Models for Context-Free Languages

Context-Free Grammar (CFG)

Gist: A *grammar* is based on a finite set of grammatical rules, by which it generates strings of its language.

Illustration: **Start nonterminal** S

Grammar G:

Nonterminals: $A, B, \boxed{S}$

Terminals: a, b, c, d

Rules:

$S \rightarrow AB,$
$A \rightarrow aAb,$
$A \rightarrow ab,$
$B \rightarrow bBa,$
$B \rightarrow ba$

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$\Downarrow$

$\overbrace{AB}$

Rule: $S \to AB$

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Terminals:	a, b, c, d
Rules:	$S \to AB,$ $A \to aAb,$ $A \to ab,$ $B \to bBa,$ $B \to ba$

S

Rule: $S \to AB$

$\overbrace{AB}$

Rule: $A \to ab$

$\overbrace{ab}B$

Rule: $B \to bBa$

$ab\overbrace{bBa}$

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Rule: $B \rightarrow bBa$

$ab\overbrace{bBa}$

Rule: $B \rightarrow ba$

$abb\overbrace{ba}a \in L(G)$

Context-Free Grammar: Definition

Definition: *A context-free grammar* (CFG) is a quadruple $G = (N, T, P, S)$, where

- N is an alphabet of *nonterminals*
- T is an alphabet of *terminals*, $N \cap T = \emptyset$
- P is a finite set of *rules* of the form $A \to x$, where $A \in N$, $x \in (N \cup T)^*$
- $S \in N$ is the *start nonterminal*

Mathematical Note on Rules:

- Strictly mathematically, P is a relation from N to $(N \cup T)^*$
- Instead of $(A, x) \in P$, we write $A \to x \in P$

- $A \to x$ means that A can be replaced with x
- $A \to \varepsilon$ is called ***ε-rule***

Convention

- $A, \ldots, F, S$: nonterminals
- S : the start nonterminal
- $a, \ldots, d$: terminals
- $U, \ldots, Z$: members of $(N \cup T)$
- $u, \ldots, z$: members of $(N \cup T)^*$
- π : sequence of productions

A subset of rules of the form:

$$A \rightarrow x_1, A \rightarrow x_2, \ldots, A \rightarrow x_n$$

can be simply written as:

$$A \rightarrow x_1 \mid x_2 \mid \ldots \mid x_n$$

Derivation Step

Gist: A change of a string by a rule.

Definition: Let $G = (N, T, P, S)$ be a CFG. Let $\boldsymbol{u}, \boldsymbol{v} \in (N \cup T)^*$ and $\boldsymbol{p} = \boldsymbol{A} \rightarrow \boldsymbol{x} \in P$. Then, $\boldsymbol{uAv}$ *directly derives* $\boldsymbol{uxv}$ *according to* $\boldsymbol{p}$ in G, written as $\boldsymbol{uAv} \Rightarrow \boldsymbol{uxv}$ [$\boldsymbol{p}$] or, simply, $\boldsymbol{uAv} \Rightarrow \boldsymbol{uxv}$.

Note: If $\boldsymbol{uAv} \Rightarrow \boldsymbol{uxv}$ in G, we also say that G makes a *derivation step* from $\boldsymbol{uAv}$ to $\boldsymbol{uxv}$.

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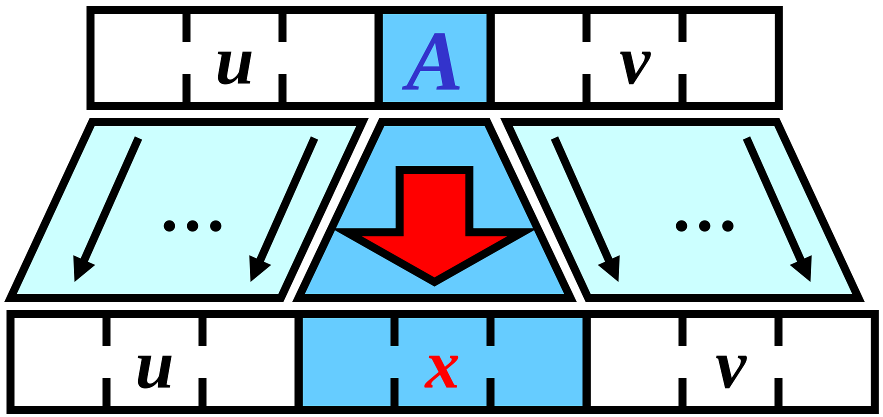
Rule: $\boldsymbol{A} \rightarrow \boldsymbol{x}$



Sequence of Derivation Steps 1/2

Gist: Several consecutive derivation steps.

Definition: Let $u \in (N \cup T)^*$. G makes a *zero-step derivation* from u to u; in symbols,

$$u \Rightarrow^0 u\ [\varepsilon] \text{ or, simply, } u \Rightarrow^0 u$$

Definition: Let $u_0,\ldots,u_n \in (N \cup T)^*$, $n \geq 1$, and $u_{i\text{-}1} \Rightarrow u_i\ [p_i]$, $p_i \in P$, for all $i = 1,\ldots, n$; that is

$$u_0 \Rightarrow u_1\ [p_1] \Rightarrow u_2\ [p_2] \ldots \Rightarrow u_n\ [p_n]$$

Then, G makes n *derivation steps* from u_0 to u_n,

$$u_0 \Rightarrow^n u_n\ [p_1 \ldots p_n] \text{ or, simply, } u_0 \Rightarrow^n u_n$$

Sequence of Derivation Steps 2/2

If $u_0 \Rightarrow^n u_n$ [π] for some $n \geq 1$, then u_0 *properly derives* u_n in G, written as $u_0 \Rightarrow^+ u_n$ [π].

If $u_0 \Rightarrow^n u_n$ [π] for some $n \geq 0$, then u_0 *derives* u_n in G, written as $u_0 \Rightarrow^* u_n$ [π].

Example: Consider

aAb $\Rightarrow$ ***aaBbb*** [**1**: ***A*** $\rightarrow$ ***aBb***], and

aaBbb $\Rightarrow$ ***aacbb*** [**2**: ***B*** $\rightarrow$ ***c***].

Then,

aAb $\Rightarrow^2$ ***aacbb*** [**1 2**],

aAb $\Rightarrow^+$ ***aacbb*** [**1 2**],

aAb $\Rightarrow^*$ ***aacbb*** [**1 2**]

Generated Language

Gist: ***G generates* a terminal string *w* by a sequence of derivation steps from *S* to *w***

Definition: Let $G = (N, T, P, S)$ be a CFG. The *language generated by G*, $L(G)$, is defined as

$$L(G) = \{w: w \in T^*, S \Rightarrow^* w\}$$

Illustration:

$G = (N, T, P, S)$, let $w = a_1a_2\ldots a_n$; $a_i \in T$ for $i = 1..n$

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$G = (N, T, P, S)$, let $w = a_1a_2\ldots a_n$; $a_i \in T$ for $i = 1..n$

if $S \Rightarrow \ldots \Rightarrow \ldots \Rightarrow \underbrace{a_1a_2\ldots a_n}_{w}$ **then** $w \in L(G)$;

otherwise, $w \notin L(G)$

Context-Free Language (CFL)

Gist: A language generated by a CFG.

Definition: Let L be a language. L is a *context-free language* (CFL) if there exists a context-free grammar that generates L.

