C)\big), \Gamma_{11}\Big),$$

   add $y \to z$ to P;

   (d) for each

   $$u = \text{reorder}\Big(\big(\alpha(x), \alpha(C)\big), \Gamma_{11}\Big),$$

   add $y \to u$ to P.
5. For each $a \in T$, add
   $(\bar{a}) \to (a)$ to P.

Basic Idea.
The state grammar $\bar{G}$ is simulated by the propagating scattered context grammar G that performs maximal derivations. Productions introduced in (1) are used to

generate a sentence $w \in L(\bar{G})$, where $|w| \leq 2$, while productions introduced in (2) start the simulation of a derivation of a sentence $w \in \bar{G}$, where $|w| \geq 3$. Let $(A, p) \to (x, q)$ be a production of $\bar{G}$ that is applicable to a sentential form $(w_1 A w_2, p)$ generated by $\bar{G}$. The sentential form $(w_1 A w_2, p)$ in $\bar{G}$ corresponds to the sentential form $\alpha(w_1)\langle A, p\rangle\alpha(w_2)$ in G. Consider the application of a production $(A, p) \to (x, q)$ in $\bar{G}$. G simulates this application as follows. First, G checks whether the production is applied to the leftmost nonterminal of the sentential form for the given state p. If not, a production from (3) is applicable. This production is applied because it has the highest priority of all productions, and its application introduces the symbol X into the sentential form, which blocks the derivation. The successful derivation proceeds by applying a production from (4a), (4b), and (4c), which non-deterministically selects the next nonterminal to be rewritten, and appends the new state to it. The production that finishes the derivation of a sentence in $\bar{G}$ is simulated by a production from (4d), which removes the compound nonterminal $\langle \dots \rangle$ from the sentential form. Finally, every symbol $\bar{a}$, where $a \in T$, is rewritten to a.

Formal Proof.
We establish the theorem by Claims 1 through 4 stated below.

Claim 1. Every $x \in L(\bar{G})$, where $|x| \leq 2$, is generated by G as follows:

$$S \;{}_{max}\!\Rightarrow_G x\ [p_1],$$

where p_1 is one of the productions introduced in (1). □

Claim 2. Every

$$(\bar{S}, p_0) \Rightarrow^+_{\bar{G}} (x, q),$$

where $q \in K$, $x \in T^+$, and

$$3 \leq |x| \leq \max\Big(\{3\} \cup \big\{|y| : (B, p) \to (y, p') \in \bar{P}\big\}\Big),$$

is generated by G as follows:

$$S \;{}_{max}\!\Rightarrow_G x\ [p_{2a}],$$

where p_{2a} is one of the productions introduced in (2a). □

Claim 3. Every

$$(\bar{S}, p_0) \Rightarrow^+_{\bar{G}} (x, q) \Rightarrow^+_{\bar{G}} (u, r),$$

where q, $r \in K$, $u \in T^+$, $x = v_0 A w_0$, $A \in \bar{V} - T$, v_0, $w_0 \in \bar{V}^*$, and

$$3 \leq |x| \leq \max\Big(\{3\} \cup \big\{|y| : (B, p) \to (y, p') \in \bar{P}\big\}\Big),$$

can only be generated by G as follows:

$$\begin{aligned} S \;{}_{max}\!\Rightarrow_G\ & \alpha(v_0)\langle A, q\rangle\alpha(w_0) && [p_{2b}] \\ {}_{max}\!\Rightarrow^*_G\ & y && [\Xi_4] \\ {}_{max}\!\Rightarrow_G\ & z && [p_{4d}] \\ {}_{max}\!\Rightarrow^{|u|}_G\ & u && [\Xi_5], \end{aligned}$$

where $y \in Z^*YZ^*$, $z = \alpha(u)$; p_{2b} and p_{4d} denote one of the productions introduced in (2b) and (4d), respectively, Ξ_4 is a sequence of productions from (4a), (4b), and (4c), and Ξ_5 is a sequence of productions from (5).

Proof. Observe that the productions from (1) and (2) are the only productions containing S on their left-hand sides and no other productions contain S on their right-hand sides. To generate a sentence u, $|u| \geq 3$, the derivation has to start as

$$S \;{}_{max}\!\Rightarrow_G \alpha(v_0)\langle A, q\rangle\alpha(w_0)\ [p_{2b}],$$

and productions from (1) and (2) are not used during the rest of the derivation.

Further, observe that none of the productions introduced in (3) can be applied during a successful derivation because no production rewrites the nonterminal X that appears on the right-hand side of every production from (3).

To generate a sentence over T, all symbols from $\bar{V} - T$ have to be removed from the sentential form, and only productions from (4) can perform this removal because only these productions contain symbols from $\bar{V} - T$ on their left-hand sides. Furthermore, productions (4a), (4b), and (4c) contain one symbol from Y both on their left-hand sides and their right-hand sides while productions from (4d) contain a symbol from Y only on their left-hand sides. Therefore, after the application of a production from (4d), none of the productions from (4) is applicable. Because for each production p_4 and p_5 introduced in (4) and (5), respectively, it holds that $\text{len}(p_4) > \text{len}(p_5)$, no production from (5) is used while some production from (4) is applicable. As a result, the corresponding part of the derivation can be expressed as follows:

$$\begin{aligned}\alpha(v_0)\langle A, q\rangle\alpha(w_0) \;{}_{max}\!\Rightarrow_G^* &\; y\ [\Xi_4]\\ {}_{max}\!\Rightarrow_G &\; z\ [p_{4d}].\end{aligned}$$

At this point, $z = \alpha(u)$ in a successful derivation. Productions from (5) replace every $\bar{a} \in \text{alph}(z)$ with a in $|u|$ steps, so we obtain

$$z \;{}_{max}\!\Rightarrow_G^{|u|} u\ [\Xi_5].$$

Putting together all the previous observations, we obtain Claim 3. □

Claim 4. In a successful derivation, every

$$\begin{aligned}&\alpha(v_0)\langle B_0, q_0\rangle\alpha(w_0)\\ {}_{max}\!\Rightarrow_G\ &\alpha(v_1)\langle B_1, q_1\rangle\alpha(w_1)\ [p_0]\\ &\vdots\\ {}_{max}\!\Rightarrow_G\ &\alpha(v_n)\langle B_n, q_n\rangle\alpha(w_n)\ [p_{n-1}]\end{aligned}$$

is performed in G if and only if

$$\begin{aligned} & (v_0 B_0 w_0, q_0) \\ \Rightarrow_{\bar{G}} & (v_1 B_1 w_1, q_1) \; \big[(B_0, q_0) \to (x_1, q_1)\big] \\ & \vdots \\ \Rightarrow_{\bar{G}} & (v_n B_n w_n, q_n) \; \big[(B_{n-1}, q_{n-1}) \to (x_n, q_n)\big] \end{aligned}$$

is performed in $\bar{G}$, where v_i, $w_i \in \bar{V}^*$, $B_i \in \bar{V} - T$, $q_i \in K$, for all $0 \leq i \leq n$, for some $n \geq 0$, $x_1, \ldots, x_n \in \bar{V}^+$, and $p_0, \ldots, p_{n-1}$ are productions introduced in (4a), (4b), and (4c).

Proof. We start by proving the only-if part of the claim.

Only If. We show that

$$\alpha(v_0)\langle B_0, q_0\rangle\alpha(w_0) \;{}_{max}\Rightarrow^m_G \alpha(v_m)\langle B_m, q_m\rangle\alpha(w_m)$$

implies

$$(v_0 B_0 w_0, q_0) \Rightarrow^m_{\bar{G}} (v_m B_m w_m, q_m)$$

by induction on $m \geq 0$.

Basis. Let $m = 0$. Then,

$$\alpha(v_0)\langle B_0, q_0\rangle\alpha(w_0) \;{}_{max}\Rightarrow^0_G \alpha(v_0)\langle B_0, q_0\rangle\alpha(w_0)$$

and clearly,

$$(v_0 B_0 w_0, q_0) \Rightarrow^0_{\bar{G}} (v_0 B_0 w_0, q_0).$$

Induction Hypothesis. Suppose that the claim holds for all k-step derivations, where $k \leq m$, for some $m \geq 0$.

