Properties of Regular Languages

Pumping Lemma for RLs

Gist: Pumping lemma demonstrates an infinite iteration of some substring in RLs.

• Let L be a RL. Then, there is $k \geq 1$ such that
if $z \in L$ and $|z| \geq k$, **then** there exist u,v,w: $z = uvw$,
1) $v \neq \varepsilon$ **2)** $|uv| \leq k$ **3)** for each $m \geq 0$, $uv^m w \in L$

Example: for RE $r = ab^*c$, $L(r)$ is ***regular.***
There is $k = 3$ such that **1)**, **2)** and **3)** holds.

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There is $k = 3$ such that **1)**, **2)** and **3)** holds.

• for $z = \mathbf{abc}$: $z \in L(r)$ & $|z| \geq 3$: $uv^0w = \mathbf{ab^0c} = \mathbf{ac} \in L(r)$
$uv^1w = \mathbf{ab^1c} = \mathbf{abc} \in L(r)$
$uv^2w = \mathbf{ab^2c} = \mathbf{abbc} \in L(r)$
⋮

u v w

$v \neq \varepsilon$, $|uv| = 2 \leq 3$

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There is $k = 3$ such that **1), 2)** and **3)** holds.

• for $z = abc$: $z \in L(r)$ & $|z| \geq 3$: ($u = a$, $v = b$, $w = c$)

$v \neq \varepsilon$, $|uv| = 2 \leq 3$

$uv^0w = ab^0c = ac \in L(r)$
$uv^1w = ab^1c = abc \in L(r)$
$uv^2w = ab^2c = abbc \in L(r)$
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• for $z = abbc$: $z \in L(r)$ & $|z| \geq 3$: ($u = a$, $v = b$, $w = bc$)

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$uv^0w = ab^0bc = abc \in L(r)$
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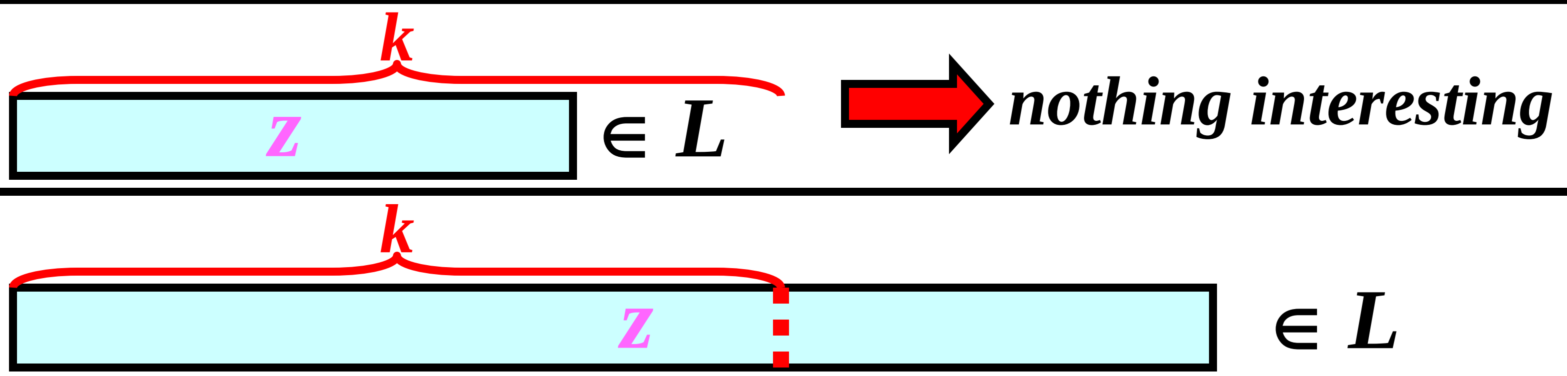

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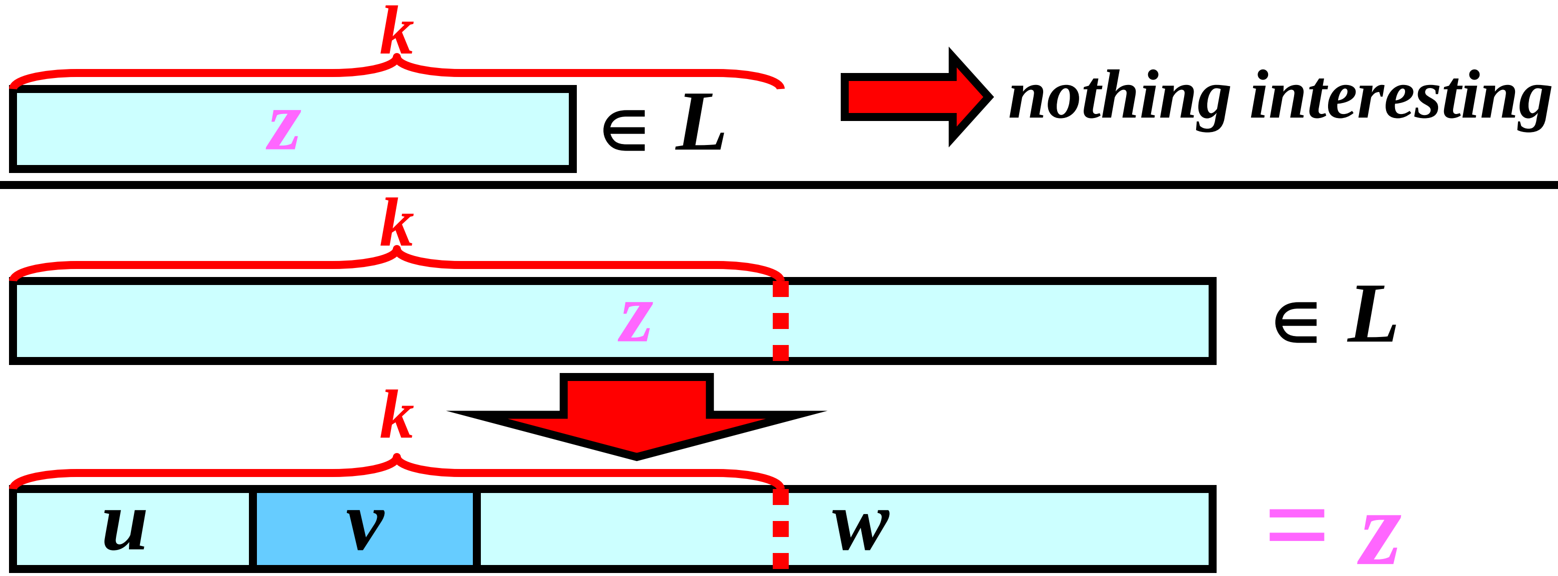

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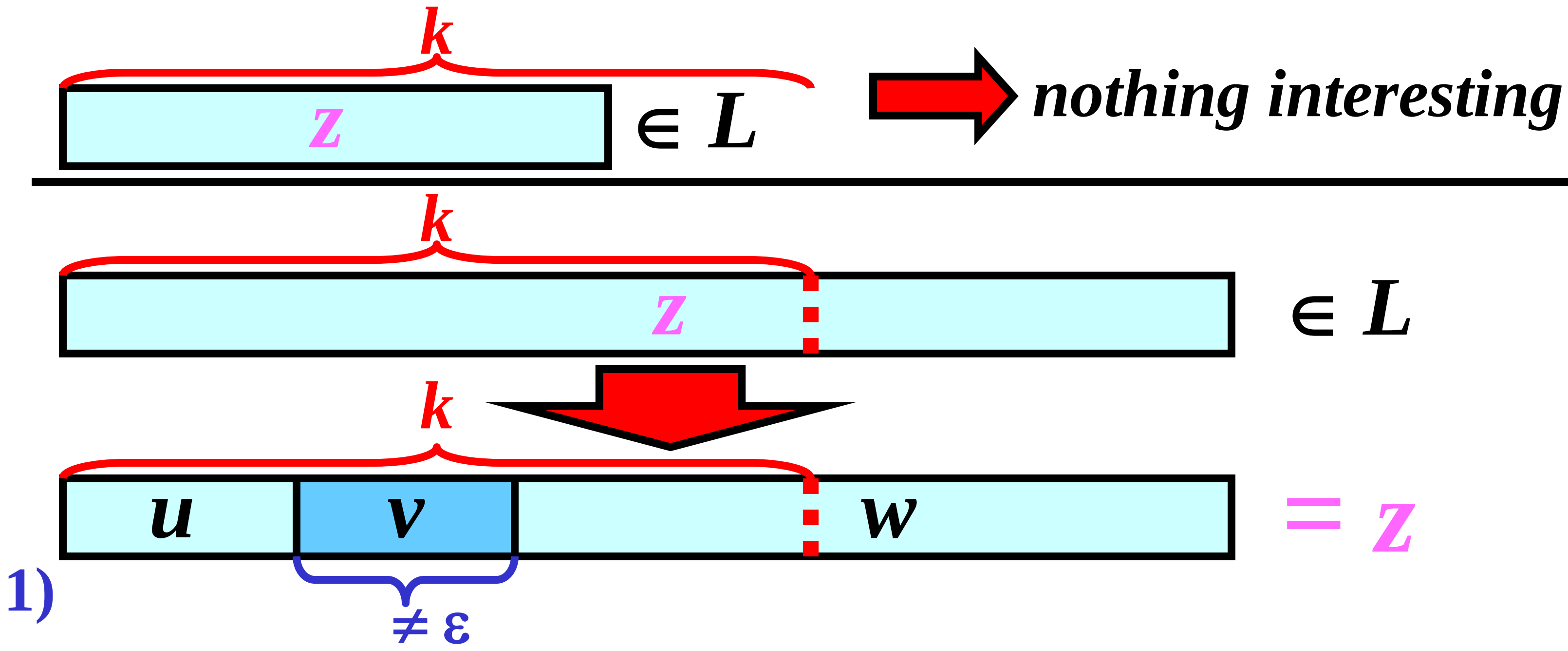