Example:

$G = (N, T, P, \boldsymbol{S})$, where $N = \{\boldsymbol{S}\}$, $T = \{\boldsymbol{a}, \boldsymbol{b}\}$,

$P = \{1\text{: } \boldsymbol{S} \to \boldsymbol{aSb}, 2\text{: } \boldsymbol{S} \to \boldsymbol{\varepsilon}\}$

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$S \Rightarrow \varepsilon$ [2] $\to$ $L(G)$

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$S \Rightarrow \varepsilon$ [2]

$S \Rightarrow aSb$ [1] $\Rightarrow ab$ [2]

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$L(G)$

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$\vdots$

$L(G) = \{a^n b^n : n \geq 0\}$

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$S \Rightarrow \varepsilon \quad [2]$

$S \Rightarrow aSb\ [1] \Rightarrow ab \quad [2]$

$S \Rightarrow aSb\ [1] \Rightarrow aaSbb\ [1] \Rightarrow aabb\ [2]$

$\vdots$

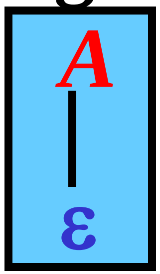
$L(G) = \{a^n b^n : n \geq 0\}$

$L = \{a^n b^n : n \geq 0\}$ is a CFL.

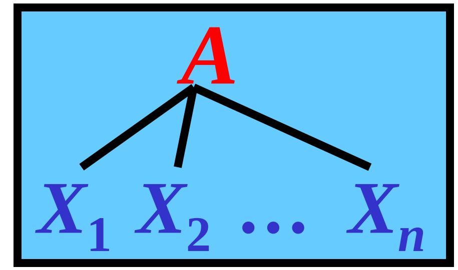
Rule Tree

- **Rule tree graphically represents a rule**

1) $A \rightarrow \varepsilon$:

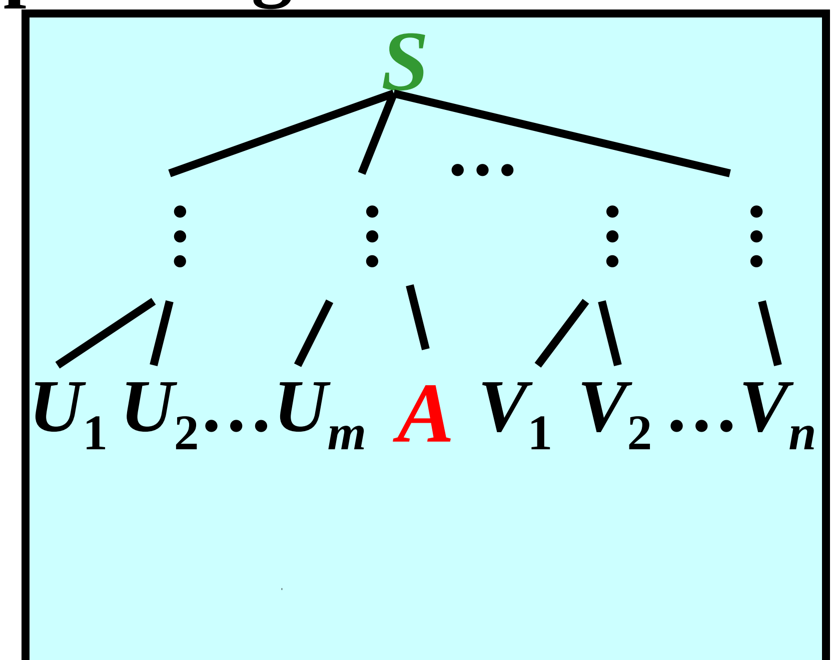


2) $A \rightarrow X_1X_2\ldots X_n$:



- **Derivation tree corresponding to a derivation**

$$S \Rightarrow \ldots$$
$$\vdots$$
$$\Rightarrow U_1U_2\ldots U_m A V_1V_2\ldots V_n$$



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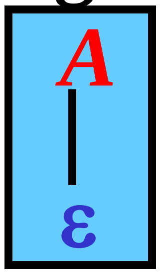

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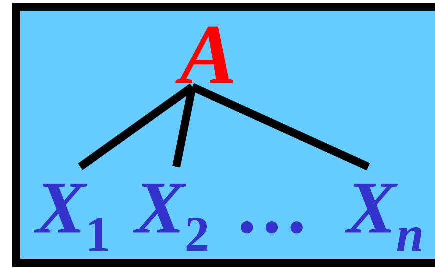


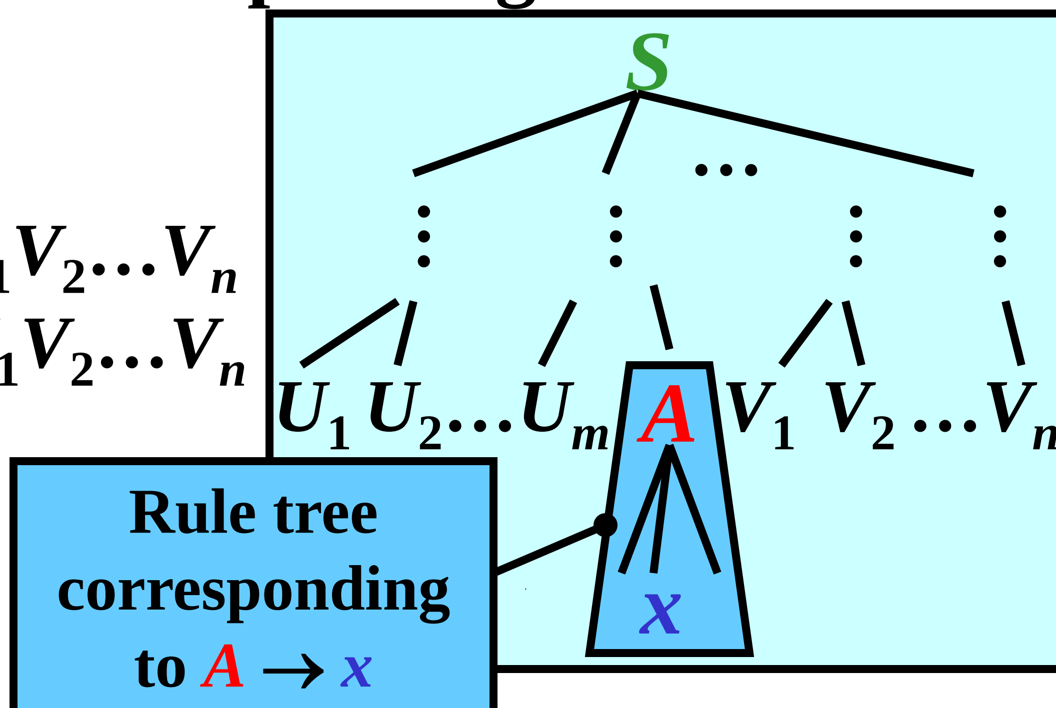
- **Derivation tree corresponding to a derivation**

$S \Rightarrow \ldots$

$\vdots$

$\Rightarrow U_1U_2\ldots U_m A V_1V_2\ldots V_n$

$\Rightarrow U_1U_2\ldots U_m x V_1V_2\ldots V_n$



Derivation Tree: Example

$G = (N, T, P, E)$, where $N = \{E, F, T\}$, $T = \{i, +, *, (,)\}$,

$P = \{$ 1: $E \rightarrow E+T$, 2: $E \rightarrow T$, 3: $T \rightarrow T*F$,
4: $T \rightarrow F$, 5: $F \rightarrow (E)$, 6: $F \rightarrow i$ $\}$