Induction Step. Let us consider a derivation

$$\alpha(v_0)\langle B_0, q_0\rangle\alpha(w_0) \;{}_{max}\Rightarrow^{m+1}_G \alpha(v_{m+1})\langle B_{m+1}, q_{m+1}\rangle\alpha(w_{m+1}).$$

Since $m + 1 \geq 1$, there is some

$$\alpha(v_m)\langle B_m, q_m\rangle\alpha(w_m) \in \big((\bar{V} - T) \cup Z\big)^* Y \big((\bar{V} - T) \cup Z\big)^*$$

and a production p_m such that

$$\begin{aligned} \alpha(v_0)\langle B_0, q_0\rangle\alpha(w_0) \;{}_{max}\Rightarrow^m_G & \; \alpha(v_m)\langle B_m, q_m\rangle\alpha(w_m) \\ {}_{max}\Rightarrow_G & \; \alpha(v_{m+1})\langle B_{m+1}, q_{m+1}\rangle\alpha(w_{m+1}) \; [p_m]. \end{aligned}$$

By the induction hypothesis, there is a derivation

$$(v_0 B_0 w_0, q_0) \Rightarrow^m_{\bar{G}} (v_m B_m w_m, q_m).$$

As p_m is a production introduced in (4a) through (4c), it has one of the following forms depending on the placement of B_{m+1}:

- $\big(B_{m+1}, \langle B_m, q_m\rangle\big) \to \big(\langle B_{m+1}, q_{m+1}\rangle, \alpha(x_{m+1})\big)$, for $v_m = v'_m B_{m+1} v''_m$;
- $\big(\langle B_m, q_m\rangle, B_{m+1}\big) \to \big(\alpha(x_{m+1}), \langle B_{m+1}, q_{m+1}\rangle\big)$, for $w_m = w'_m B_{m+1} w''_m$;
- $\big(\langle B_m, q_m\rangle, \alpha(A)\big) \to \big(\alpha(x'_{m+1})\langle B_{m+1}, q_{m+1}\rangle \alpha(x''_{m+1}), \alpha(A)\big)$, or $\big(\alpha(A), \langle B_m, q_m\rangle\big) \to \big(\alpha(A), \alpha(x'_{m+1})\langle B_{m+1}, q_{m+1}\rangle \alpha(x''_{m+1})\big)$, for $x_{m+1} = x'_{m+1} B_{m+1} x''_{m+1}$,

where $A \in \bar{V}$ and $x_{m+1}, x'_{m+1}, x''_{m+1} \in \bar{V}^*$. Their construction is based on $\bar{P}$, so there is a production $(B_m, q_m) \to (x_{m+1}, q_{m+1}) \in \bar{P}$.

Simulating a derivation of G by $\bar{G}$, we now prove that for the given state q_m, the leftmost nonterminal in the sentential form is rewritten in G. We make this proof by contradiction. Suppose that there is a production $p'_m \in P$ from (4) that rewrites some $B'_m \in \bar{V} - T$ in a state q_m, and $B'_m \in \text{alph}(v_m)$. Then, there exists $(B'_m, q_m) \to (x'_{m+1}, q'_{m+1}) \in \bar{P}$ and, as a result, there also exist productions from (3), which are based on $(B_m, q_m) \to (x_{m+1}, q_{m+1})$ and $(B'_m, q_m) \to (x'_{m+1}, q'_{m+1})$. These productions have the following forms:

$$\begin{aligned}\big(B'_m, \langle B_m, q_m\rangle, \alpha(A)\big) &\to (X, X, X),\\ \big(B'_m, \alpha(A), \langle B_m, q_m\rangle\big) &\to (X, X, X),\\ \big(\alpha(A), B'_m, \langle B_m, q_m\rangle\big) &\to (X, X, X),\end{aligned}$$

where $A \in \bar{V}$. Because $\big|\alpha(v_m)\langle B_m, q_m\rangle\alpha(w_m)\big| \geq 3$, one of these productions is applicable. As productions introduced in (3) have higher priority than productions introduced in (4), one of them is applied, which introduces X to the sentential form. However, this symbol can never be removed from the sentential form, so the derivation is unsuccessful in this case.

As a result, the leftmost nonterminal in the state q_m is rewritten in G, so $(B_m, q_m) \to (x_{m+1}, q_{m+1})$ is used in $\bar{G}$, and we obtain

$$(v_m B_m w_m, q_m) \Rightarrow_{\bar{G}} (v_{m+1} B_{m+1} w_{m+1}, q_{m+1}) \ \big[(B_m, q_m) \to (x_{m+1}, q_{m+1})\big].$$

If. We demonstrate that

$$(v_0 B_0 w_0, q_0) \Rightarrow^m_{\bar{G}} (v_m B_m w_m, q_m)$$

implies

$$\alpha(v_0)\langle B_0, q_0\rangle\alpha(w_0) \ {}_{max}\!\Rightarrow^m_G \alpha(v_m)\langle B_m, q_m\rangle\alpha(w_m)$$

by induction on m.

Basis. Let $m = 0$. Then,

$$(v_0 B_0 w_0, q_0) \Rightarrow^0_{\bar{G}} (v_0 B_0 w_0, q_0).$$

Clearly,

$$\alpha(v_0)\langle B_0, q_0\rangle\alpha(w_0) \ {}_{max}\!\Rightarrow^0_G \alpha(v_0)\langle B_0, q_0\rangle\alpha(w_0).$$

Induction Hypothesis. Suppose that the claim holds for all k-step derivations, where $k \leq m$, for some $m \geq 0$.

Induction Step. Consider a derivation

$$(v_0B_0w_0, q_0) \Rightarrow_{\bar{G}}^{m+1} (v_{m+1}B_{m+1}w_{m+1}, q_{m+1}).$$

Since $m+1 \geq 1$, there is some $(v_mB_mw_m, q_m)$, where v_m, $w_m \in \bar{V}^*$, $B_m \in \bar{V} - T$, and a production $(B_m, q_m) \to (x_{m+1}, q_{m+1})$ such that

$$\begin{aligned}(v_0B_0w_0, q_0) &\Rightarrow_{\bar{G}}^{m} (v_mB_mw_m, q_m)\\ &\Rightarrow_{\bar{G}} (v_{m+1}B_{m+1}w_{m+1}, q_{m+1}) \left[(B_m, q_m) \to (x_{m+1}, q_{m+1})\right].\end{aligned}$$

By the induction hypothesis, there is a derivation

$$\alpha(v_0)\langle B_0, q_0\rangle\alpha(w_0) \ {}_{\max}\!\Rightarrow_G^m \alpha(v_m)\langle B_m, q_m\rangle\alpha(w_m).$$

Because $(B_m, q_m) \to (x_{m+1}, q_{m+1})$ rewrites the leftmost rewritable symbol B_m in a given state q_m, there is no production $(B'_m, q_m) \to (x'_{m+1}, q'_{m+1})$ satisfying $B'_m \in \text{alph}(v_m)$. As a result, none of the productions from (3) is applicable.

For each $(B_m, q_m) \to (x_{m+1}, q_{m+1}) \in \bar{P}$, there are productions of the following forms in G whose application depends on the placement of B_{m+1}:

- $\big(B_{m+1}, \langle B_m, q_m\rangle\big) \to \big(\langle B_{m+1}, q_{m+1}\rangle, \alpha(x_{m+1})\big)$, for $v_m = v'_mB_{m+1}v''_m$;
- $\big(\langle B_m, q_m\rangle, B_{m+1}\big) \to \big(\alpha(x_{m+1}), \langle B_{m+1}, q_{m+1}\rangle\big)$, for $w_m = w'_mB_{m+1}w''_m$;
- $\big(\langle B_m, q_m\rangle, \alpha(A)\big) \to \big(\alpha(x'_{m+1})\langle B_{m+1}, q_{m+1}\rangle\alpha(x''_{m+1}), \alpha(A)\big)$, or $\big(\alpha(A), \langle B_m, q_m\rangle\big) \to \big(\alpha(A), \alpha(x'_{m+1})\langle B_{m+1}, q_{m+1}\rangle\alpha(x''_{m+1})\big)$, for $x_{m+1} = x'_{m+1}B_{m+1}x''_{m+1}$,

where $A \in \bar{V}$ and x_{m+1}, x'_{m+1}, $x''_{m+1} \in \bar{V}^*$. As $\left|\alpha(v_m)\langle B_m, q_m\rangle\alpha(w_m)\right| \geq 3$, one of them is applicable in G, so we obtain

$$\alpha(v_m)\langle B_m, q_m\rangle\alpha(w_m) \ {}_{\max}\!\Rightarrow_G \alpha(v_{m+1})\langle B_{m+1}, q_{m+1}\rangle\alpha(w_{m+1}).$$

□

By Claims 1 through 4, it follows that $\mathscr{L}(CS) \subseteq \mathscr{L}(PSC, max)$. As propagating scattered context grammars do not contain erasing productions, their derivations can be simulated by context-sensitive grammars. As a result, $\mathscr{L}(PSC, max) \subseteq \mathscr{L}(CS)$. Therefore, $\mathscr{L}(CS) = \mathscr{L}(PSC, max)$. ■

Theorem 5.14. $\mathscr{L}(CS) = \mathscr{L}(PSC, min)$.