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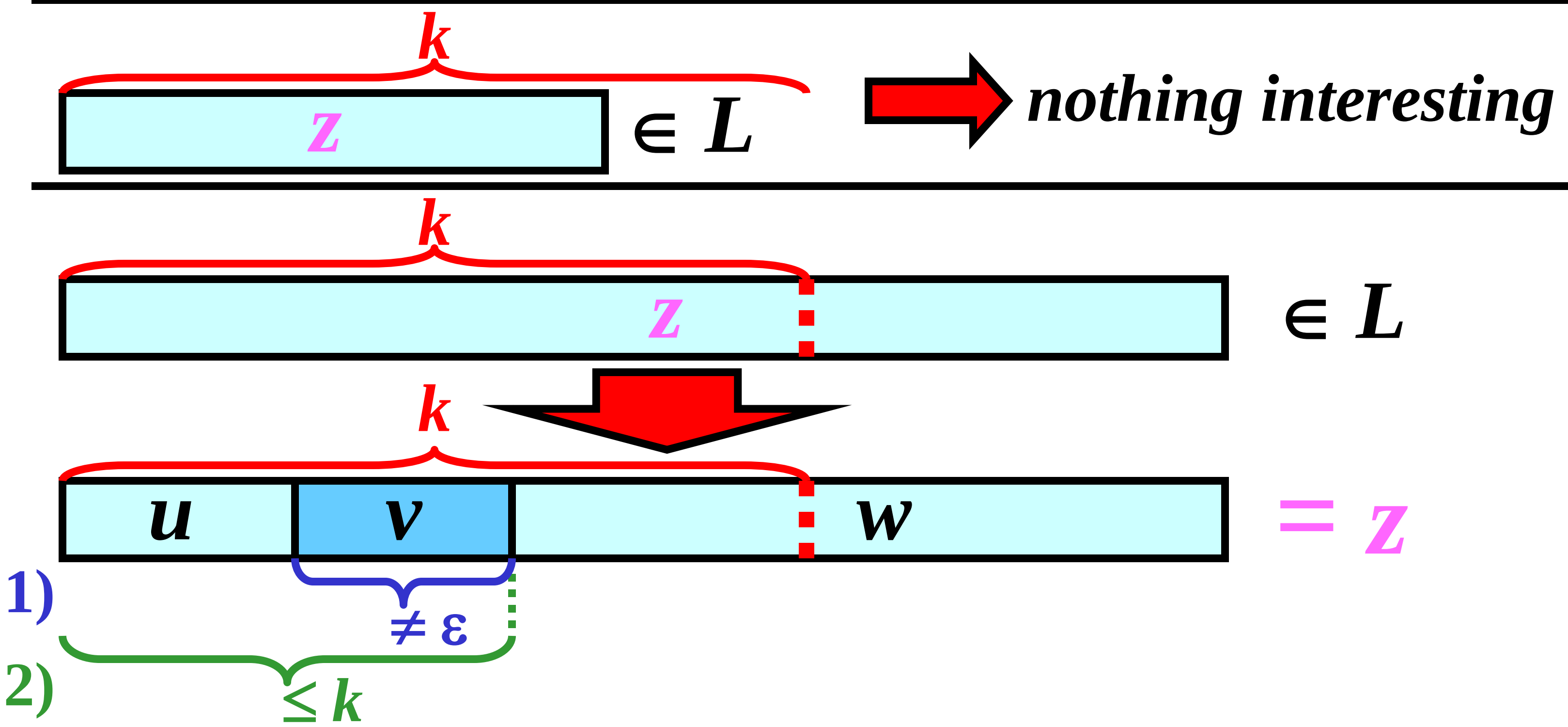

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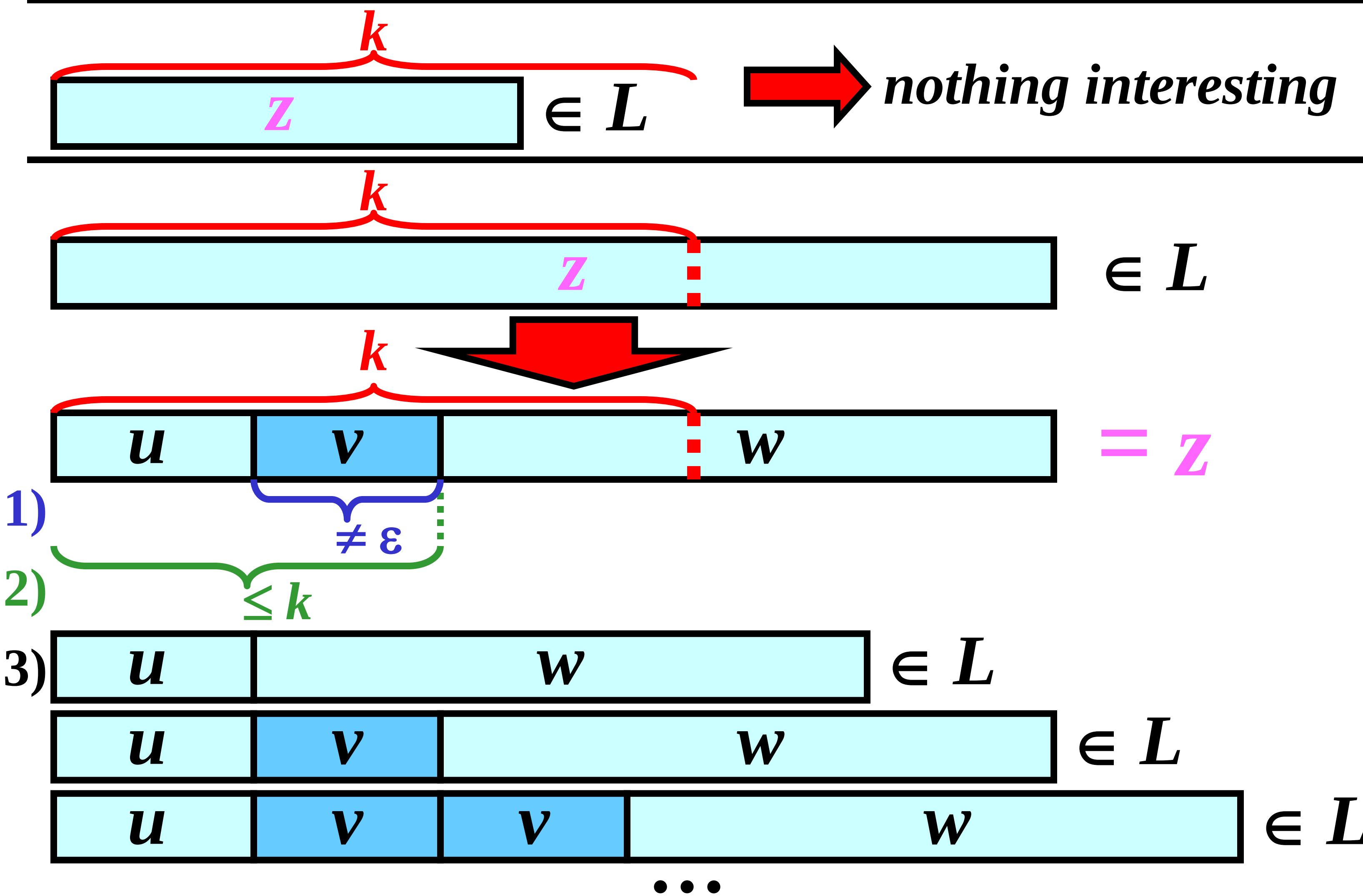
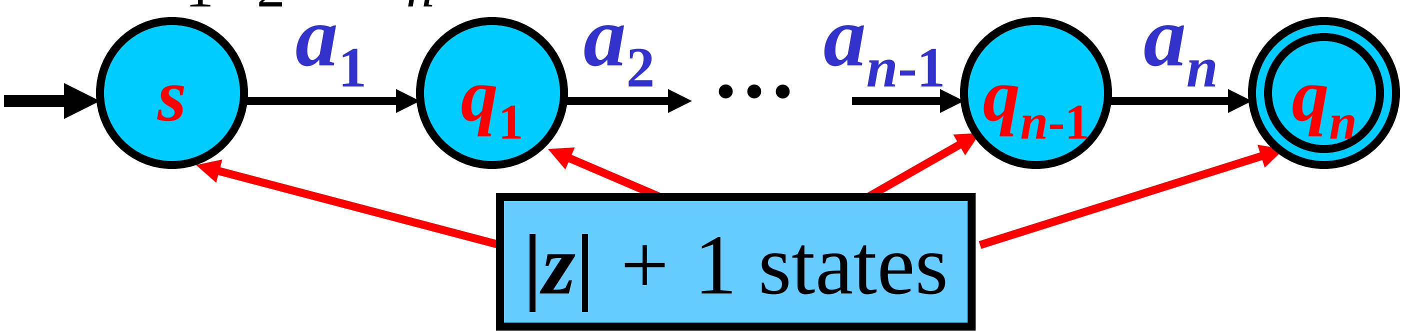


Proof of Pumping Lemma 1/3

- Let L be a regular language. Then, there exists **DFA** $M = (Q, \Sigma, R, s, F)$, and $L = L(M)$.
- For $z \in L(M)$, M makes $|z|$ moves and M visits $|z| + 1$ states:
- for $z = a_1 a_2 \ldots a_n$:



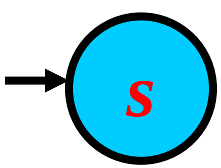
$$\overbrace{s a_1 a_2 \ldots a_n}^{|z|} \vdash q_1 a_2 \ldots a_n \vdash \ldots \vdash q_{n-1} a_n \vdash q_n$$

Proof of Pumping Lemma 2/3

- Let $\boldsymbol{k}$ = card(Q) (the number of states).

For each $z \in L$ and $|z| \geq k$, M visits $k + 1$ or more states. As $k + 1 >$ card(Q), there exists a state $\boldsymbol{q}$ that M visits at least twice.