Derivation: **Derivation tree:**

Derivation Tree: Example

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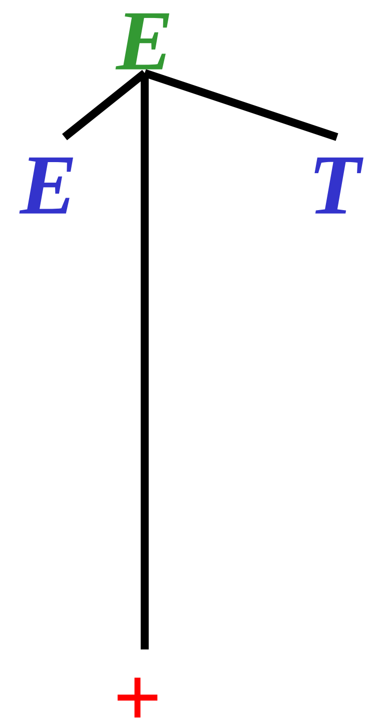
Derivation:

$\underline{E} \Rightarrow E + \underline{T}$ [1]

Derivation tree:



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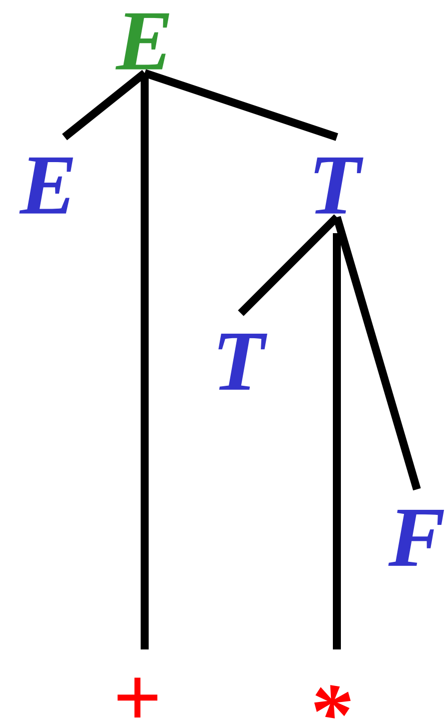
Derivation:

$\underline{E} \Rightarrow E + \underline{T}$ [1]

$\Rightarrow E + \underline{T} * F$ [3]

Derivation tree:



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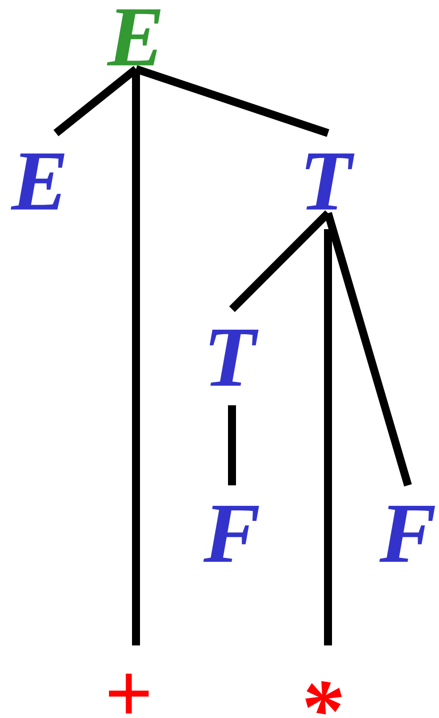
Derivation tree:



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Derivation:

$\underline{E} \Rightarrow E + \underline{T}$ [1]
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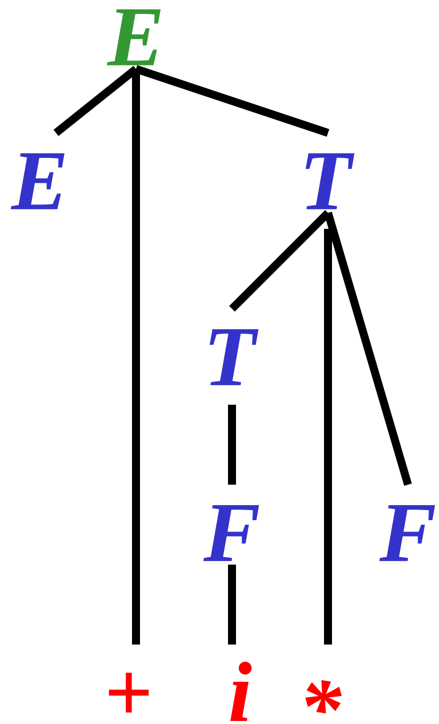
$\Rightarrow E + \underline{F} * F$ [4]
$\Rightarrow \underline{E} + i * F$ [6]

Derivation tree:



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Derivation:

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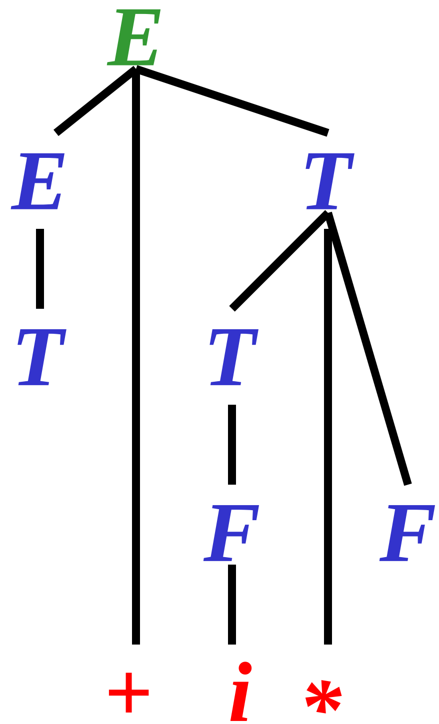
$\Rightarrow E + \underline{F} * F$ [4]
$\Rightarrow \underline{E} + i * F$ [6]
$\Rightarrow T + i * \underline{F}$ [2]

Derivation tree:



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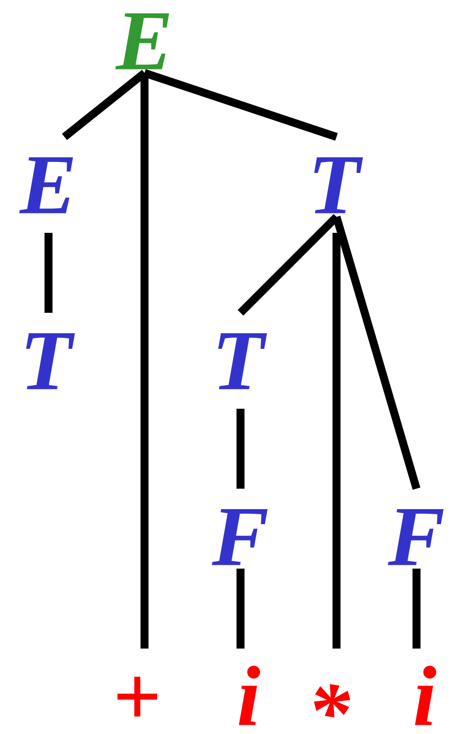