Proof. Let L be a context-sensitive language described by a state grammar $\bar{G} = (\bar{V}, T, K, \bar{P}, \bar{S}, p_0)$. Set

$$Y = \left\{\langle A, q\rangle : A \in \bar{V} - T, q \in K\right\}$$

and $Z = \{\bar{a} : a \in T\}$. Define the homomorphism α form $\bar{V}^*$ to $\big((\bar{V} - T) \cup Z\big)^*$ as $\alpha(A) = A$, for all $A \in \bar{V} - T$, and $\alpha(a) = \bar{a}$, for all $a \in T$. Set $V = \bar{V} \cup Y \cup Z \cup \{S, X\}$.

Define the propagating scattered context grammar $G' = (V, T, P', S)$, where P' is constructed as follows:

1. For each $x \in L(\bar{G})$, where $|x| \leq 3$, add
   $(S) \to (x)$ to P'.

2. For each

$$(x,q) \in \Big\{(x,q) : (\bar{S}, p_0) \Rightarrow^+_{\bar{G}} (x,q), \text{ for some } q \in K, \\ \text{and } 4 \leq |x| \leq \max\big(\{4\} \cup \{|y| : (B,p) \to (y,p') \in \bar{P}\}\big)\Big\},$$

   where

   (a) $x \in T^*$, add
   $(S) \to (x)$ to P';

   (b) $x = x_1 A x_2$, $A \in \bar{V} - T$, and $x_1, x_2 \in \bar{V}^*$, add
   $(S) \to \big(\alpha(x_1)\langle A, q\rangle \alpha(x_2)\big)$ to P'.

3. For each $(A, p) \to (x, q)$ and $(B, p) \to (y, r) \in \bar{P}$, add
   $\big(B, \langle A, p\rangle\big) \to (X, X)$ to P'.

4. For each $(A, p) \to (x, q) \in \bar{P}$, $B \in \bar{V} - T$, D, $E \in \bar{V}$, $\Gamma_{21} \in \text{perm}(2,1)$, $\Gamma_{12} \in \text{perm}(1,2)$, and

$$\begin{aligned}
u &= \text{reorder}\Big(\big(B, \langle A,p\rangle, \alpha(D)\big), \Gamma_{21}\Big), \\
u' &= \text{reorder}\Big(\big(\langle B,q\rangle, \alpha(x), \alpha(D)\big), \Gamma_{21}\Big), \\
r &= \text{reorder}\Big(\big(\langle A,p\rangle, B, \alpha(D)\big), \Gamma_{21}\Big), \\
r' &= \text{reorder}\Big(\big(\alpha(x), \langle B,q\rangle, \alpha(D)\big), \Gamma_{21}\Big), \\
y &= \text{reorder}\Big(\big(\langle A,p\rangle, \alpha(D), \alpha(E)\big), \Gamma_{12}\Big),
\end{aligned}$$

   add

   (a) $u \to u'$,

   (b) $r \to r'$ to P';

   (c) if $x = vBw$ and v, $w \in \bar{V}^*$, for each

$$z = \text{reorder}\Big(\big(\alpha(v)\langle B,q\rangle\alpha(w), \alpha(D), \alpha(E)\big), \Gamma_{12}\Big),$$

   add $y \to z$ to P'.

   (d) For each

$$u = \text{reorder}\Big(\big(\alpha(x), \alpha(D), \alpha(E)\big), \Gamma_{12}\Big),$$

   add $y \to u$ to P'.

5. For each a, b, c, $d \in T$, add
   (a) $(\bar{a}, \bar{b}, \bar{c}, \bar{d}) \to (a, \bar{b}, \bar{c}, \bar{d})$,
   (b) $(\bar{a}, \bar{b}, \bar{c}, \bar{d}) \to (a, b, c, d)$ to P'.

Claim 1. Every

$$(\bar{S}, p_0) \Rightarrow^+_{\bar{G}} (x, q) \Rightarrow^+_{\bar{G}} (u, r),$$

where q, $r \in K$, $u \in T^+$, $x = v_0 A w_0$, $A \in \bar{V} - T$, v_0, $w_0 \in \bar{V}^*$,

$$4 \leq |x| \leq \max\Big(\{4\} \cup \big\{|y| : (B, p) \to (y, p') \in \bar{P}\big\}\Big),$$

can only be generated by G' as follows:

$$\begin{aligned} S \ {}_{min}{\Rightarrow}_G \ & \alpha(v_0)\langle A, q\rangle\alpha(w_0) \ [p_{2b}] \\ {}_{min}{\Rightarrow}^*_G \ & y \qquad [\Xi_4] \\ {}_{min}{\Rightarrow}_G \ & z \qquad [p_{4d}] \\ {}_{min}{\Rightarrow}^{|u|-4}_G \ & v \qquad [\Xi_5] \\ {}_{min}{\Rightarrow}_G \ & u \qquad [p_{5b}], \end{aligned}$$

where $y \in Z^* Y Z^*$, $z = \alpha(u)$, $v \in (T \cup Z)^+$, p_{2b}, p_{4d}, and p_{5b} represent productions introduced in (2b), (4d), and (5b), respectively, Ξ_4 is a sequence of productions from (4a), (4b), and (4c), and Ξ_5 is a sequence of productions from (5a).

Proof. The proof of the form of the beginning of the derivation,

$$\begin{aligned} S \ {}_{min}{\Rightarrow}_G \ & \alpha(v_0)\langle A, q\rangle\alpha(w_0) \ [p_{2b}] \\ {}_{min}{\Rightarrow}^*_G \ & y \qquad [\Xi_4] \\ {}_{min}{\Rightarrow}_G \ & z \qquad [p_{4d}], \end{aligned}$$

is analogous to the proof of Claim 3 (in terms of minimal derivations) and, therefore, left to the reader.

Recall that z satisfies $z = \alpha(u)$. Each of the productions from (5a) replaces one occurrence of $\bar{a}$ with a, for some $a \in T$, and finally, the application of a production from (5b) replaces the remaining four nonterminals with their terminal variants. Therefore,

$$\begin{aligned} z \ {}_{min}{\Rightarrow}^{|u|-4}_G \ & v \ [\Xi_5] \\ {}_{min}{\Rightarrow}_G \ & u \ [p_{5b}], \end{aligned}$$

so the claim holds. □

Notice that $\text{len}(p_3) < \text{len}(p_4) < \text{len}(p_5)$, for each production p_3, p_4, and p_5 introduced in (3), (4), and (5), respectively, so the priorities of the productions from the individual steps are the same as in the case of grammars that use maximal derivations. As a result, Claim 1, 2, and 4 can be straightforwardly rephrased in terms of minimal derivations, and we leave this simple task to the reader. Therefore, $\mathscr{L}(CS) \subseteq \mathscr{L}(PSC, min)$, and for the same reason as in the proof of Theorem 5.13, $\mathscr{L}(PSC, min) \subseteq \mathscr{L}(CS)$. Thus, $\mathscr{L}(CS) = \mathscr{L}(PSC, min)$. ∎

Before closing this section, let us point out a potential use of the obtained results in the future. Simply and briefly stated, we have demonstrated that both of the modified versions of propagating scattered context grammars, which make either maximal or minimal derivations, characterize the family of context-sensitive languages. Consequently, if in the future formal language theory proves that these versions are as powerful as ordinary propagating scattered context grammars, it also achieves "yes" as the answer to the long-standing open problem whether context-sensitive and propagating scattered context grammars are equivalent.