- For z exist u, v, w such that $z = \boldsymbol{uvw}$:



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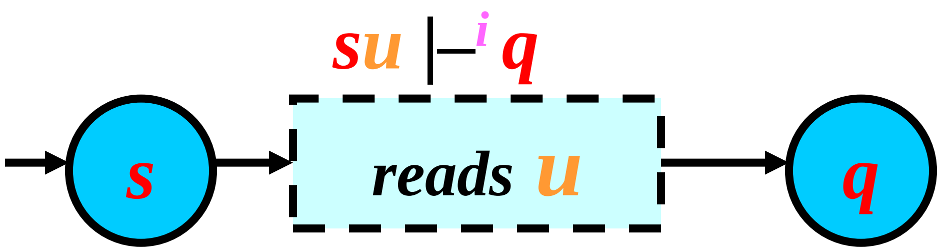

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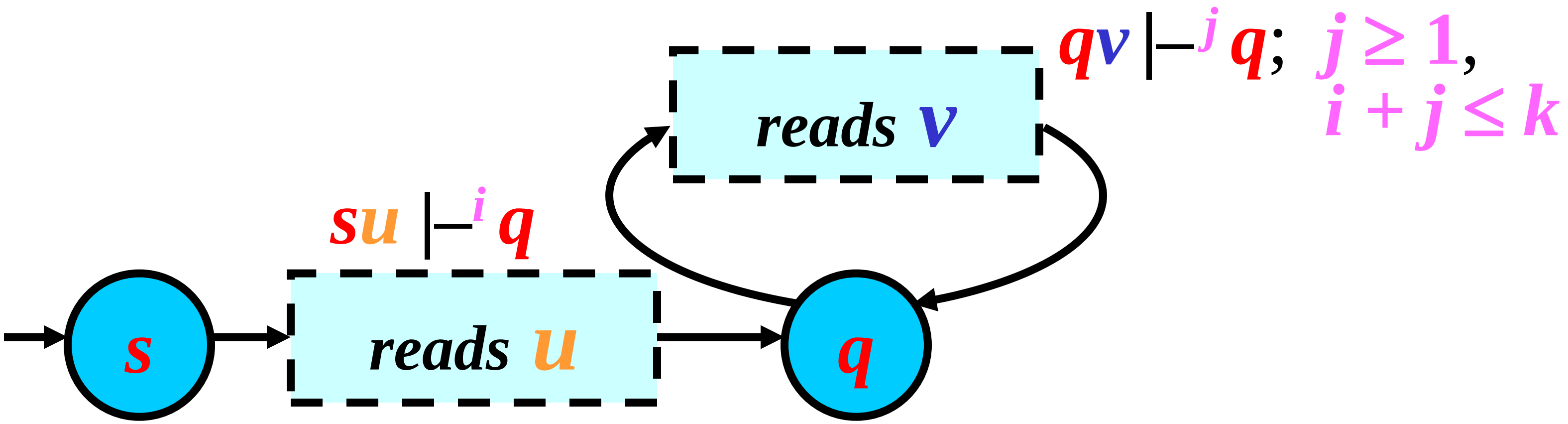

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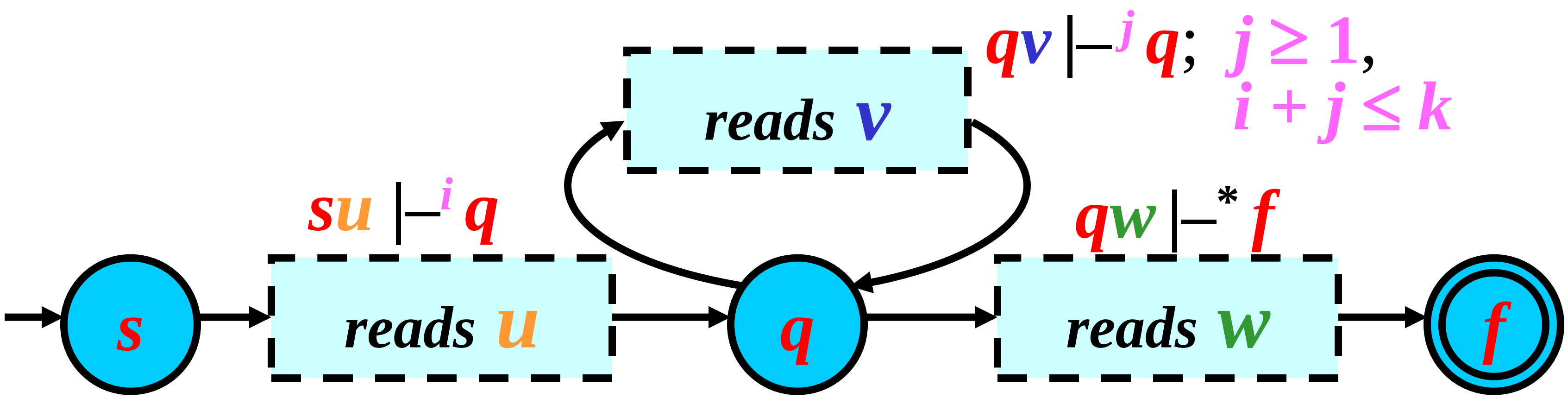

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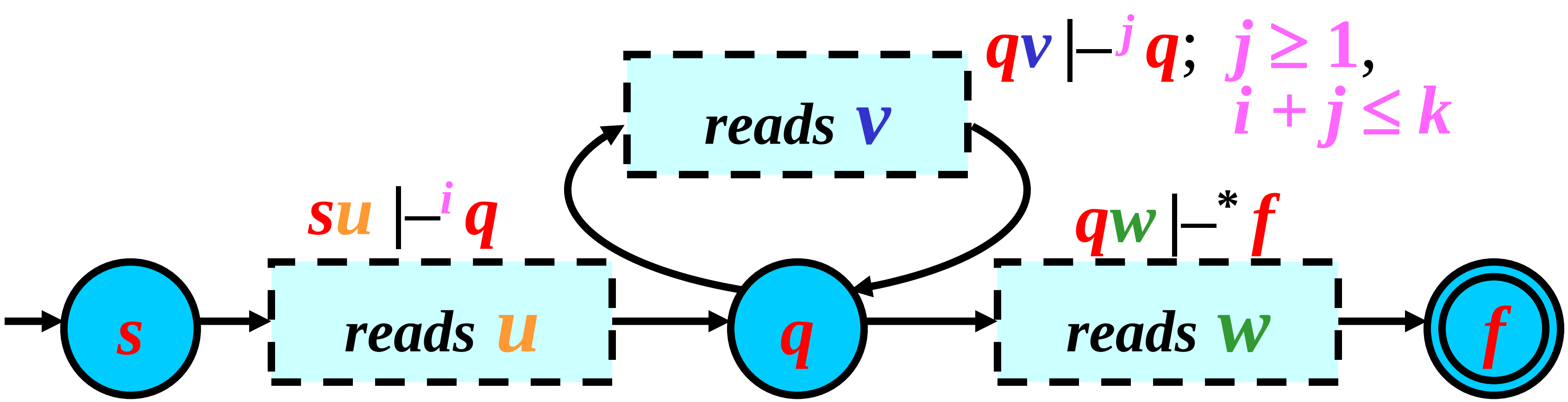

Summary:

$$sz = suvw \vdash^i qvw \vdash^j qw \vdash^* f,\ f \in F$$

Proof of Pumping Lemma 3/3

- There exist moves:

(1.) $su \vdash^{i} q$; (2.) $qv \vdash^{j} q$; (3.) $qw \vdash^{*} f$, $f \in F$, so

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- for $m = 0$, $uv^{m}w = uv^{0}w = uw$,

suw

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  $suw \vdash^i qw$ (1.)

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- for each $m > 0$,

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$suw \vdash^{i} qw \vdash^{*} f$, $f \in F$ (moves 1., 3.)

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$suv^m w \vdash^{i} qv^m w \vdash^{j} qv^{m-1} w \vdash^{j} \ldots \vdash^{j} qw \vdash^{*} f$, $f \in F$ (moves 1., 2., 2., 2., 3.)

Summary:

1) $qv \vdash^{j} q$, $j \geq 1$; therefore, $|v| \geq \mathbf{1}$, so $v \neq \varepsilon$

2) $suv \vdash^{i} qv \vdash^{j} q$, $i + j \leq k$; therefore, $|uv| \leq k$

3) For each $m \geq 0$: $suv^m w \vdash^{*} f$, $f \in F$, therefore $uv^m w \in L$

QED

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false assumption (→ back to "Assume that L is regular")

⇒ Therefore, **L is not regular**

Pumping Lemma: Example

Prove that $L = \{a^n b^n : n \geq 0\}$ is not regular:

1) Assume that L is regular. Let $k \geq 1$ be the pumping lemma constant for L.

2) Let $z = a^k b^k$: $a^k b^k \in L$, $|z| = |a^k b^k| = 2k \geq k$

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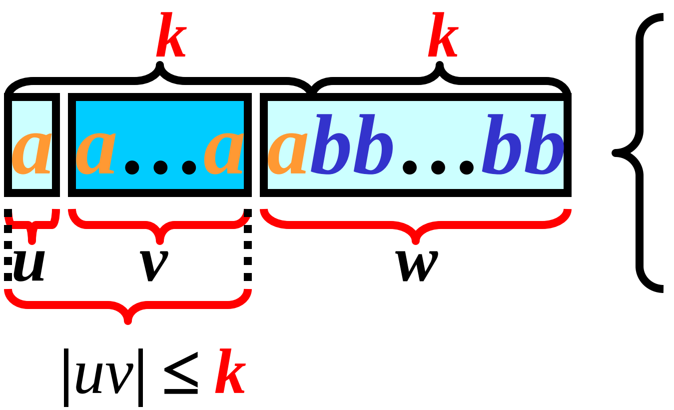

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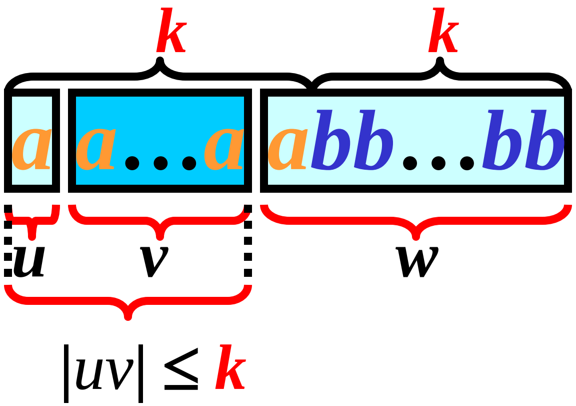

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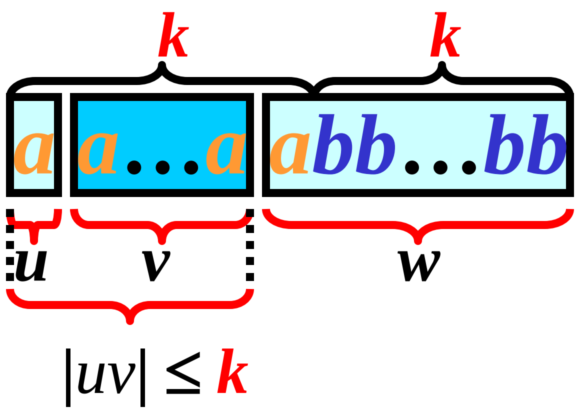