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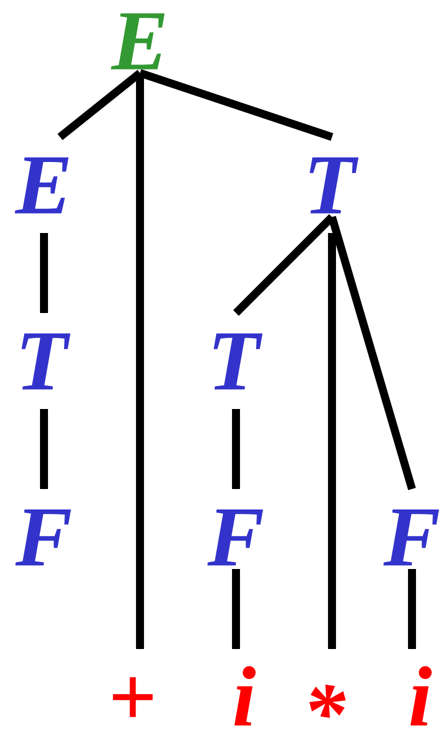
Derivation:

$$
\begin{aligned}
\underline{E} &\Rightarrow E + \underline{T} && [1] \\
&\Rightarrow E + \underline{T} * F && [3] \\
&\Rightarrow E + \underline{F} * F && [4] \\
&\Rightarrow \underline{E} + i * F && [6] \\
&\Rightarrow T + i * \underline{F} && [2] \\
&\Rightarrow \underline{T} + i * i && [6] \\
&\Rightarrow \underline{F} + i * i && [4]
\end{aligned}
$$

Derivation tree:



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Derivation:

$\underline{E} \Rightarrow E + \underline{T}$ [1]
$\Rightarrow E + \underline{T} * F$ [3]
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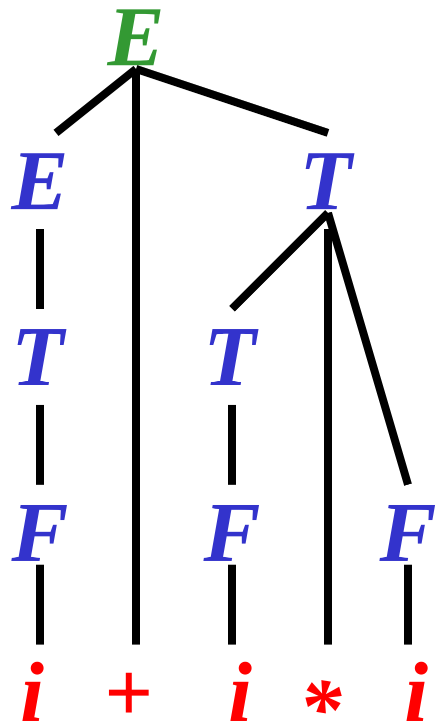

Leftmost Derivation

Gist: During a *leftmost derivation step*, the leftmost nonterminal is rewritten.

Definition: Let $G = (N, T, P, S)$ be a CFG, let $\boldsymbol{u} \in T^*$, $\boldsymbol{v} \in (N \cup T)^*$. Let $\boldsymbol{p} = \boldsymbol{A} \rightarrow \boldsymbol{x} \in P$ be a rule. Then, $\boldsymbol{uAv}$ directly derives $\boldsymbol{uxv}$ in the *leftmost way* according to $\boldsymbol{p}$ in G, written as

$$\boldsymbol{uAv} \Rightarrow_{lm} \boldsymbol{uxv}\ [\boldsymbol{p}]$$

Note: We define $\Rightarrow_{lm}^+$ and $\Rightarrow_{lm}^*$ by analogy with $\Rightarrow^+$ and $\Rightarrow^*$, respectively.

Leftmost Derivation: Example

$G = (N, T, P, \boldsymbol{E})$, where $N = \{\boldsymbol{E}, \boldsymbol{F}, \boldsymbol{T}\}$, $T = \{\boldsymbol{i}, \mathbf{+}, \mathbf{*}, \mathbf{(}, \mathbf{)}\}$,

$P = \{$ **1**: $\boldsymbol{E} \rightarrow \boldsymbol{E}\mathbf{+}\boldsymbol{T}$, **2**: $\boldsymbol{E} \rightarrow \boldsymbol{T}$, **3**: $\boldsymbol{T} \rightarrow \boldsymbol{T}\mathbf{*}\boldsymbol{F}$,
4: $\boldsymbol{T} \rightarrow \boldsymbol{F}$, **5**: $\boldsymbol{F} \rightarrow \mathbf{(}\boldsymbol{E}\mathbf{)}$, **6**: $\boldsymbol{F} \rightarrow \boldsymbol{i}$ $\}$

Leftmost derivation:

Derivation tree:

Leftmost Derivation: Example

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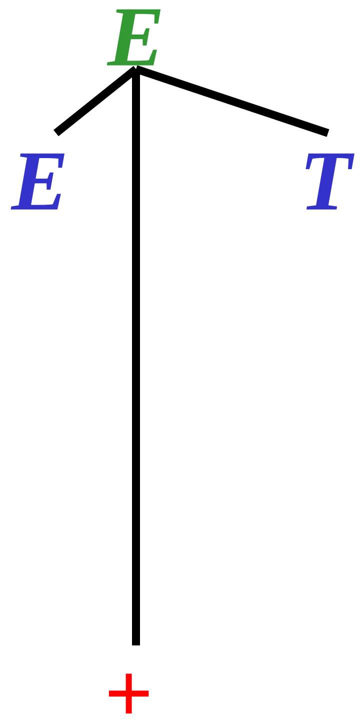
Leftmost derivation:

$\underline{E} \Rightarrow_{lm} \underline{E} + T$ [1]

Derivation tree:



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Leftmost derivation:

$\underline{E} \Rightarrow_{lm} \underline{E} + T$ [1]

$\Rightarrow_{lm} \underline{T} + T$ [2]

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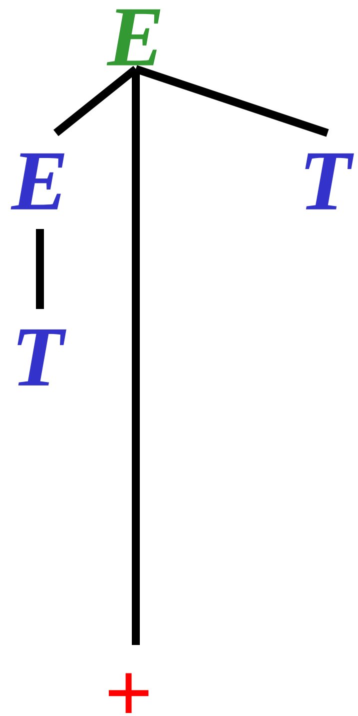

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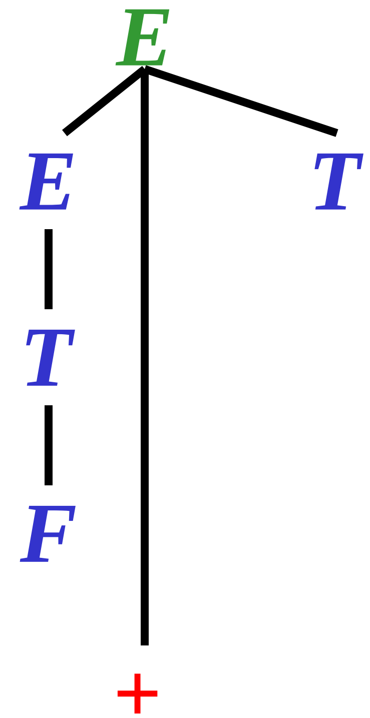
Leftmost derivation:

$\underline{E} \Rightarrow_{lm} \underline{E} + T$ [1]

$\Rightarrow_{lm} \underline{T} + T$ [2]

$\Rightarrow_{lm} \underline{F} + T$ [4]

Derivation tree:



Leftmost Derivation: Example

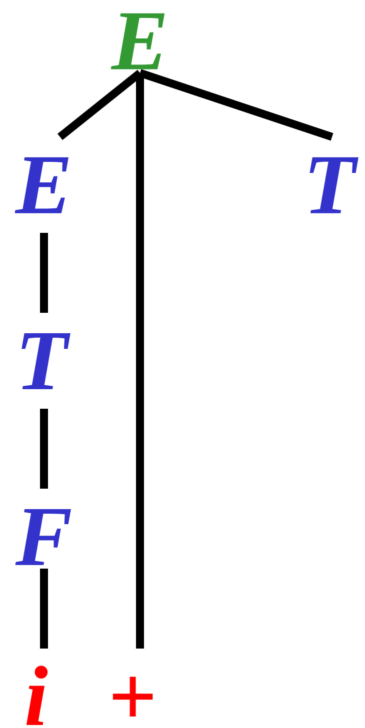
$G = (N, T, P, \mathbf{E})$, where $N = \{\mathbf{E}, \mathbf{F}, \mathbf{T}\}$, $T = \{\mathbf{i}, \mathbf{+}, \mathbf{*}, \mathbf{(}, \mathbf{)}\}$,

$P = \{$ 1: $\mathbf{E} \rightarrow \mathbf{E+T}$, 2: $\mathbf{E} \rightarrow \mathbf{T}$, 3: $\mathbf{T} \rightarrow \mathbf{T*F}$,
4: $\mathbf{T} \rightarrow \mathbf{F}$, 5: $\mathbf{F} \rightarrow \mathbf{(E)}$, 6: $\mathbf{F} \rightarrow \mathbf{i}$ $\}$

Leftmost derivation:

$$\underline{\mathbf{E}} \Rightarrow_{lm} \underline{\mathbf{E}} + \mathbf{T} \qquad [1]$$
$$\Rightarrow_{lm} \underline{\mathbf{T}} + \mathbf{T} \qquad [2]$$
$$\Rightarrow_{lm} \underline{\mathbf{F}} + \mathbf{T} \qquad [4]$$
$$\Rightarrow_{lm} \mathbf{i} + \underline{\mathbf{T}} \qquad [6]$$

Derivation tree:



Leftmost Derivation: Example

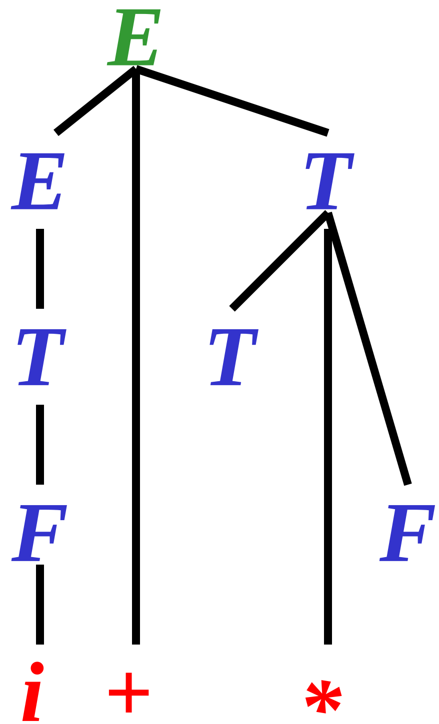
$G = (N, T, P, E)$, where $N = \{E, F, T\}$, $T = \{i, +, *, (,)\}$,

$P = \{$ 1: $E \rightarrow E+T$, 2: $E \rightarrow T$, 3: $T \rightarrow T*F$,
4: $T \rightarrow F$, 5: $F \rightarrow (E)$, 6: $F \rightarrow i$ $\}$

Leftmost derivation:

$$\begin{aligned} \underline{E} &\Rightarrow_{lm} \underline{E} + T && [1] \\ &\Rightarrow_{lm} \underline{T} + T && [2] \\ &\Rightarrow_{lm} \underline{F} + T && [4] \\ &\Rightarrow_{lm} i + \underline{T} && [6] \\ &\Rightarrow_{lm} i + \underline{T} * F && [3] \end{aligned}$$

Derivation tree:



Leftmost Derivation: Example

$G = (N, T, P, E)$, where $N = \{E, F, T\}$, $T = \{i, +, *, (,)\}$,

$P = \{$ 1: $E \rightarrow E+T$, 2: $E \rightarrow T$, 3: $T \rightarrow T*F$,
4: $T \rightarrow F$, 5: $F \rightarrow (E)$, 6: $F \rightarrow i$ $\}$

Leftmost derivation:

$\underline{E} \Rightarrow_{lm} \underline{E} + T$ [1]

$\Rightarrow_{lm} \underline{T} + T$ [2]

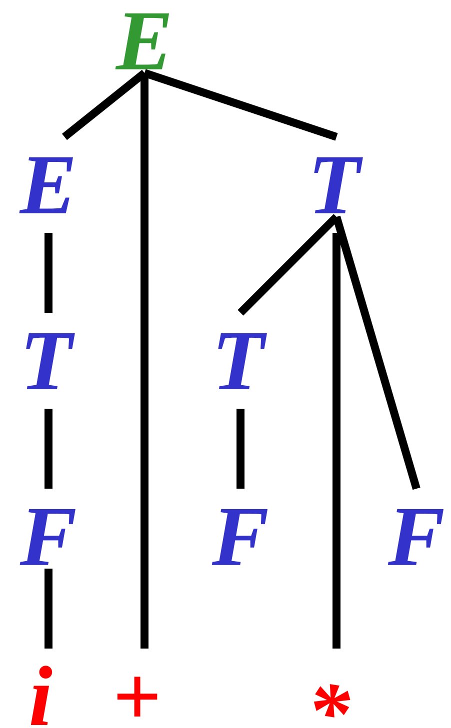
$\Rightarrow_{lm} \underline{F} + T$ [4]

$\Rightarrow_{lm} i + \underline{T}$ [6]

$\Rightarrow_{lm} i + \underline{T} * F$ [3]

$\Rightarrow_{lm} i + \underline{F} * F$ [4]

Derivation tree:



Leftmost Derivation: Example

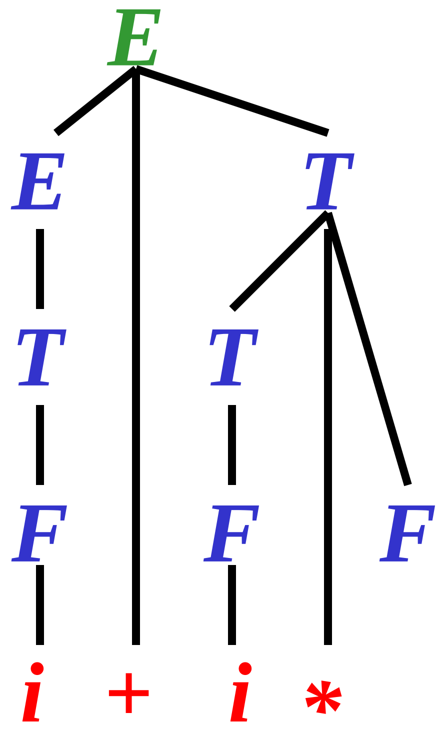
$G = (N, T, P, E)$, where $N = \{E, F, T\}$, $T = \{i, +, *, (,)\}$,

$P = \{$ 1: $E \rightarrow E+T$, 2: $E \rightarrow T$, 3: $T \rightarrow T*F$,
4: $T \rightarrow F$, 5: $F \rightarrow (E)$, 6: $F \rightarrow i$ $\}$