5.4 Unordered Scattered Context Grammars

Unordered scattered context grammars belong to the most natural modifications of scattered context grammars. As indicated by their name, the order of the context-free components in an unordered scattered context production is unimportant; that is, these components can be applied by simultaneously rewriting their left-hand sides no matter in what order these left-hand sides appear in the current sentential form. This important modification was independently introduced and studied in [21] and [26]. After their introduction, [15, 27] discussed special derivation forms of these grammars.

Definition 5.15. An *unordered scattered context grammar* is a quadruple $G = (V, T, P, S)$, where V, T, P, and S are defined as in the case of a scattered context grammar. If there is a permutation $\pi \in \text{perm}(n)$, for some $n \geq 1$, such that

$$p = \text{reorder}\big((A_1, \dots, A_n), \pi\big) \rightarrow \text{reorder}\big((x_1, \dots, x_n), \pi\big) \in P,$$

and

$$u = u_1 A_1 \dots u_n A_n u_{n+1},$$
$$v = u_1 x_1 \dots u_n x_n u_{n+1},$$

where $u_i \in V^*$, for all $1 \leq i \leq n+1$, then G makes a derivation step from u to v according to p, symbolically written as

$$u \Rightarrow_G v\ [p],$$

or, simply, $u \Rightarrow_G v$. In the standard way, define the transitive closure and the reflexive and transitive closure of $\Rightarrow_G$, and the language generated by G. If every production

$$(A_1, \dots, A_k) \rightarrow (x_1, \dots, x_k) \in P$$

satisfies $x_i \in V^+$, for all $1 \leq i \leq k$, G is a *propagating unordered scattered context grammar*. The language families generated by unordered scattered context grammars and propagating unordered scattered context grammars are denoted by $\mathscr{L}(SC, un)$ and $\mathscr{L}(PSC, un)$, respectively.

Example 5.16 (see [19])**.** Consider the unordered scattered context grammar

$$G = \big(\{S, A, a, b, c\}, \{a, b, c\}, P, S\big),$$

where

$$\begin{aligned} P = \big\{&(S) \to (AAA), \\ &(A, A, A) \to (aA, bA, cA), \\ &(A, A, A) \to (\varepsilon, \varepsilon, \varepsilon)\big\}. \end{aligned}$$

This grammar generates the language

$$L(G) = \big\{x_1 \dots x_{3n} : n \geq 0, \{x_j, x_{n+j}, x_{2n+j}\} = \{a, b, c\}, \text{ for all } 1 \leq j \leq n\big\}$$

while the generated language of the scattered context grammar with the same set of productions is $\{a^n b^n c^n : n \geq 0\}$.

Propagating unordered scattered context grammars are as powerful as programmed grammars without erasing productions and appearance checking as demonstrated both in [21] and [26]. Next, we establish two lemmas that straightforwardly imply this result.

Lemma 5.17. $\mathscr{L}(P) \subseteq \mathscr{L}(PSC, un)$.

Proof. Observe that the construction used in the proof of Lemma 3.22 can be used to prove this lemma as well; however, the constructed grammar $\bar{G}$ is a propagating unordered scattered context grammar in this case. ∎

Lemma 5.18. $\mathscr{L}(PSC, un) \subseteq \mathscr{L}(P)$.

Proof. Let $G = (V, T, P, S)$ be a propagating unordered scattered context grammar. Set

$$N = \big\{\langle x_i\rangle : (A_1, \dots, A_n) \to (x_1, \dots, x_n) \in P, 1 \leq i \leq n\big\}.$$

Define the programmed grammar $\bar{G} = (V \cup N, T, \bar{P}, S)$, where $\bar{P}$ is defined as follows:

1. Let Q_2 denote the set of all productions introduced in step (2) of this construction (see below). For each

$$p = (A_1, \dots, A_n) \to (x_1, \dots, x_n) \in P,$$

add

$$\begin{aligned} p_1 &= \big(A_1 \to \langle x_1\rangle, \{p_2\}\big), \\ &\ \ \vdots \\ p_{n-1} &= \big(A_{n-1} \to \langle x_{n-1}\rangle, \{p_n\}\big), \\ p_n &= \big(A_n \to \langle x_n\rangle, \{r_1\}\big), \\ r_1 &= \big(\langle x_1\rangle \to x_1, \{r_2\}\big), \\ &\ \ \vdots \\ r_{n-1} &= \big(\langle x_{n-1}\rangle \to x_{n-1}, \{r_n\}\big), \\ r_n &= \big(\langle x_n\rangle \to x_n, Q_2\big) \text{ to } \bar{P}. \end{aligned}$$

2. Let Q_1 denote the set of all productions of the form $p_1 = \big(A_1 \to \langle x_1 \rangle, \{p_2\}\big)$ introduced in step (1) of this construction. For each $A \in V - T$, add

$$(A \to A, Q_1) \text{ to } \bar{P}.$$

To give an insight into the construction above, we informally explain how $\bar{G}$ simulates the application of a production from G. The application of every production of the form $(A_1, \ldots, A_n) \to (x_1, \ldots, x_n)$ is simulated by $2n$ derivation steps in $\bar{G}$. First, all left-hand sides of its context-free components, $A_1, \ldots, A_n$, are rewritten to $\langle x_1 \rangle, \ldots, \langle x_n \rangle$, respectively, which are symbols that encode the real right-hand sides of the context-free productions. As a result, in this phase of the simulation, $\bar{G}$ cannot rewrite $x_1, \ldots, x_n$, which would lead to an illegal simulation. Second, all the symbols $\langle x_1 \rangle, \ldots, \langle x_n \rangle$ are rewritten back to the original right-hand sides, $x_1, \ldots, x_n$, respectively. Finally, to start the simulation of the following production of G, a production from (2) is used, which non-deterministically selects the next production of the propagating unordered scattered context grammar to be simulated. The simulation is successfully completed when the resulting sentential form w does not contain any nonterminal. Clearly, $w \in L(\bar{G})$ if and only if $w \in L(G)$. ■

Putting together Lemmas 5.17 and 5.18, we obtain the following result:

Theorem 5.19. $\mathscr{L}(PSC, un) = \mathscr{L}(P)$. ■

From Theorems 3.24 and 5.19, we obtain:

Corollary 5.20. $\mathscr{L}(PSC, un) \subset \mathscr{L}(PSC)$. ■

Obviously, the constructions described in Lemmas 3.22 and 5.18 can be applied to unordered scattered context grammars with erasing productions as well. In this way, we obtain the following corollary.

Corollary 5.21. $\mathscr{L}(SC, un) = \mathscr{L}(P, \varepsilon)$. ■

From Theorem 3.20, and Corollaries 5.21 and 2.29, we obtain:

Corollary 5.22. $\mathscr{L}(SC, un) \subset \mathscr{L}(SC)$. ■

In the conclusion of this section, we summarize the most important results achieved in the papers dealing with unordered scattered context grammars. A normal form similar to 2-limited unordered scattered context grammars was established in [26] by using a technique that resembles the proof of Theorem 3.4. Unordered scattered context grammars in which productions are applied in the leftmost way were studied in [15]. This paper demonstrates that these grammars are less powerful than phrase-structure grammars, but more powerful than ordinary unordered scattered context grammars. Finally, [27] summarizes all results concerning regulated versions of scattered context grammars. In essence, this study demonstrates that all well-known mechanisms of derivation regulation, such as appearance checking or unconditional transfer, applied to scattered context grammars result in the generative power that coincides with the power of programmed grammars regulated in the same way. To put it more generally, under regulating restrictions, unordered scattered context grammars behave just like most regulated grammars do.

5.5 Linear Scattered Context Grammars

Indisputably, linear and right-linear grammars represent more than significant special cases of context-free grammars. As context-free grammars underlie scattered context grammars, it is natural and important to discuss special cases of scattered context grammars based upon linear and right-linear grammars—the subject of the present section. These special cases of scattered context grammars first apply a context-free starting production to generate an initial string consisting of n nonterminals, where n is a positive integer. After this initial application, however, only scattered context productions with linear or right-linear components are applied. First, this section proves that linear versions of scattered context grammars with n-nonterminal initial strings are as powerful as linear simple matrix grammars of degree n. Then, it demonstrates that right-linear versions of scattered context grammars with n-nonterminal initial strings are equivalent to right-linear simple matrix grammars of degree n. In addition, the present section derives several corollaries from these two main results and discusses the generative power of scattered context grammars with context-sensitive and phrase-structure productions.