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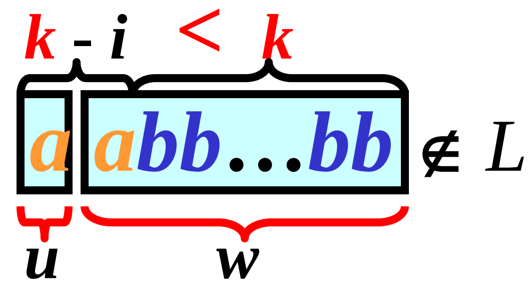

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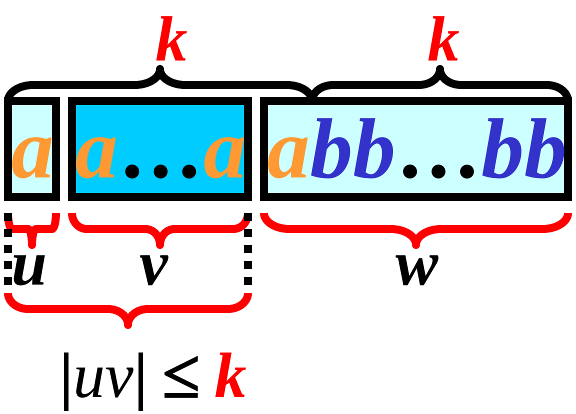

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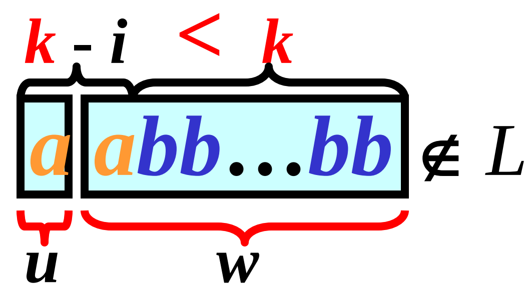

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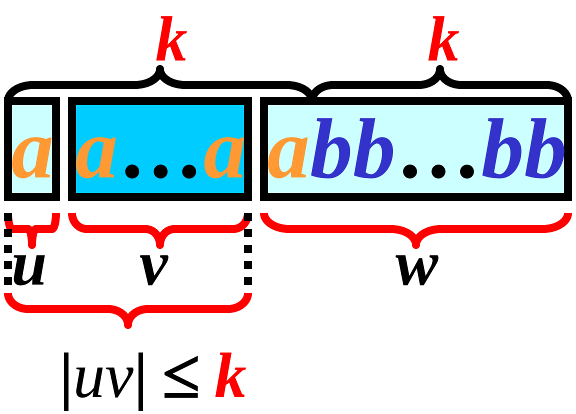

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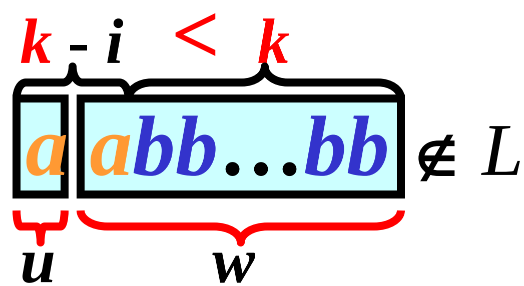

Contradiction!

4) Therefore, L is not regular

Note on Use of Pumping Lemma

- **Pumping lemma:**

if *L* is regular **then** ⇒ exist $k \geq 1$ and ...

Main application of the pumping lemma:

- proof by contradiction that *L* is **not** regular.

Note on Use of Pumping Lemma

- **Pumping lemma:**

if [*L* is regular] **then** ⇒ [exist $k \geq 1$ and ...]

Main application of the pumping lemma:

- proof by contradiction that *L* is **not** regular.

- **However, the next implication is incorrect:**

~~**if** [exist $k \geq 1$ and ...] **then** ⇒ [*L* is regular]~~

- **We cannot use the pumping lemma to prove that *L* is regular.**

Pumping Lemma: Application II. 1/3

- We can use the pumping lemma to prove some other theorems.

Illustration:

- Let M be a DFA and k be the pumping lemma constant (k is the number of states in M). Then, $L(M)$ is infinite $\Leftrightarrow$ there exists $z \in L(M)$, $k \leq |z| < 2k$

Proof:

1) there exists $z \in L(M)$, $k \leq |z| < 2k \Rightarrow L(M)$ is infinite**:**

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if $z \in L(M)$, $k \leq |z|$, then by PL:

$z = uvw$, $v \neq \varepsilon$, and for each $m \geq 0$: $uv^m w \in L(M)$

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$L(M)$ is infinite

Pumping Lemma: Application II. 2/3

2) $L(M)$ is infinite $\Rightarrow$ there exists $z \in L(M)$, $k \leq |z| < 2k$**:**

- We prove by contradiction, that

$L(M)$ is infinite —**a)**→ there exists $z \in L(M)$, $|z| \geq k$

—**b)**↓ there exists $z \in L(M)$, $k \leq |z| < 2k$

---

a) Prove by contradiction that

- $\boldsymbol{L(M)}$ **is infinite** $\boldsymbol{\Rightarrow}$ **there exists** $\boldsymbol{z \in L(M), |z| \geq k}$

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a) Prove by contradiction that

- **$L(M)$ is infinite $\Rightarrow$ there exists $z \in L(M)$, $|z| \geq k$**

Assume that **$L(M)$ is infinite** and there exists no $z \in L(M)$, $|z| \geq k$

Pumping Lemma: Application II. 2/3

2) $L(M)$ is infinite $\Rightarrow$ there exists $z \in L(M)$, $k \leq |z| < 2k$**:**

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$L(M)$ is infinite **a)** $\rightarrow$ there exists $z \in L(M)$, $|z| \geq k$

b) $\downarrow$

there exists $z \in L(M)$, $k \leq |z| < 2k$

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Assume that ***L(M)* is infinite** and there exists no $z \in L(M)$, $|z| \geq k$

$\downarrow$

for all $z \in L(M)$ holds $|z| < k$

Pumping Lemma: Application II. 2/3

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Pumping Lemma: Application II. 2/3

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Pumping Lemma: Application II. 3/3

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Pumping Lemma: Application II. 3/3

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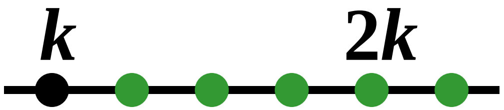

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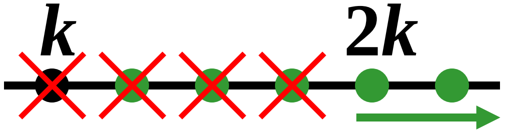

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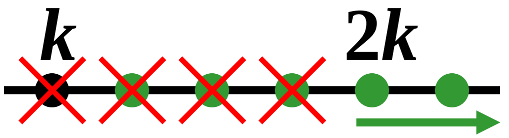

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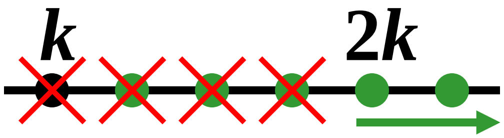

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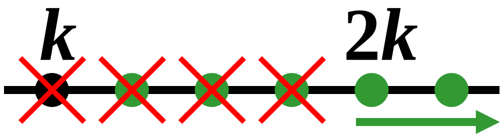

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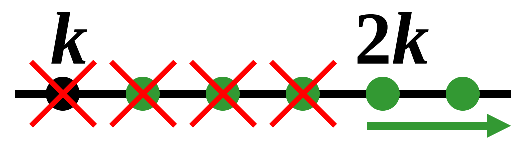

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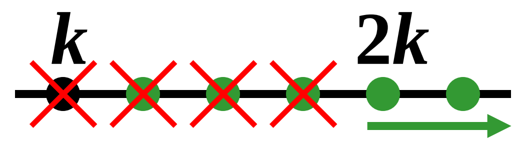

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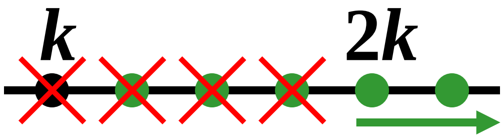

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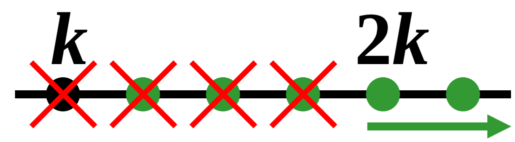

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Contradiction !

Closure properties 1/2

Definition: The family of regular languages is closed under an operation ***o*** if the language resulting from the application of ***o*** to any regular languages is also regular.

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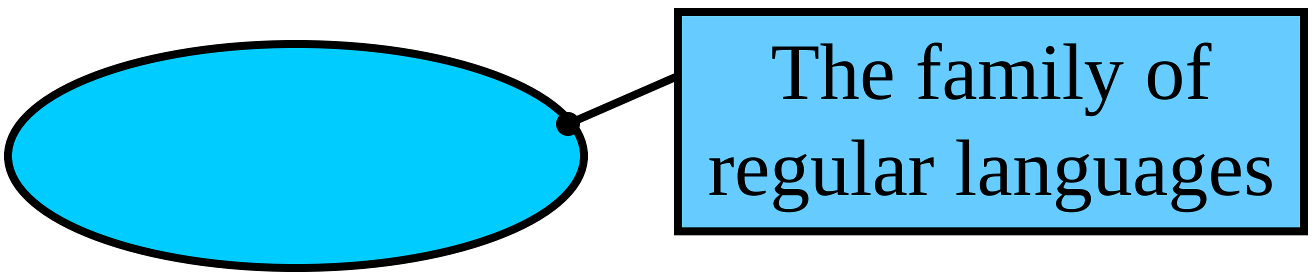