Leftmost derivation:

$$\underline{E} \Rightarrow_{lm} \underline{E} + T \quad [1]$$
$$\Rightarrow_{lm} \underline{T} + T \quad [2]$$
$$\Rightarrow_{lm} \underline{F} + T \quad [4]$$
$$\Rightarrow_{lm} i + \underline{T} \quad [6]$$
$$\Rightarrow_{lm} i + \underline{T} * F \quad [3]$$
$$\Rightarrow_{lm} i + \underline{F} * F \quad [4]$$
$$\Rightarrow_{lm} i + i * \underline{F} \quad [6]$$

Derivation tree:



Leftmost Derivation: Example

$G = (N, T, P, E)$, where $N = \{E, F, T\}$, $T = \{i, +, *, (,)\}$,

$P = \{$ 1: $E \rightarrow E+T$, 2: $E \rightarrow T$, 3: $T \rightarrow T*F$, 4: $T \rightarrow F$, 5: $F \rightarrow (E)$, 6: $F \rightarrow i$ $\}$

Leftmost derivation:

$\underline{E} \Rightarrow_{lm} \underline{E} + T$ [1]

$\Rightarrow_{lm} \underline{T} + T$ [2]

$\Rightarrow_{lm} \underline{F} + T$ [4]

$\Rightarrow_{lm} i + \underline{T}$ [6]

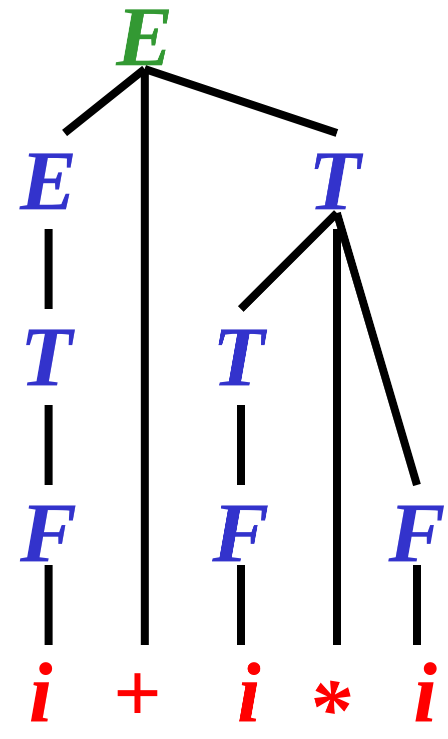
$\Rightarrow_{lm} i + \underline{T} * F$ [3]

$\Rightarrow_{lm} i + \underline{F} * F$ [4]

$\Rightarrow_{lm} i + i * \underline{F}$ [6]

$\Rightarrow_{lm} i + i * i$ [6]

Derivation tree:



Rightmost Derivation

Gist: During a *rightmost derivation step*, the rightmost nonterminal is rewritten.

Definition: Let $G = (N, T, P, S)$ be a CFG, let $\boldsymbol{u} \in (N \cup T)^*$, $\boldsymbol{v} \in T^*$. Let $\boldsymbol{p} = \boldsymbol{A} \rightarrow \boldsymbol{x} \in P$ be a rule. Then, $\boldsymbol{uAv}$ directly derives $\boldsymbol{uxv}$ in the *rightmost way* according to $\boldsymbol{p}$ in G, written as

$$\boldsymbol{uAv} \Rightarrow_{rm} \boldsymbol{uxv}\ [\boldsymbol{p}]$$

Note: We define $\Rightarrow_{rm}{}^+$ and $\Rightarrow_{rm}{}^*$ by analogy with $\Rightarrow^+$ and $\Rightarrow^*$, respectively.

Rightmost Derivation: Example

$G = (N, T, P, E)$, where $N = \{E, F, T\}$, $T = \{i, +, *, (,)\}$,

$P = \{$ 1: $E \rightarrow E+T$, 2: $E \rightarrow T$, 3: $T \rightarrow T*F$,
4: $T \rightarrow F$, 5: $F \rightarrow (E)$, 6: $F \rightarrow i$ $\}$

Rightmost derivation: **Derivation tree:**

Rightmost Derivation: Example

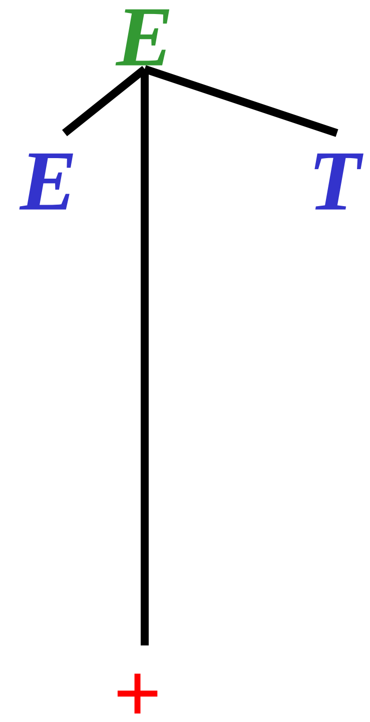
$G = (N, T, P, E)$, where $N = \{E, F, T\}$, $T = \{i, +, *, (,)\}$,

$P = \{$ 1: $E \rightarrow E+T$, 2: $E \rightarrow T$, 3: $T \rightarrow T*F$,
4: $T \rightarrow F$, 5: $F \rightarrow (E)$, 6: $F \rightarrow i$ $\}$

Rightmost derivation:

$\underline{E} \Rightarrow_{rm} E + \underline{T}$ [1]

Derivation tree:



Rightmost Derivation: Example

$G = (N, T, P, E)$, where $N = \{E, F, T\}$, $T = \{i, +, *, (,)\}$,

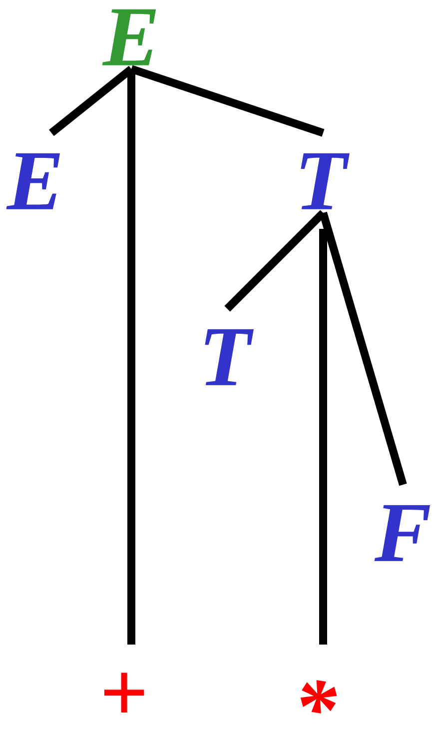
$P = \{$ 1: $E \to E+T$, 2: $E \to T$, 3: $T \to T*F$,
4: $T \to F$, 5: $F \to (E)$, 6: $F \to i$ $\}$

Rightmost derivation:

$\underline{E} \Rightarrow_{rm} E + \underline{T}$ [1]

$\Rightarrow_{rm} E + T * \underline{F}$ [3]

Derivation tree:



Rightmost Derivation: Example

$G = (N, T, P, E)$, where $N = \{E, F, T\}$, $T = \{i, +, *, (,)\}$,

$P = \{$ 1: $E \rightarrow E+T$, 2: $E \rightarrow T$, 3: $T \rightarrow T*F$,
4: $T \rightarrow F$, 5: $F \rightarrow (E)$, 6: $F \rightarrow i$ $\}$