We start by defining linear and right-linear scattered context grammars formally.

Definition 5.23. A *linear scattered context grammar* is a scattered context grammar $G = (V, T, P, S)$, where P is a finite set of productions of the following two forms:

1. $(S) \to (x_1 A_1 \dots x_k A_k x_{k+1})$, where $A_i \in (V - T) - \{S\}$, $x_j \in T^*$, for all $1 \leq i \leq k$, $1 \leq j \leq k+1$;
2. $(A_1, \dots, A_k) \to (z_1, \dots, z_k)$, where $A_i \in (V - T) - \{S\}$, and
   - either $z_i = x_i B_i y_i$, where $x_i, y_i \in T^*$, $B_i \in (V - T) - \{S\}$,
   - or $z_i \in T^*$, for all $1 \leq i \leq k$ and some $k \geq 1$.

A linear scattered context grammar is *of degree n* if $(S) \to (x_1 A_1 \dots x_n A_n x_{n+1}) \in P$ satisfies $n \geq m$, for all $(S) \to (y_1 A_1 \dots y_m A_m y_{m+1}) \in P$. The family of languages generated by linear scattered context grammars of degree n is denoted by $\mathscr{L}(SC, LIN, n)$, and

$$\mathscr{L}(SC, LIN) = \bigcup_{n=1}^{\infty} \mathscr{L}(SC, LIN, n).$$

Definition 5.24. A *right-linear scattered context grammar* is a linear scattered context grammar $G = (V, T, P, S)$, where P is a finite set of productions of the following two forms:

1. $(S) \to (x_1A_1 \dots x_kA_k)$, where $A_i \in (V - T) - \{S\}$, $x_i \in T^*$, for all $1 \le i \le k$ and some $k \ge 1$;
2. $(A_1, \dots, A_k) \to (z_1, \dots, z_k)$, where $A_i \in (V - T) - \{S\}$, and
   - either $z_i = x_iB_i$, where $x_i \in T^*$, $B_i \in (V - T) - \{S\}$,
   - or $z_i \in T^*$, for all $1 \le i \le k$ and some $k \ge 1$.

The family of languages generated by right-linear scattered context grammars of degree n is denoted by $\mathscr{L}(SC, RLIN, n)$, and

$$\mathscr{L}(SC, RLIN) = \bigcup_{n=1}^{\infty} \mathscr{L}(SC, RLIN, n).$$

To prove that $\mathscr{L}(SM, LIN, n) = \mathscr{L}(SC, LIN, n)$, for every $n \ge 1$, we first give two preliminary lemmas.

Lemma 5.25. *For every $n \ge 1$, $\mathscr{L}(SM, LIN, n) \subseteq \mathscr{L}(SC, LIN, n)$.*

Proof. Let $\bar{G} = (\bar{V}_1, \dots, \bar{V}_n, T, \bar{P}, \bar{S})$ be a linear simple matrix grammar of degree n. Set $N = \{\langle p, i\rangle : p \in \bar{P}, 1 \le i \le n\}$. Define the linear scattered context grammar of degree n,

$$G = \left(\bar{V}_1 \cup \dots \cup \bar{V}_n \cup N \cup \{S\}, T, P, S\right),$$

where P is defined as follows:

1. For each $(\bar{S}) \to (x_1A_1 \dots x_nA_nx_{n+1}) \in \bar{P}$, where $x_i \in T^*$, for all $1 \le i \le n+1$, and $p = (A_1, \dots, A_n) \to (y_1, \dots, y_n) \in \bar{P}$, add $(S) \to \left(x_1\langle p, 1\rangle x_2A_2 \dots x_nA_nx_{n+1}\right)$ to P.
2. For each $p = (A_1, \dots, A_i, \dots, A_n) \to (x_1B_1y_1, \dots, x_iB_iy_i, \dots, x_nB_ny_n) \in \bar{P}$, where $x_j, y_j \in T^*$, $A_j, B_j \in \bar{V}_j - T$, for all $1 \le j \le n$, and
   (a) for each $i < n$, add $\left(\langle p, i\rangle, A_{i+1}\right) \to \left(x_iB_iy_i, \langle p, i+1\rangle\right)$ to P;
   (b) for each $q = (B_1, \dots, B_n) \to (z_1, \dots, z_n) \in \bar{P}$, add
      i. $\left(B_1, \langle p, n\rangle\right) \to \left(\langle q, 1\rangle, x_nB_ny_n\right)$ to P;
      ii. if $n = 1$, add $\left(\langle p, 1\rangle\right) \to \left(x_1\langle q, 1\rangle y_1\right)$ to P.
3. For each $p = (A_1, \dots, A_i, \dots, A_n) \to (x_1, \dots, x_i, \dots, x_n) \in \bar{P}$, where $x_j \in T^*$, for all $1 \le j \le n$, and
   (a) for each $i < n$, add $\left(\langle p, i\rangle, A_{i+1}\right) \to \left(x_i, \langle p, i+1\rangle\right)$ to P;
   (b) add $\left(\langle p, n\rangle\right) \to (x_n)$ to P.

Every production introduced in (1) simulates the initial production of $\bar{G}$ and, in addition, selects the next production p to be simulated. After its application, we obtain the sentential form of the form

$$w_1\langle p,1\rangle w_2A_2\dots w_nA_nw_{n+1},$$

where $w_i \in T^*$, for all $1 \le i \le n$, and

$$p = (A_1,\dots,A_n) \to (z_1,\dots,z_n) \in \bar{P}.$$

Consider any derivation

$$w_1A_1w_2A_2\dots w_nA_nw_{n+1} \Rightarrow_{\bar{G}} w_1x_1B_1y_1\dots w_nx_nB_ny_nw_{n+1}\ [p],$$

where

$$p = (A_1,\dots,A_n) \to (x_1B_1y_1,\dots,x_nB_ny_n),$$

x_i, $y_i \in T^*$, and A_i, $B_i \in \bar{V}_i - T$, for all $1 \le i \le n$. This derivation is simulated by G in n derivation steps by first applying a production from (2a) $n-1$ times and, finally, applying a production from (2bi), so

$$\begin{aligned}
&w_1\langle p,1\rangle w_2A_2\dots w_nA_nw_{n+1}\\
\Rightarrow_G\ &w_1x_1B_1y_1w_2\langle p,2\rangle\dots w_nA_nw_{n+1}\\
&\vdots\\
\Rightarrow_G\ &w_1x_1B_1y_1w_2x_2B_2y_2\dots w_n\langle p,n\rangle w_{n+1}\\
\Rightarrow_G\ &w_1x_1\langle q,1\rangle y_1w_2x_2B_2y_2\dots w_nx_nB_ny_nw_{n+1},
\end{aligned}$$

where

$$q = (B_1,\dots,B_n) \to (z_1,\dots,z_n) \in \bar{P}.$$

Observe that no nonterminal A_i can be skipped by a production from (2a) because the sentential form contains exactly n nonterminals and the form of the productions from (2a) requires their n applications during every simulation. For the same reason, a production from (2bi) has to select the first nonterminal in a sentential form of G. If $n = 1$, a production from (2bii) is used instead of a production from (2a) or (2bi). Finally, a production of the form $(A_1,\dots,A_n) \to (x_1,\dots,x_n) \in \bar{P}$, where $x_i \in T^*$, for all $1 \le i \le n$, is simulated by productions from (3a) and (3b) that perform the simulation analogously to the productions from (2a) and (2bi), respectively. By removing the symbol from N from the sentential form, (3b) finishes the derivation. ■

As the number of components in every production of G constructed in the proof of Lemma 5.25 is at most 2, we state the following corollary.