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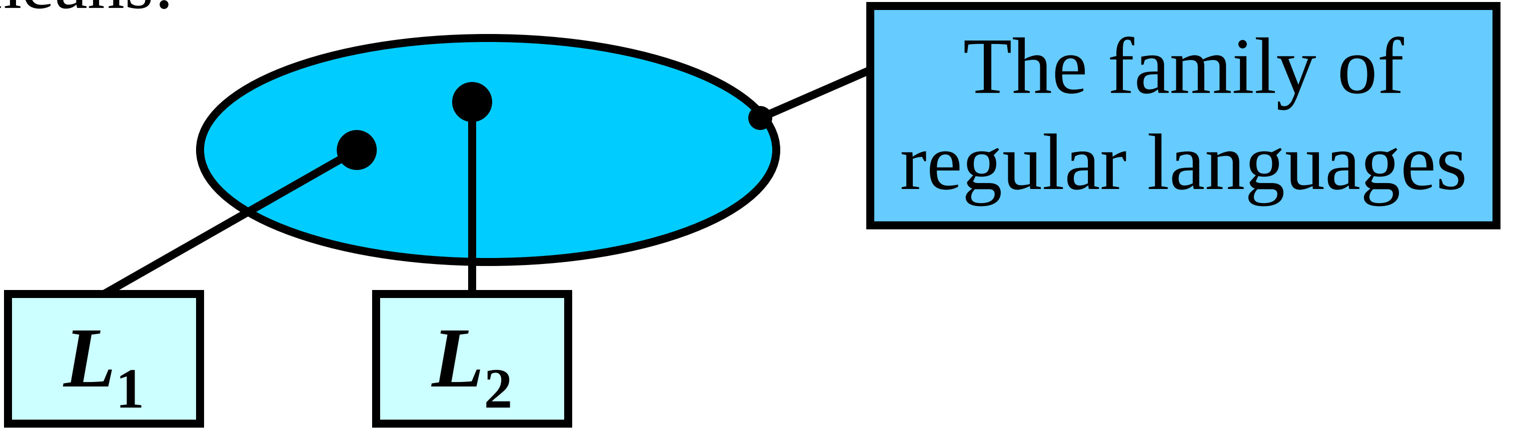

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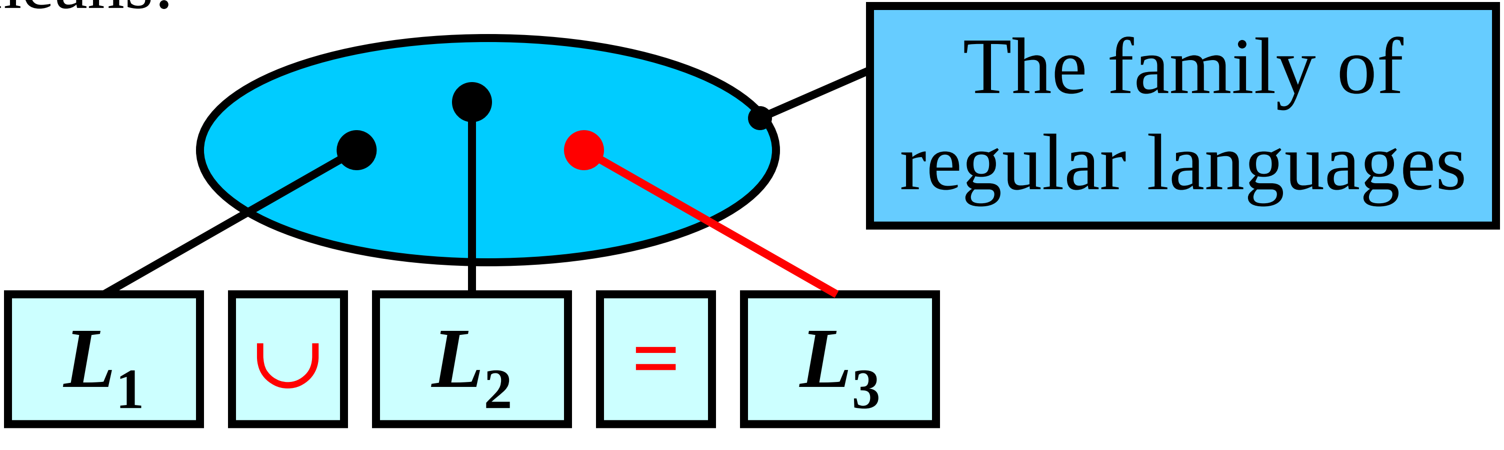


Closure properties 2/2

Theorem: The family of regular languages is closed under **union**, **concatenation**, **iteration**.

Proof:

- Let $\boldsymbol{L_1}$, $\boldsymbol{L_2}$ be two **regular languages**
- Then, there exist two REs r_1, r_2: $L(r_1) = \boldsymbol{L_1}$, $L(r_2) = \boldsymbol{L_2}$;
- By the definition of regular expressions:
  - $r_1.r_2$ is a RE denoting $\boldsymbol{L_1 L_2}$
  - $r_1 + r_2$ is a RE denoting $\boldsymbol{L_1 \cup L_2}$
  - r_1^* is a RE denoting $\boldsymbol{L_1^*}$
- Every RE denotes regular language, so $\boldsymbol{L_1 L_2}$, $\boldsymbol{L_1 \cup L_2}$, $\boldsymbol{L_1^*}$ are a **regular languages**

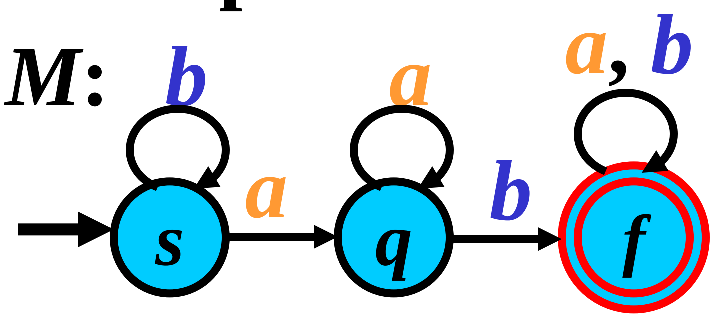
Algorithm: FA for Complement

- **Input:** Complete FA: $M = (Q, \Sigma, R, s, F)$
- **Output:** Complete FA: $M' = (Q, \Sigma, R, s, F')$,

$$L(M') = \overline{L(M)}$$

- **Method:**
- $F' := Q - F$

Example:



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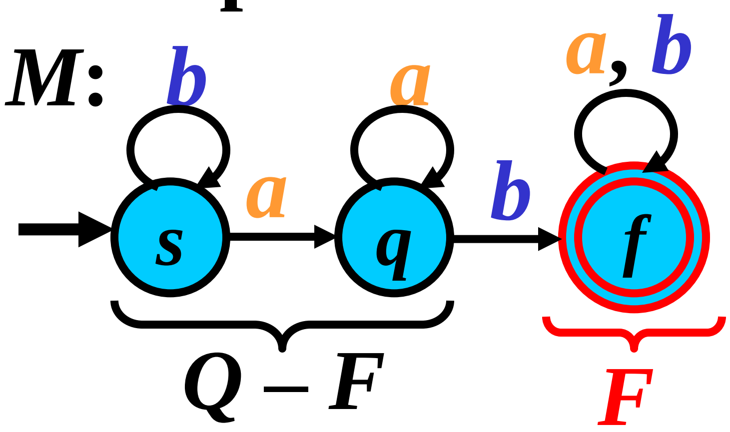

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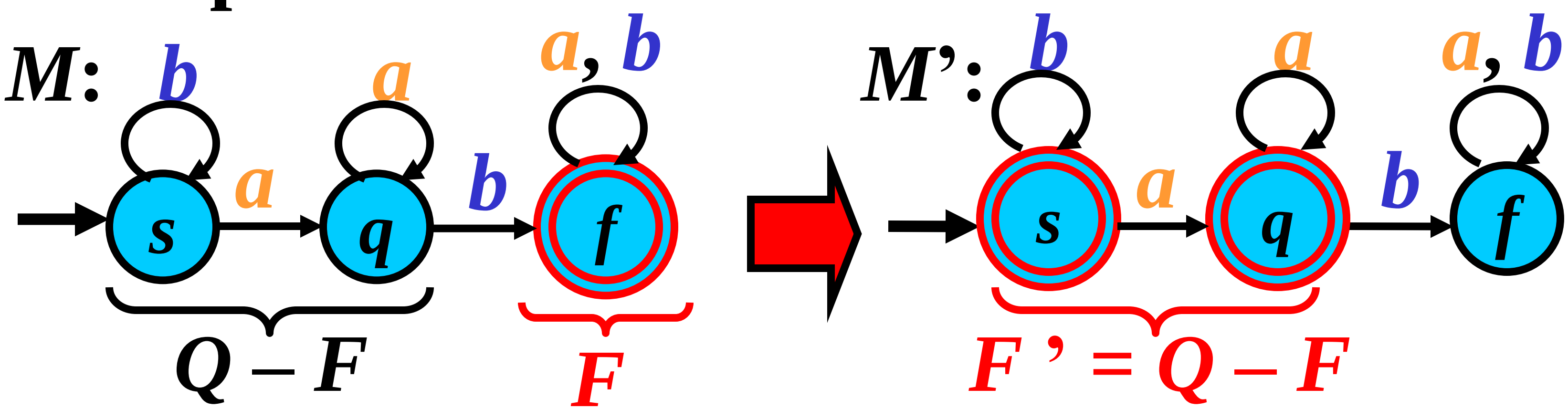


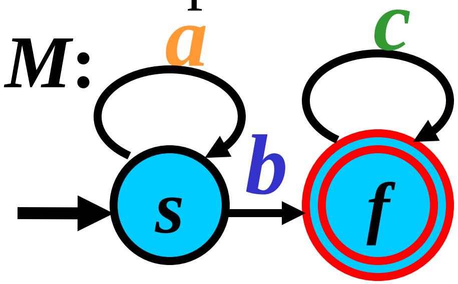
$L(M) = \{x$: *ab* is a *substring* of $x\}$; $L(M') = \{x$: *ab* is no *substring* of $x\}$

FA for Complement: Problem

- Previous algorithm requires a **complete** FA
- If *M* is incomplete FA, then *M* must be converted to a complete FA before we use the previous algorithm

Example:

Incomplete DFA:



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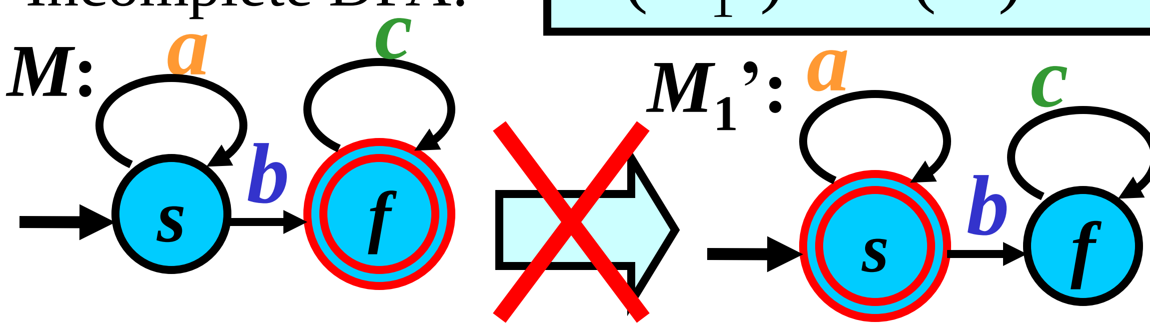
Incomplete DFA:

$L(M_1') \neq \overline{L(M)}$**!** - $c \notin L(M)$, $c \notin L(M_1')$



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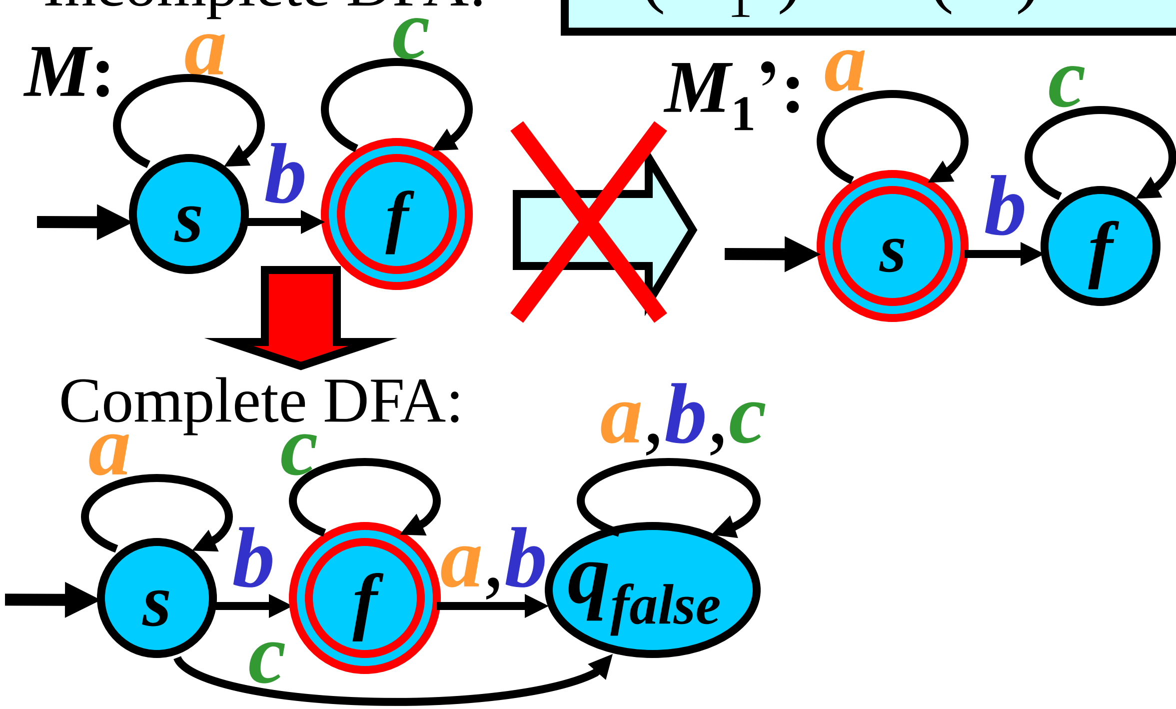

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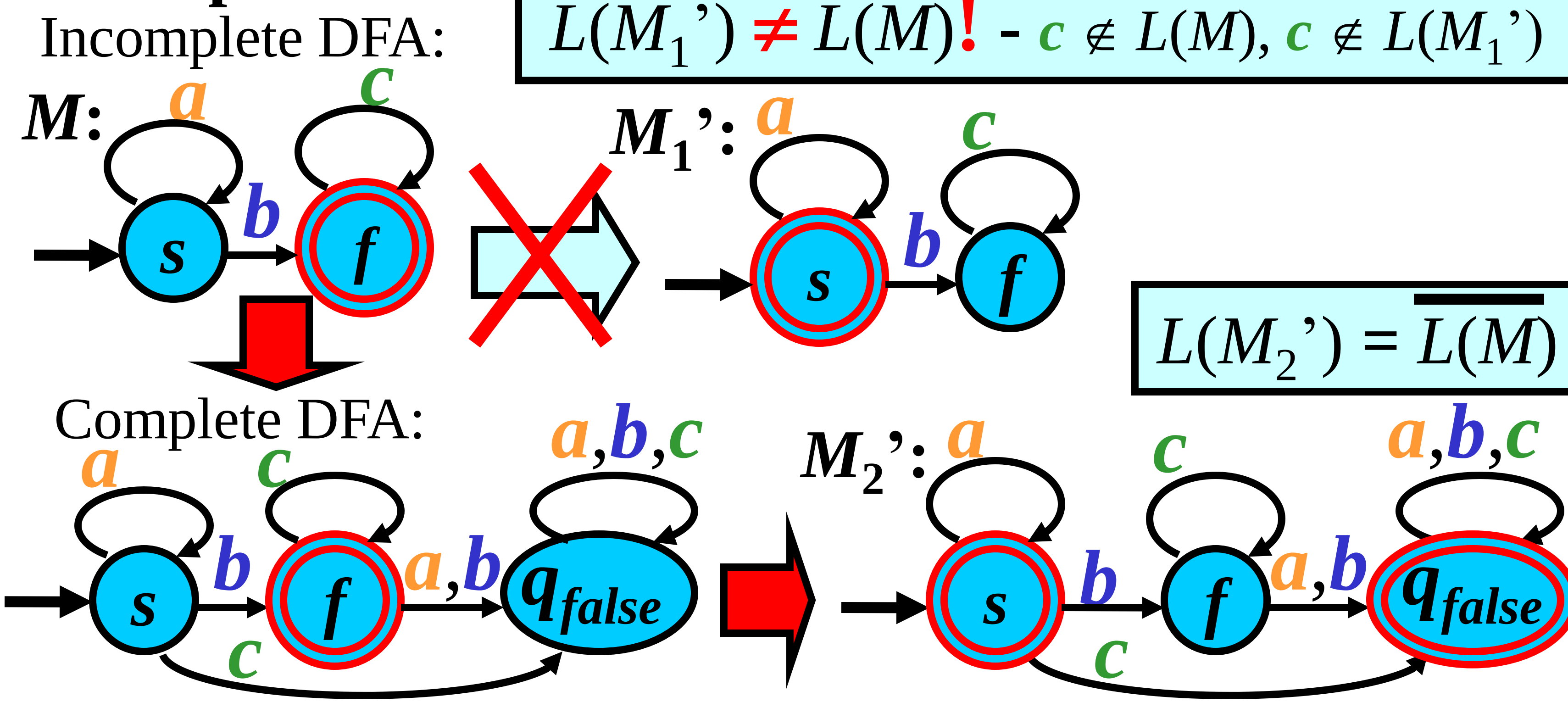


$L(M_2') = \overline{L(M)}$

Closure properties: Complement

Theorem: The family of regular languages is closed under **complement**.

Proof:

- Let L be a **regular language**
- Then, there exists a complete DFA M: $L(M) = L$
- We can construct a complete DFA M': $L(M') = \overline{L}$ by using the previous algorithm
- Every FA defines a regular language, so $\overline{L}$ is a **regular language**

Closure properties: Intersection

Theorem: The family of regular languages is closed under **intersection**.

Proof:

- Let L_1, L_2 be two **regular languages**
- $\overline{L_1}$, $\overline{L_2}$ are **regular languages**

(the family of regular languages is closed under complement)

- $\overline{L_1} \cup \overline{L_2}$ is a **regular language**

(the family of regular languages is closed under union)

- $\overline{\overline{L_1} \cup \overline{L_2}}$ is a **regular language**

(the family of regular languages is closed under complement)

- $L_1 \cap L_2 = \overline{\overline{L_1} \cup \overline{L_2}}$ is a **regular language** (DeMorgan's law)

Boolean Algebra of Languages

Definition: Let a family of languages be closed under union, intersection, and complement. Then, this family represents a ***Boolean algebra of languages***.

Theorem: The family of regular languages is a Boolean algebra of languages.

Proof:

- The family of regular languages is closed under union, intersection, and complement.

Main Decidable Problems

1. Membership problem:

- **Instance:** FA M, $w \in \Sigma^*$; **Question:** $w \in L(M)$**?**

2. Emptiness problem:

- **Instance:** FA M; **Question:** $L(M) = \emptyset$**?**

3. Finiteness problem:

- **Instance:** FA M; **Question:** Is $L(M)$ finite**?**

4. Equivalence problem:

- **Instance:** FA M_1, M_2; **Question:** $L(M_1) = L(M_2)$**?**

Algorithm: Membership Problem

- **Input:** DFA $M = (Q, \Sigma, R, s, F)$; $w \in \Sigma^*$
- **Output:** **YES** if $w \in L(M)$
  NO if $w \notin L(M)$

- **Method:**
- **if** $sw \vdash^* f,\ f \in F$ **then** write ('**YES**')
  else write ('**NO**')

Summary:

The membership problem for FAs is decidable

Algorithm: Emptiness Problem

- **Input:** FA $M = (Q, \Sigma, R, s, F)$;
- **Output:** **YES** if $L(M) = \emptyset$
  NO if $L(M) \neq \emptyset$

- **Method:**
- **if** s is nonterminating **then** write ('**YES**')
  else write ('**NO**')

Summary:

The emptiness problem for FAs is decidable

Algorithm: Finiteness Problem

- **Input:** DFA $M = (Q, \Sigma, R, s, F)$;
- **Output:** **YES** if $L(M)$ is finite
  NO if $L(M)$ is infinite

- **Method:**
- Let $k = \text{card}(Q)$
- **if** there exist $z \in L(M)$, $k \le |z| < 2k$ **then** write ('**NO**')
  else write ('**YES**')