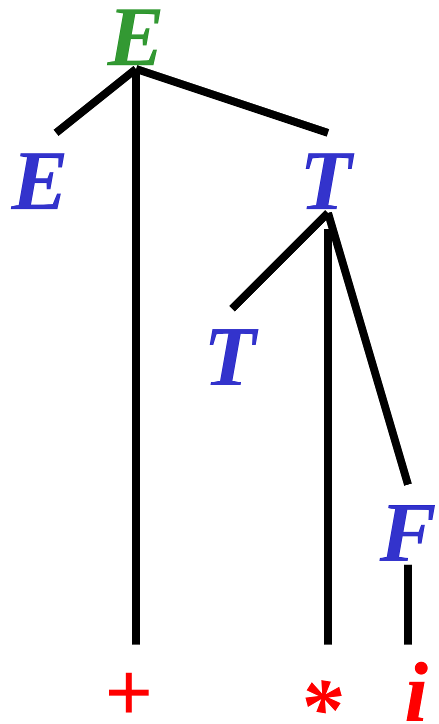
Rightmost derivation:

$\underline{E} \Rightarrow_{rm} E + \underline{T}$ [1]

$\Rightarrow_{rm} E + T * \underline{F}$ [3]

$\Rightarrow_{rm} E + \underline{T} * i$ [6]

Derivation tree:



Rightmost Derivation: Example

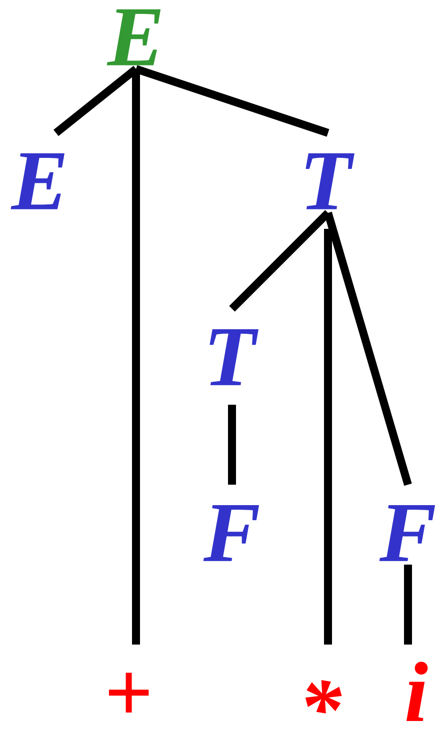
$G = (N, T, P, E)$, where $N = \{E, F, T\}$, $T = \{i, +, *, (,)\}$,

$P = \{$ 1: $E \rightarrow E+T$, 2: $E \rightarrow T$, 3: $T \rightarrow T*F$,
4: $T \rightarrow F$, 5: $F \rightarrow (E)$, 6: $F \rightarrow i$ $\}$

Rightmost derivation:

$\underline{E} \Rightarrow_{rm} E + \underline{T}$ [1]
$\Rightarrow_{rm} E + T * \underline{F}$ [3]
$\Rightarrow_{rm} E + \underline{T} * i$ [6]
$\Rightarrow_{rm} E + \underline{F} * i$ [4]

Derivation tree:



Rightmost Derivation: Example

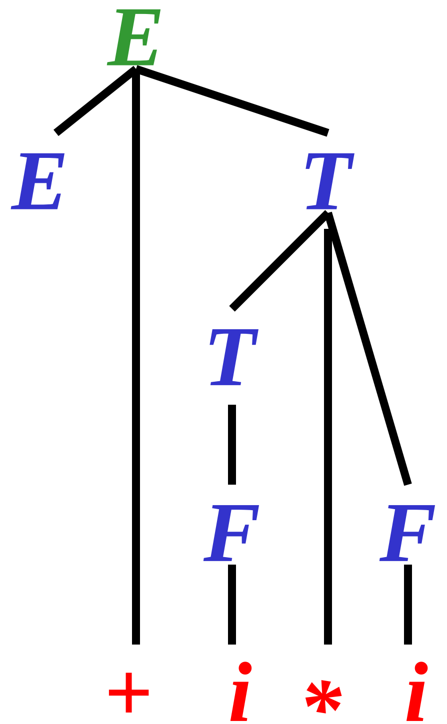
$G = (N, T, P, E)$, where $N = \{E, F, T\}$, $T = \{i, +, *, (,)\}$,

$P = \{$ 1: $E \rightarrow E+T$, 2: $E \rightarrow T$, 3: $T \rightarrow T*F$,
4: $T \rightarrow F$, 5: $F \rightarrow (E)$, 6: $F \rightarrow i$ $\}$

Rightmost derivation:

$$\underline{E} \Rightarrow_{rm} E + \underline{T} \quad [1]$$
$$\Rightarrow_{rm} E + T * \underline{F} \quad [3]$$
$$\Rightarrow_{rm} E + \underline{T} * i \quad [6]$$
$$\Rightarrow_{rm} E + \underline{F} * i \quad [4]$$
$$\Rightarrow_{rm} \underline{E} + i * i \quad [6]$$

Derivation tree:



Rightmost Derivation: Example

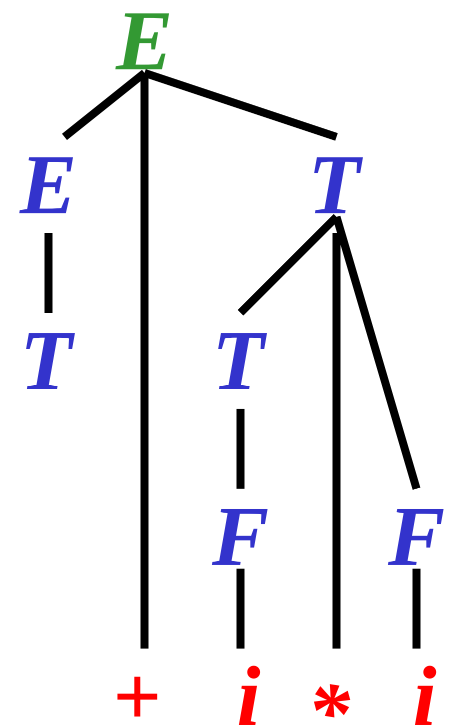
$G = (N, T, P, E)$, where $N = \{E, F, T\}$, $T = \{i, +, *, (,)\}$,

$P = \{$ 1: $E \to E+T$, 2: $E \to T$, 3: $T \to T*F$,
4: $T \to F$, 5: $F \to (E)$, 6: $F \to i$ $\}$

Rightmost derivation:

$$
\begin{aligned}
\underline{E} &\Rightarrow_{rm} E + \underline{T} && [1] \\
&\Rightarrow_{rm} E + T * \underline{F} && [3] \\
&\Rightarrow_{rm} E + \underline{T} * i && [6] \\
&\Rightarrow_{rm} E + \underline{F} * i && [4] \\
&\Rightarrow_{rm} \underline{E} + i * i && [6] \\
&\Rightarrow_{rm} \underline{T} + i * i && [2]
\end{aligned}
$$

Derivation tree:



Rightmost Derivation: Example

$G = (N, T, P, E)$, where $N = \{E, F, T\}$, $T = \{i, +, *, (,)\}$,

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Rightmost derivation:

$\underline{E} \Rightarrow_{rm} E + \underline{T}$ [1]

$\Rightarrow_{rm} E + T * \underline{F}$ [3]

$\Rightarrow_{rm} E + \underline{T} * i$ [6]