Corollary 5.26. *For every linear simple matrix grammar $\bar{G}$ of degree n, there is a linear scattered context grammar G of degree n such that $L(\bar{G}) = L(G)$, and* $\mathrm{mcs}(G) = 1$. ■

Lemma 5.27. *For every $n \geq 1$, $\mathscr{L}(SC, LIN, n) \subseteq \mathscr{L}(SM, LIN, n)$.*

Proof. Let $\bar{G} = (\bar{V}, T, \bar{P}, \bar{S})$ be a linear scattered context grammar of degree n. Set

$$V_1 = \Big\{\langle a, 1\rangle : a \in \big(\bar{V} - \{\bar{S}\}\big) \cup \{\varepsilon\}\Big\} \cup T,$$
$$\vdots$$
$$V_n = \Big\{\langle a, n\rangle : a \in \big(\bar{V} - \{\bar{S}\}\big) \cup \{\varepsilon\}\Big\} \cup T.$$

For all $1 \leq i \leq n$, set $\alpha(xay, i) = x\langle a, i\rangle y$, where $a \in \bar{V} - \{\bar{S}\}$, $x, y \in T^*$, and $\alpha(\varepsilon, i) = \langle \varepsilon, i\rangle$. Define the linear simple matrix grammar $G = (V_1, \ldots, V_n, T, P, S)$ of degree n, where P is defined as follows:

1. For each $(\bar{S}) \rightarrow (x_1 A_1 \ldots x_k A_k x_{k+1}) \in \bar{P}$, where $k \leq n$, add $(S) \rightarrow \big(x_1\langle A_1, 1\rangle \ldots x_k\langle A_k, k\rangle x_{k+1}\langle \varepsilon, k+1\rangle \ldots \langle \varepsilon, n\rangle\big)$ to P.
2. For each $(A_1, \ldots, A_k) \rightarrow (z_1, \ldots, z_k) \in \bar{P}$, where $k \leq n$, $A_i \in (\bar{V} - T) - \{\bar{S}\}$, for all $1 \leq i \leq k$, $c_1, \ldots, c_{n-k} \in \big(\bar{V} - \{\bar{S}\}\big) \cup \{\varepsilon\}$, $\Gamma \in \mathrm{perm}(k, n-k)$, and

   $$(d_1, \ldots, d_n) = \mathrm{reorder}\big((A_1, \ldots, A_k, c_1, \ldots, c_{n-k}), \Gamma\big),$$
   $$(u_1, \ldots, u_n) = \mathrm{reorder}\big((z_1, \ldots, z_k, c_1, \ldots, c_{n-k}), \Gamma\big),$$

   add $\big(\langle d_1, 1\rangle, \ldots, \langle d_n, n\rangle\big) \rightarrow \big(\alpha(u_1, 1), \ldots, \alpha(u_n, n)\big)$ to P.
3. For each $a_i \in T \cup \{\varepsilon\}$, for all $1 \leq i \leq n$, add $\big(\langle a_1, 1\rangle, \ldots, \langle a_n, n\rangle\big) \rightarrow \big(a_1, \ldots, a_n\big)$ to P.

Productions from (1) simulate productions of $\bar{G}$ of the form

$$(\bar{S}) \rightarrow (x_1 A_1 \ldots x_k A_k x_{k+1}),$$

where $k \leq n$, so that each $A_i \in (\bar{V} - T) - \{\bar{S}\}$ is converted to $\langle A_i, i\rangle \in V_i - T$, for all $1 \leq i \leq k$, and the string $\langle \varepsilon, k+1\rangle \ldots \langle \varepsilon, n\rangle$, which is erased in the last step of the derivation, is added to the end of the resulting sentential form so that the sentential form contains n nonterminals. Consider the sentential form of the form

$$w_1\langle B_1, 1\rangle \ldots w_m\langle B_m, m\rangle w_{m+1}\langle \varepsilon, m+1\rangle \ldots \langle \varepsilon, n\rangle,$$

where $w_j \in T^*$ and $\langle B_i, i\rangle \in V_i - T$, for all $1 \leq j \leq m+1$, $1 \leq i \leq m$. Each $\langle B_i, i\rangle$ may be of the form

- $\langle \varepsilon, i\rangle$, which indicates that the ith nonterminal was deleted in $\bar{G}$;
- $\langle a, i\rangle$, where $a \in T$, which indicates that the ith nonterminal was rewritten to a in $\bar{G}$;
- $\langle A, i\rangle$, where $A \in (\bar{V} - T) - \{\bar{S}\}$.

The application of a production $(A_1, \ldots, A_k) \rightarrow (z_1, \ldots, z_k) \in \bar{P}$, where $k \leq n$, $A_i \in (\bar{V} - T) - \{\bar{S}\}$, for all $1 \leq i \leq k$, can be simulated in G if $B_{j_1} \ldots B_{j_k} = A_1 \ldots A_k$, where $j_i < j_{i+1}$, $1 \leq j_i \leq m$, for all $1 \leq i \leq k$. The productions of G constructed in (2) permute $A_1, \ldots, A_k$ (preserving their order) with symbols from $\big(\bar{V} - \{\bar{S}\}\big) \cup \{\varepsilon\}$ (not preserving their order), and convert them to the corresponding symbols from $(V_1 \cup \cdots \cup V_n) - T$. The same is performed with the right-hand side of the production for the same permutation. As a result, every symbol $\langle B_i, i\rangle$ remains unchanged if $\langle B_i, i\rangle = \langle \varepsilon, i\rangle$, $\langle B_i, i\rangle = \langle a, i\rangle$, where $a \in T$, or the simulated production is not applied to B_i. Otherwise, $\langle B_i, i\rangle$ is rewritten to $x\langle C, i\rangle y$, $x\langle a, i\rangle y$, or $\langle \varepsilon, i\rangle$, where $x, y \in T^*$, $a \in T$, and $C \in (\bar{V} - T) - \{\bar{S}\}$, depending on the right-hand side of the simulated context-free component applied to B_i. Finally, a production from (3) finishes the derivation by rewriting all $\langle a, i\rangle$, where $a \in T$, to a and erasing all $\langle \varepsilon, i\rangle$. ■

The main result of this section follows next.

Theorem 5.28. *For every $n \geq 1$,*

$$\mathscr{L}(SC, LIN, n) = \mathscr{L}(SM, LIN, n),$$
$$\mathscr{L}(SC, LIN) = \mathscr{L}(SM, LIN).$$

Proof. This theorem follows from Lemmas 5.25 and 5.27. ■

A similar result can be proved for right-linear scattered context grammars as well.

Theorem 5.29. *For every $n \geq 1$,*

$$\mathscr{L}(SC, RLIN, n) = \mathscr{L}(SM, RLIN, n),$$
$$\mathscr{L}(SC, RLIN) = \mathscr{L}(SM, RLIN).$$

Proof. The proof is analogous to the proof of Theorem 5.28 and, therefore, left to the reader. ■

Theorems 5.28, 5.29, and Theorems 2.23, 2.24 imply the following two corollaries.

Corollary 5.30. *For every $n \geq 1$,*

$$\mathscr{L}(SC, LIN, n) \subset \mathscr{L}(SC, LIN, n+1),$$
$$\mathscr{L}(SC, RLIN, n) \subset \mathscr{L}(SC, RLIN, n+1),$$
$$\mathscr{L}(SC, RLIN, n) \subset \mathscr{L}(SC, LIN, n).$$

■

Corollary 5.31.

$$\mathscr{L}(CF) - \mathscr{L}(SC, LIN) \neq \emptyset,$$
$$\mathscr{L}(CF) - \mathscr{L}(SC, RLIN) \neq \emptyset,$$

and

$$\mathscr{L}(SC, RLIN) \subset \mathscr{L}(SC, LIN) \subset \mathscr{L}(PSC).$$

■

We have proved that linear scattered context grammars are equivalent to linear simple matrix grammars. The main difference of these grammars consists in the way of applying their productions. While in linear simple matrix grammars every production contains exactly n components, each of which rewrites symbols over its own alphabet, the number of components in the productions of linear scattered context grammars may vary, and all the components share the same alphabet. In addition, we have proved that the generative power of linear scattered context grammars depends on the number of nonterminals appearing in the starting production, but it is independent of the number of components in scattered context productions (see Corollary 5.26). As a result, linear scattered context grammars are more convenient for describing languages than linear simple matrix grammars because the total number of their nonterminals is lower; in addition, their productions can capture only true context dependencies while avoiding unnecessary rewriting of certain symbols.