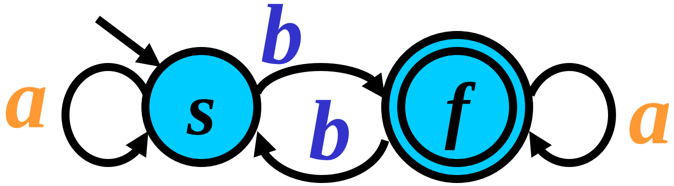
Note: This algorithm is based on
$L(M)$ is infinite $\Leftrightarrow$ there exists z: $z \in L(M)$, $k \le |z| < 2k$

Summary:

The finiteness problem for FAs is decidable

Decidable Problems: Example

***M*:**



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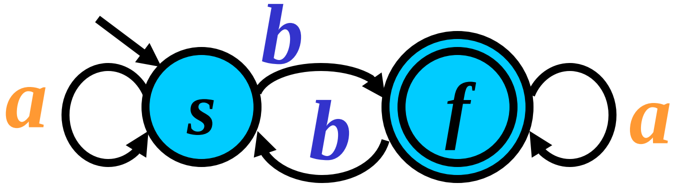

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sab |– ***sb*** |– ***f***, ***f*** ∈ ***F***

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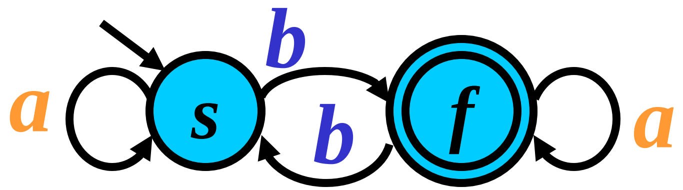

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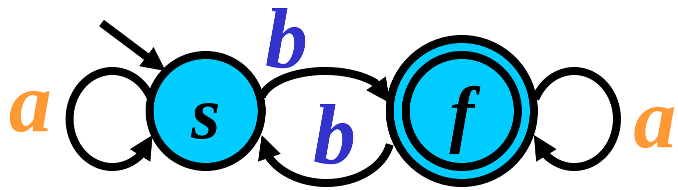

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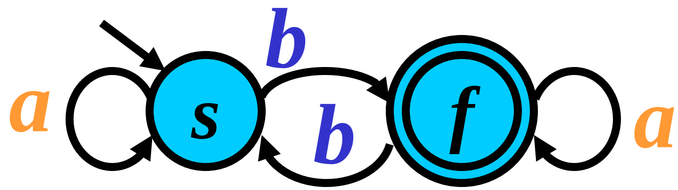

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$Q_0 = \{f\}$

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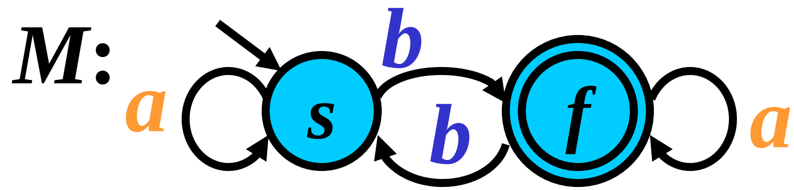

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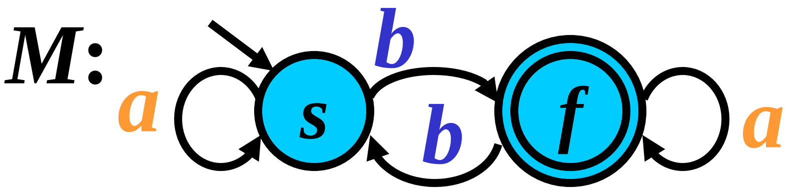
$Q_0 = \{f\}$

1. $qa' \rightarrow f$; $q \in Q$; $a' \in \Sigma$: $sb \rightarrow f$, $fa \rightarrow f$

$Q_1 = \{f\} \cup \{s, f\} = \{f, s\}$ … s is terminating

Decidable Problems: Example



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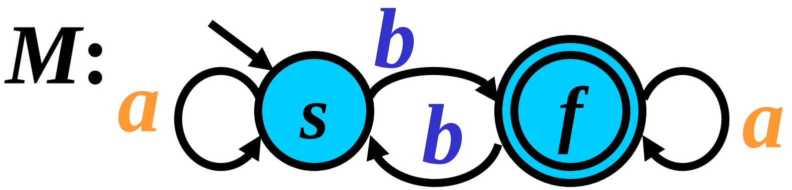
$Q_0 = \{f\}$

1. $qa' \rightarrow f; q \in Q; a' \in \Sigma$: $sb \rightarrow f, fa \rightarrow f$

$Q_1 = \{f\} \cup \{s, f\} = \{f, s\}$ **…** s is terminating

Answer: **NO** because s is terminating

Decidable Problems: Example



Question: $ab \in L(M)$ **?**

$sab \vdash sb \vdash f, f \in F$

Answer: **YES** because $sab \vdash^* f, f \in F$

Question: $L(M) = \emptyset$ **?**

$Q_0 = \{f\}$

1. $qa' \to f$; $q \in Q$; $a' \in \Sigma$: $sb \to f$, $fa \to f$

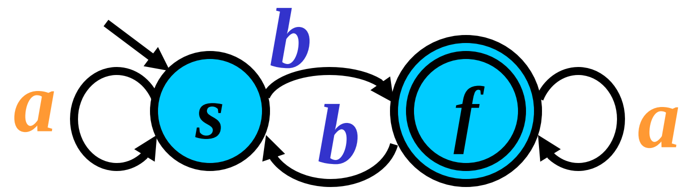
$Q_1 = \{f\} \cup \{s, f\} = \{f, s\}$ **…** s is terminating

Answer: **NO** because s is terminating

Question: Is $L(M)$ finite?

Decidable Problems: Example

***M*:**



Question: ***ab*** $\in L(M)$ **?**

sab $\vdash$ ***sb*** $\vdash$ ***f***, ***f*** $\in$ ***F***

Answer: YES because ***sab*** $\vdash^*$ ***f***, ***f*** $\in$ ***F***

Question: $L(M) = \emptyset$ **?**

$Q_0 = \{f\}$

1. $qa' \rightarrow f$; $q \in Q$; $a' \in \Sigma$: $sb \rightarrow f$, $fa \rightarrow f$

$Q_1 = \{f\} \cup \{s, f\} = \{f, s\}$ … ***s*** is terminating

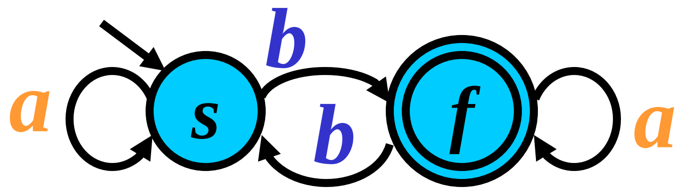
Answer: NO because ***s*** is terminating

Question: Is $L(M)$ finite**?** $k = \text{card}(Q) = 2$

All strings $z \in \Sigma^*$: $2 \le |z| < 4$: ***aa***, ***bb***, ***ab***, **…**

Decidable Problems: Example

***M*:**



Question: $ab \in L(M)$ **?**

$sab \vdash sb \vdash f$, $f \in F$

Answer: YES because $sab \vdash^* f$, $f \in F$

Question: $L(M) = \emptyset$ **?**

$Q_0 = \{f\}$

1. $qa' \rightarrow f$; $q \in Q$; $a' \in \Sigma$: $sb \rightarrow f$, $fa \rightarrow f$

$Q_1 = \{f\} \cup \{s, f\} = \{f, s\}$ … ***s*** is terminating

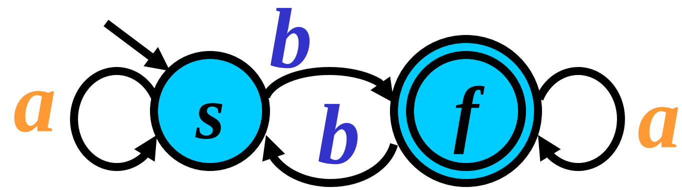
Answer: NO because ***s*** is terminating

Question: Is $L(M)$ finite**?** $k = \text{card}(Q) = 2$

All strings $z \in \Sigma^*$: $2 \leq |z| < 4$**:** ***aa***, ***bb***, ***ab*** $\in L(M)$, …

Decidable Problems: Example

M:



Question: $ab \in L(M)$ **?**

$sab \vdash sb \vdash f$, $f \in F$

Answer: YES because $sab \vdash^* f$, $f \in F$

Question: $L(M) = \emptyset$ **?**

$Q_0 = \{f\}$

1. $qa' \rightarrow f$; $q \in Q$; $a' \in \Sigma$: $sb \rightarrow f$, $fa \rightarrow f$

$Q_1 = \{f\} \cup \{s, f\} = \{f, s\}$ … s is terminating

Answer: NO because s is terminating

Question: Is $L(M)$ finite**?** $k = \text{card}(Q) = 2$

All strings $z \in \Sigma^*$: $2 \leq |z| < 4$: aa, bb, $ab \in L(M)$, …

Answer: NO because there exist $z \in L(M)$, $k \leq |z| < 2k$

Algorithm: Equivalence Problem

- **Input:** Two minimum state FA, M_1 and M_2
- **Output:** **YES** if $L(M_1) = L(M_2)$
  NO if $L(M_1) \neq L(M_2)$

- **Method:**
- **if** M_1 coincides with M_2 except for the name of states
  then write ('**YES**')
  else write ('**NO**')

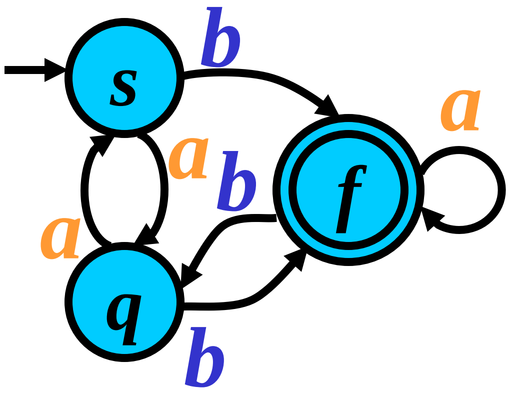
Summary:

The equivalence problem for FA is decidable

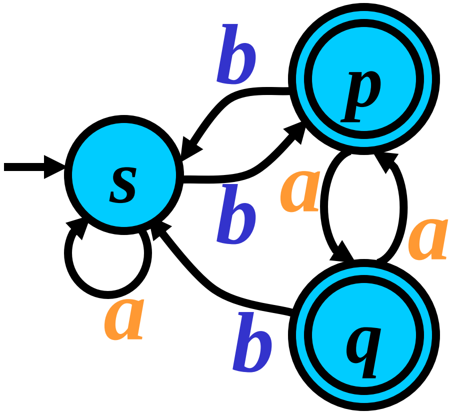
Equivalence Problem: Example

Question: $L(M_1) = L(M_2)$?