Interestingly enough, when we restrict scattered context grammars and simple matrix grammars to their linear and right-linear variants, the power of the resulting grammars is identical; as opposed to this identity, ordinary scattered context grammars are more powerful than simple matrix grammars (see Theorem 3.27).

Taking into account the Chomsky hierarchy, we add a concluding note about scattered context grammars with context-sensitive and phrase-structure productions. As obvious, scattered context grammars with context-sensitive productions generate the family of context-sensitive languages while the others characterize the family of phrase-structure languages. Therefore, as these scattered context grammars are as powerful as the grammars underlying them, their use is of little or no interest.

5.6 Extended Propagating Scattered Context Grammars

In classical formal language theory, monotone grammars, which generate the family of context-sensitive languages, represent phrase-structure grammars in which every production has its right-hand side at least as long as its left-hand side. In this section, we discuss analogically restricted scattered context grammars. That is, by concatenating all the strings occurring on the right-hand side of any n-component production, we obtain a string of length n or more. Clearly, all propagating scattered context grammars satisfy this requirement, but not all scattered context grammars satisfying this requirement are propagating because they allow the empty string to appear on the right-hand side of their productions. In this sense, they represent an extension of propagating scattered context grammars, so we refer to them as extended propagating scattered context grammars. In [28], it was proved that these grammars generate the family of context-sensitive languages. Before establishing this result, we define these grammars formally.

Definition 5.32. An *extended propagating scattered context grammar* is a scattered context grammar $G = (V, T, P, S)$ in which every

$$(A_1, \ldots, A_n) \to (x_1, \ldots, x_n) \in P$$

satisfies

$$|x_1 \ldots x_n| \geq n.$$

The family of languages generated by extended propagating scattered context grammars is denoted by $\mathscr{L}(PSC, ext)$.

In what follows, we prove that $\mathscr{L}(PSC, ext) = \mathscr{L}(CS)$. We use a slightly modified construction of the proof presented in [28]. First, we prove that $\mathscr{L}(CS) \subseteq \mathscr{L}(PSC, ext)$.

Lemma 5.33. $\mathscr{L}(CS) \subseteq \mathscr{L}(PSC, ext)$.

Proof. Let $G = (V, T, P, S)$ be a context-sensitive grammar in Kuroda normal form (see Definition 2.17). Set

$$\begin{aligned}
\bar{N}_1 &= \{a' : a \in T\},\\
\bar{N}_2 &= \big\{\hat{a} : a \in (V - T) \cup \bar{N}_1\big\},\\
\bar{N}_3 &= \big\{\langle a \rangle : a \in (V - T) \cup \bar{N}_1 \cup \bar{N}_2\big\},\\
\bar{N}_4 &= \{\hat{a} : a \in T\}.
\end{aligned}$$

Define the extended propagating scattered context grammar

$$\bar{G} = \big(V \cup \bar{N}_1 \cup \bar{N}_2 \cup \bar{N}_3 \cup \bar{N}_4 \cup \{\bar{S}\}, T, \bar{P}, \bar{S}\big),$$

where $\bar{P}$ is constructed as follows:

1. For each $S \Rightarrow_G^* w$, where $w \in T^*$ and $|w| \leq 3$, add $(\bar{S}) \to (w)$ to $\bar{P}$.
2. For each $S \Rightarrow_G^* ABCD$, where A, B, C, $D \in V - T$, add $(\bar{S}) \to \big(\langle \hat{A} \rangle BC \langle D \rangle\big)$ to $\bar{P}$.
3. For each a, b, $c \in (V - T) \cup \bar{N}_1$, add
   (a) $\big(\langle a \rangle, b, \langle c \rangle\big) \to \big(\varepsilon, \langle b \rangle, c\langle a \rangle\big)$,
   (b) $\big(\langle \hat{a} \rangle, b, \langle c \rangle\big) \to \big(\varepsilon, \langle b \rangle, c\langle \hat{a} \rangle\big)$,
   (c) $\big(\langle a \rangle, \hat{b}, \langle c \rangle\big) \to \big(\varepsilon, \langle \hat{b} \rangle, c\langle a \rangle\big)$,
   (d) $\big(\langle a \rangle, b, \langle \hat{c} \rangle\big) \to \big(\varepsilon, \langle b \rangle, \hat{c}\langle a \rangle\big)$ to $\bar{P}$.

4. For each $AB \rightarrow CD \in P$, where A, B, C, $D \in V - T$, and a, $b \in (V - T) \cup \bar{N}_1$, add
   (a) $\big(\langle A\rangle, B, a, \langle b\rangle\big) \rightarrow \big(\varepsilon, \varepsilon, \langle a\rangle, bC\langle D\rangle\big)$,
   (b) $\big(\langle \hat{A}\rangle, B, a, \langle b\rangle\big) \rightarrow \big(\varepsilon, \varepsilon, \langle a\rangle, b\hat{C}\langle D\rangle\big)$,
   (c) $\big(\langle A\rangle, B, \hat{a}, \langle b\rangle\big) \rightarrow \big(\varepsilon, \varepsilon, \langle \hat{a}\rangle, bC\langle D\rangle\big)$,
   (d) $\big(\langle A\rangle, B, a, \langle \hat{b}\rangle\big) \rightarrow \big(\varepsilon, \varepsilon, \langle a\rangle, \hat{b}C\langle D\rangle\big)$ to $\bar{P}$.
5. For each $A \rightarrow BC \in P$, where A, B, $C \in V - T$, and a, $b \in (V - T) \cup \bar{N}_1$, add
   (a) $\big(\langle A\rangle, a, \langle b\rangle\big) \rightarrow \big(\varepsilon, \langle a\rangle, bB\langle C\rangle\big)$,
   (b) $\big(\langle \hat{A}\rangle, a, \langle b\rangle\big) \rightarrow \big(\varepsilon, \langle a\rangle, b\hat{B}\langle C\rangle\big)$,
   (c) $\big(\langle A\rangle, \hat{a}, \langle b\rangle\big) \rightarrow \big(\varepsilon, \langle \hat{a}\rangle, bB\langle C\rangle\big)$,
   (d) $\big(\langle A\rangle, a, \langle \hat{b}\rangle\big) \rightarrow \big(\varepsilon, \langle a\rangle, \hat{b}B\langle C\rangle\big)$ to $\bar{P}$.
6. For each $A \rightarrow a \in P$, where $A \in V - T$, $a \in T$, and b, $c \in (V - T) \cup \bar{N}_1$, add
   (a) $\big(\langle A\rangle, b, \langle c\rangle\big) \rightarrow \big(\varepsilon, \langle b\rangle, c\langle a\rangle\big)$,
   (b) $\big(\langle \hat{A}\rangle, b, \langle c\rangle\big) \rightarrow \big(\varepsilon, \langle b\rangle, c\langle \hat{a}\rangle\big)$,
   (c) $\big(\langle A\rangle, \hat{b}, \langle c\rangle\big) \rightarrow \big(\varepsilon, \langle \hat{b}\rangle, c\langle a\rangle\big)$,
   (d) $\big(\langle A\rangle, b, \langle \hat{c}\rangle\big) \rightarrow \big(\varepsilon, \langle b\rangle, \hat{c}\langle a\rangle\big)$ to $\bar{P}$.
7. For each a, $b \in T$, add
   (a) $\big(\langle \hat{a}'\rangle, \langle b'\rangle\big) \rightarrow (\hat{a}, b')$,
   (b) $(\hat{a}, b') \rightarrow (a, \hat{b})$,
   (c) $(\hat{a}) \rightarrow (a)$ to $\bar{P}$.

Basic Idea.
The simulation of G is started in $\bar{G}$ by using a production from (2). The simulation is performed by cycling through every sentential form by productions from (3) until the nonterminals to be rewritten appear at the beginning of the current sentential form. Then, depending on the production whose application is being simulated, a production from (4), (5), or (6) is used. After this, another simulation can be performed analogically. The derivation is completed by applying productions from (7).

To be able to cycle through the sentential form, the grammar marks the first and the last nonterminal. During one cycle, the current first symbol is placed to the end of the sentential form and is marked as the last; the previously second symbol is marked as the first. The simulation of an application of a production from G is simulated analogously.

In addition, the grammar records the symbol corresponding to the first symbol of the sentential form of G. The derivation is successful only if this symbol appears at the beginning of the sentential form during the final phase of the derivation.