M_1:



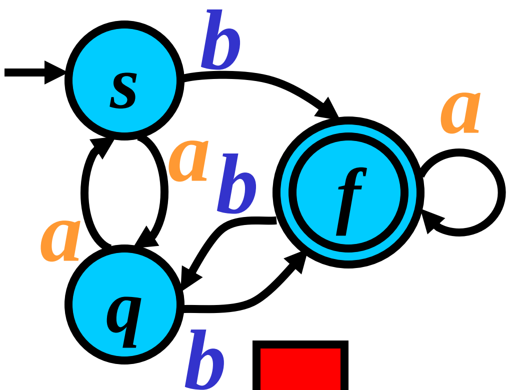
M_2:



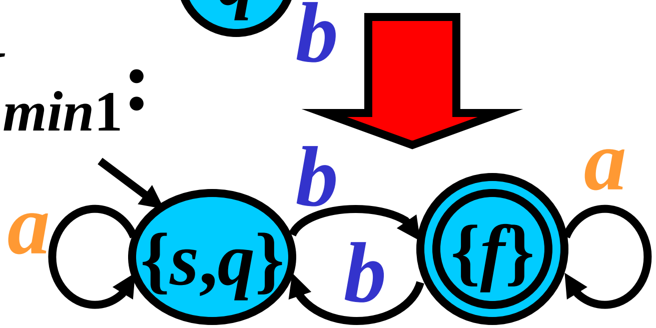
Equivalence Problem: Example

Question: $L(M_1) = L(M_2)$?

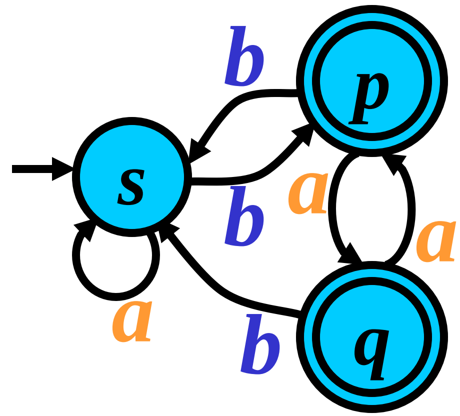
M_1:



M_{min1}:

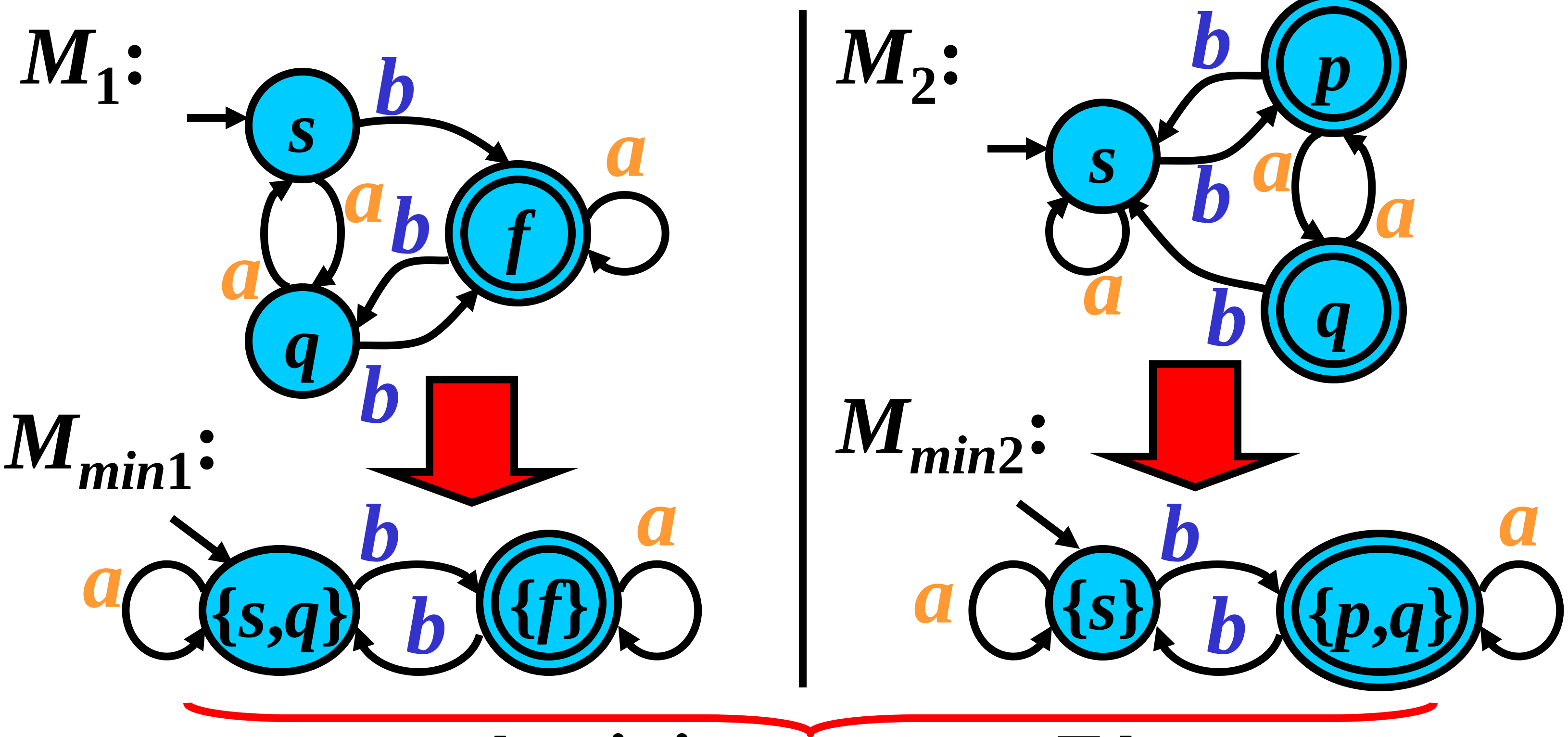


M_2:



Equivalence Problem: Example

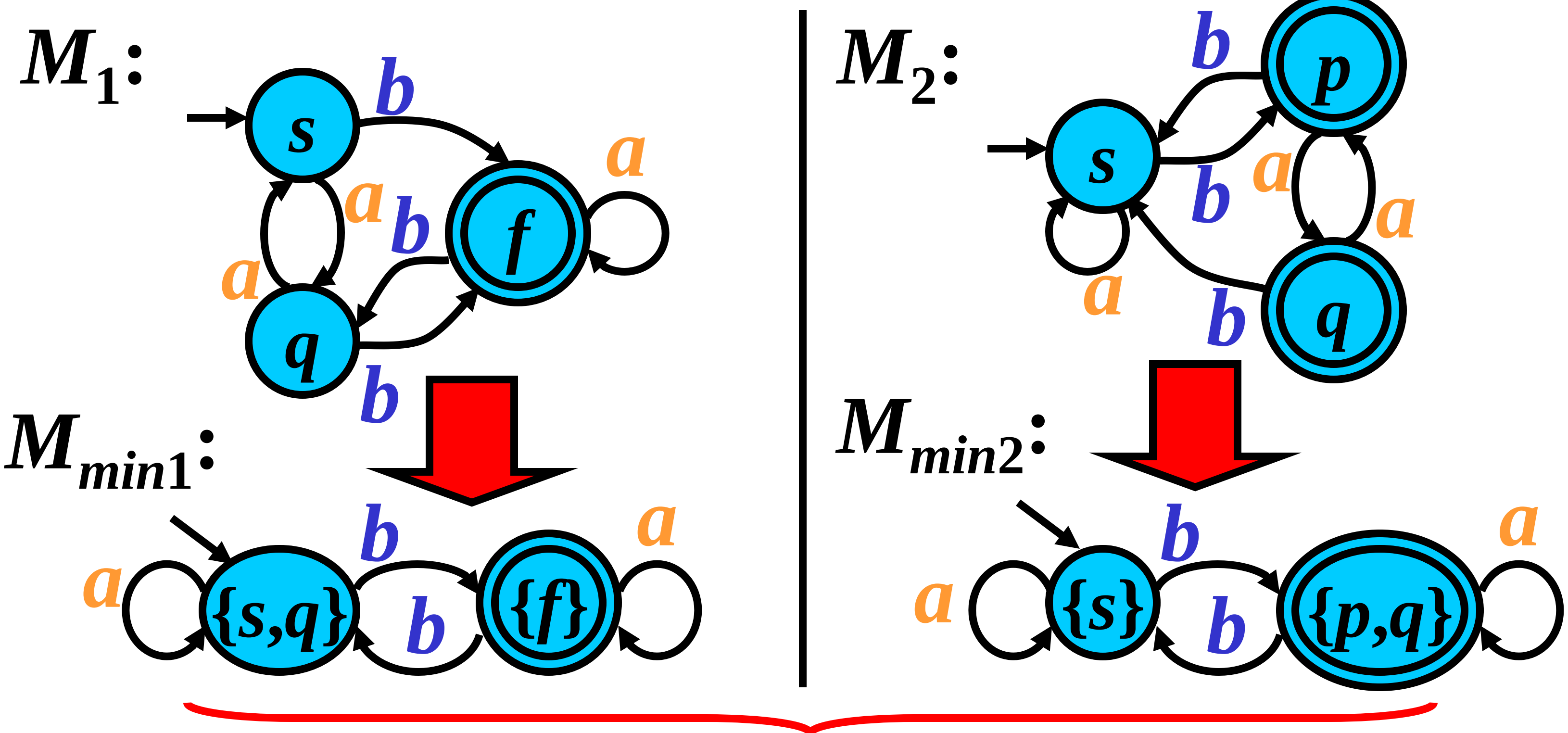
Question: $L(M_1) = L(M_2)$?



A minimum state FA

Equivalence Problem: Example

Question: $L(M_1) = L(M_2)$?



A minimum state FA

Answer: **YES** because M_{min1} coincides with M_{min2}