Formal Proof.
First, we prove Claims 1 through 4 stated below.

Claim 1. Every $w \in L(\bar{G})$, where $|w| \leq 3$, is generated by a production from (1).

Proof. As the constructed grammar is monotone, the only productions that are able to generate sentences consisting of at most 3 terminals are those from (1). □

Claim 2. Every derivation of $w \in L(\bar{G})$, where $|w| \geq 4$, can only be performed in the following way:

$$\begin{aligned} \bar{S} &\Rightarrow_{\bar{G}} x \ [p_2] \\ &\Rightarrow_{\bar{G}}^{*} y \ [\Xi_1] \\ &\Rightarrow_{\bar{G}} z \ [p_{7a}] \\ &\Rightarrow_{\bar{G}}^{*} w \ [\Xi_2], \end{aligned}$$

where $x \in \bar{N}_3(V-T)(V-T)\bar{N}_3$, every sentential form y_i in $x \Rightarrow_{\bar{G}}^{*} y$ satisfies

$$y_i \in \big((V-T) \cup \bar{N}_1 \cup \bar{N}_2 \cup \bar{N}_3\big)^{+},$$

every sentential form w_i in $z \Rightarrow_{\bar{G}}^{*} w$ satisfies $w_i \in (T \cup \bar{N}_1 \cup \bar{N}_4)^{+}$, p_2 is a production from (2), p_{7a} is a production from (7a), and Ξ_1 and Ξ_2 are sequences of productions introduced in (3) through (6) and (7), respectively.

Proof. First, observe that the only productions that can start the derivation of $w \in L(\bar{G})$, where $|w| \geq 4$, are those from (2). After the application of a production from (2), we obtain a sentential form $x \in \bar{N}_3(V-T)(V-T)\bar{N}_3$. Productions from (2) cannot be used during the rest of the derivation because $\bar{S}$ does not appear on the right-hand side of any production.

Now, productions from (3) through (6) can be used. Notice that these productions rewrite only symbols from $(V-T) \cup \bar{N}_1 \cup \bar{N}_2 \cup \bar{N}_3$, so after their application, the resulting sentential forms also contain only these symbols. In addition, each of these productions contains precisely 2 symbols from $\bar{N}_3$ both on its left-hand side and right-hand side. As a result, every sentential form contains exactly 2 symbols from $\bar{N}_3$. Productions from (7b) and (7c) cannot be used as they require a symbol from $\bar{N}_4$ to be present in the sentential form.

During this phase, a production from (7a) can be used as well. It rewrites the symbols from $\bar{N}_3$ and introduces a symbol from $\bar{N}_4$ to the sentential form. From this moment, only productions from (7b) and (7c) are applicable. As these productions rewrite only symbols from $\bar{N}_1 \cup \bar{N}_4$ and no other productions can be used, every sentential form w_i has to satisfy $w_i \in (T \cup \bar{N}_1 \cup \bar{N}_4)^{+}$ in order to generate $w \in T^{+}$. □

Claim 3. In the derivation

$$\begin{aligned} \bar{S} &\Rightarrow_{\bar{G}} x \ [p_2] \\ &\Rightarrow^*_{\bar{G}} y \ [\Xi_1] \\ &\Rightarrow_{\bar{G}} z \ [p_{7a}] \\ &\Rightarrow^*_{\bar{G}} w \ [\Xi_2] \end{aligned}$$

from Claim 2, every sentential form y_i in $x \Rightarrow^*_{\bar{G}} y$ satisfies

$$y_i \in \bar{N}_3\big((V-T)\cup\bar{N}_1\cup\bar{N}_2\big)^*\bar{N}_3$$

and every sentential form w_i in $z \Rightarrow^*_{\bar{G}} w$ satisfies $w_i \in T^*\bar{N}_4\bar{N}_1^*$.

Proof. To show that

$$y_i \in \bar{N}_3\big((V-T)\cup\bar{N}_1\cup\bar{N}_2\big)^*\bar{N}_3$$

and $w_i \in T^*\bar{N}_4\bar{N}_1^*$, first notice that

$$x \in \bar{N}_3\big((V-T)\cup\bar{N}_1\cup\bar{N}_2\big)^*\bar{N}_3.$$

Furthermore, observe that the form of the productions from (3) through (6) guarantees that a nonterminal from $\bar{N}_3$ remains at the end of y_i. In addition, productions from (3) through (6) move the nonterminal from $\bar{N}_3$ right in the sentential form. Productions from (7) replace this nonterminal with a symbol from $\bar{N}_4$, which can only move right as well. In this way, the grammar guarantees that if any nonterminal appears in front of this symbol, it can never be rewritten during the rest of the derivation, and the derivation is unsuccessful. As a result,

$$y_i \in \bar{N}_3\big((V-T)\cup\bar{N}_1\cup\bar{N}_2\big)^*\bar{N}_3$$

and $w_i \in T^*\bar{N}_4\bar{N}_1^*$. □

Claim 4. In the derivation

$$\begin{aligned} \bar{S} &\Rightarrow_{\bar{G}} x \ [p_2] \\ &\Rightarrow^*_{\bar{G}} y \ [\Xi_1] \\ &\Rightarrow_{\bar{G}} z \ [p_{7a}] \\ &\Rightarrow^*_{\bar{G}} w \ [\Xi_2] \end{aligned}$$

from Claim 2, every sentential form y_i in $x \Rightarrow^*_{\bar{G}} y$ contains precisely one symbol from $\bar{N}_2 \cup \big\{\langle a\rangle : a \in \bar{N}_2\big\}$.

Proof. Productions from (3) through (6) either do not contain any symbol from $\bar{N}_2 \cup \big\{\langle a\rangle : a \in \bar{N}_2\big\}$ or they contain precisely one, occurring both on their left-hand sides and right-hand sides. Because x contains one symbol from $\bar{N}_2 \cup \big\{\langle a\rangle : a \in \bar{N}_2\big\}$, this number is preserved after applications of productions from (3) through (6). □

Now, it is easy to see that every production of the form $AB \rightarrow CD \in P$ is simulated properly by $\bar{G}$. Indeed, when applying a production from (4), the symbols corresponding to A and B in $\bar{G}$ necessarily occur as two neighbors at the beginning of the sentential form; if some nonterminals occur between them, these nonterminals eventually appear in front of the leading nonterminal from $\bar{N}_3$, and by Claim 3, they can never be rewritten. Symbols corresponding to C and D are placed to the end of the sentential form. Productions from (4b), (4c), and (4d) handle different positions of the nonterminal from $\bar{N}_2 \cup \{\langle a \rangle : a \in \bar{N}_2\}$, and they make sure that its position remains correct after the simulation. In particular, notice that there is no production of the form

$$\left(\langle A \rangle, \hat{B}, a, \langle b \rangle\right) \rightarrow \left(\varepsilon, \varepsilon, \langle a \rangle, bC\langle \hat{D} \rangle\right).$$

This production would check the context between the first and the last nonterminal of the original sentential form of G, which is illegal in context-sensitive grammars. The simulation of $A \rightarrow BC \in P$, $A \rightarrow a \in P$, and cycling through the sentential form is performed analogously.

In order to enter the final phase by a production from (7a), in which all nonterminals are rewritten to terminals by productions from (7b) and (7c), the symbol from $\{\langle a \rangle : a \in \bar{N}_2\}$ has to appear at the beginning of the sentential form in $\bar{G}$; otherwise, the nonterminals preceding this symbol cannot be rewritten.

From these observations and Claims 1 through 4, we obtain $w \in L(G)$ if and only if $w \in L(\bar{G})$, so the theorem holds. ■

Next, we demonstrate that $\mathscr{L}(PSC, ext) \subseteq \mathscr{L}(CS)$.

Lemma 5.34. $\mathscr{L}(PSC, ext) \subseteq \mathscr{L}(CS)$.

Proof. By definition, the derivations of extended propagating scattered context grammars are monotone. Therefore, their derivations can be simulated by context-sensitive grammars. As a result, $\mathscr{L}(PSC, ext) \subseteq \mathscr{L}(CS)$. ■

Putting Lemmas 5.33 and 5.34 together, we obtain the main result of this section.

Theorem 5.35. $\mathscr{L}(PSC, ext) = \mathscr{L}(CS)$. ■