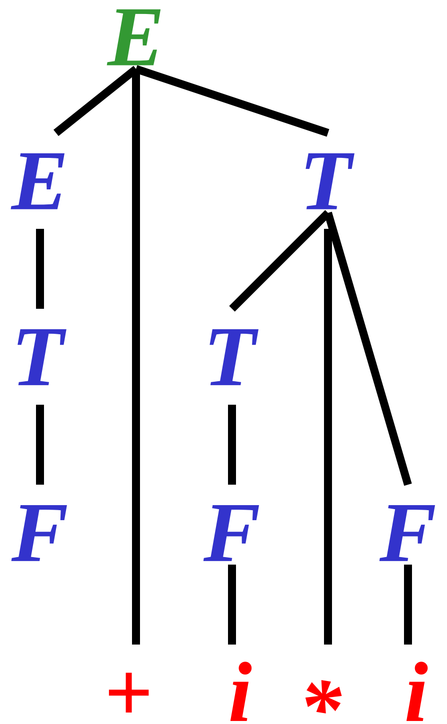
$\Rightarrow_{rm} E + \underline{F} * i$ [4]

$\Rightarrow_{rm} \underline{E} + i * i$ [6]

$\Rightarrow_{rm} \underline{T} + i * i$ [2]

$\Rightarrow_{rm} \underline{F} + i * i$ [4]

Derivation tree:



Rightmost Derivation: Example

$G = (N, T, P, E)$, where $N = \{E, F, T\}$, $T = \{i, +, *, (,)\}$,

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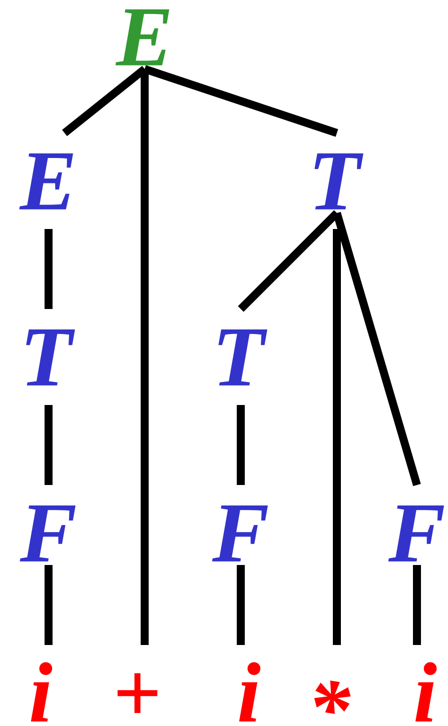
4: $T \rightarrow F$, 5: $F \rightarrow (E)$, 6: $F \rightarrow i$ $\}$

Rightmost derivation:

$$\underline{E} \Rightarrow_{rm} E + \underline{T} \quad [1]$$
$$\Rightarrow_{rm} E + T * \underline{F} \quad [3]$$
$$\Rightarrow_{rm} E + \underline{T} * i \quad [6]$$
$$\Rightarrow_{rm} E + \underline{F} * i \quad [4]$$
$$\Rightarrow_{rm} \underline{E} + i * i \quad [6]$$
$$\Rightarrow_{rm} \underline{T} + i * i \quad [2]$$
$$\Rightarrow_{rm} \underline{F} + i * i \quad [4]$$
$$\Rightarrow_{rm} i + i * i \quad [6]$$

Derivation tree:



Derivations: Summary

- Let $\boldsymbol{A} \rightarrow \boldsymbol{x} \in P$ be a rule.

1) Derivation:

Let $\boldsymbol{u}, \boldsymbol{v} \in (N \cup T)^*$: $\boldsymbol{uAv} \Rightarrow \boldsymbol{uxv}$

Note: Any nonterminal is rewritten

2) Leftmost derivation:

Let $\boldsymbol{u} \in T^*, \boldsymbol{v} \in (N \cup T)^*$: $\boldsymbol{uAv} \Rightarrow_{\mathbf{lm}} \boldsymbol{uxv}$

Note: Leftmost nonterminal is rewritten

3) Rightmost derivation:

Let $\boldsymbol{u} \in (N \cup T)^*, \boldsymbol{v} \in T^*$: $\boldsymbol{uAv} \Rightarrow_{\mathbf{rm}} \boldsymbol{uxv}$

Note: Rightmost nonterminal is rewritten

Reduction of the Number of Derivations

Gist: Without any loss of generality, we can consider only leftmost or rightmost derivations.

Theorem: Let $G = (N, T, P, S)$ be a CFG. The next three languages coincide

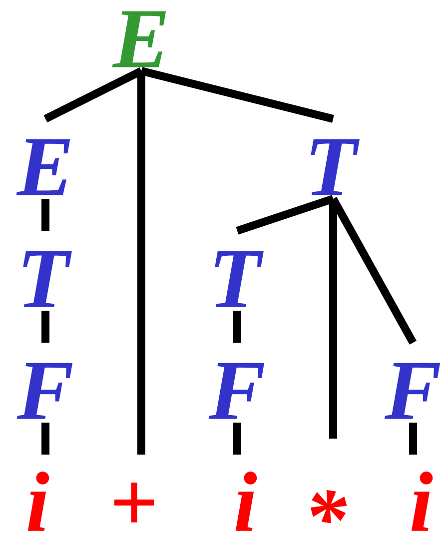
(1) $\{w: w \in T^*, S \Rightarrow_{lm}^* w\}$

(2) $\{w: w \in T^*, S \Rightarrow_{rm}^* w\}$

(3) $\{w: w \in T^*, S \Rightarrow^* w\} = L(G)$

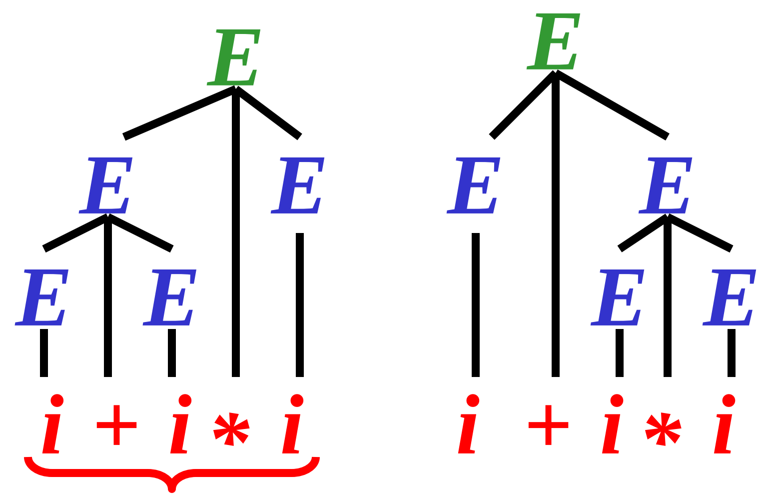
Introduction to Ambiguity

$G_{expr1} = (N, T, P, E)$, where
$N = \{E, F, T\}$, $T = \{i, +, *, (,)\}$,
$P = \{$ 1: $E \to E+T$, 2: $E \to T$,
3: $T \to T*F$, 4: $T \to F$,
5: $F \to (E)$, 6: $F \to i$ $\}$



Theory: ☹ × Practice: ☺

$G_{expr2} = (N, T, P, E)$, where
$N = \{E\}$, $T = \{i, +, *, (,)\}$,
$P = \{$ 1: $E \to E+E$, 2: $E \to E*E$,
3: $E \to (E)$, 4: $E \to i$ $\}$



Theory: ☺ × Practice: ☹

Improper during compilation

Note: $L(G_{expr1}) = L(G_{expr2})$

Grammatical Ambiguity

Definition: Let $G = (N, T, P, S)$ be a CFG. If there exists $x \in L(G)$ with more than one derivation tree, then G is *ambiguous*; otherwise, G is *unambiguous*.

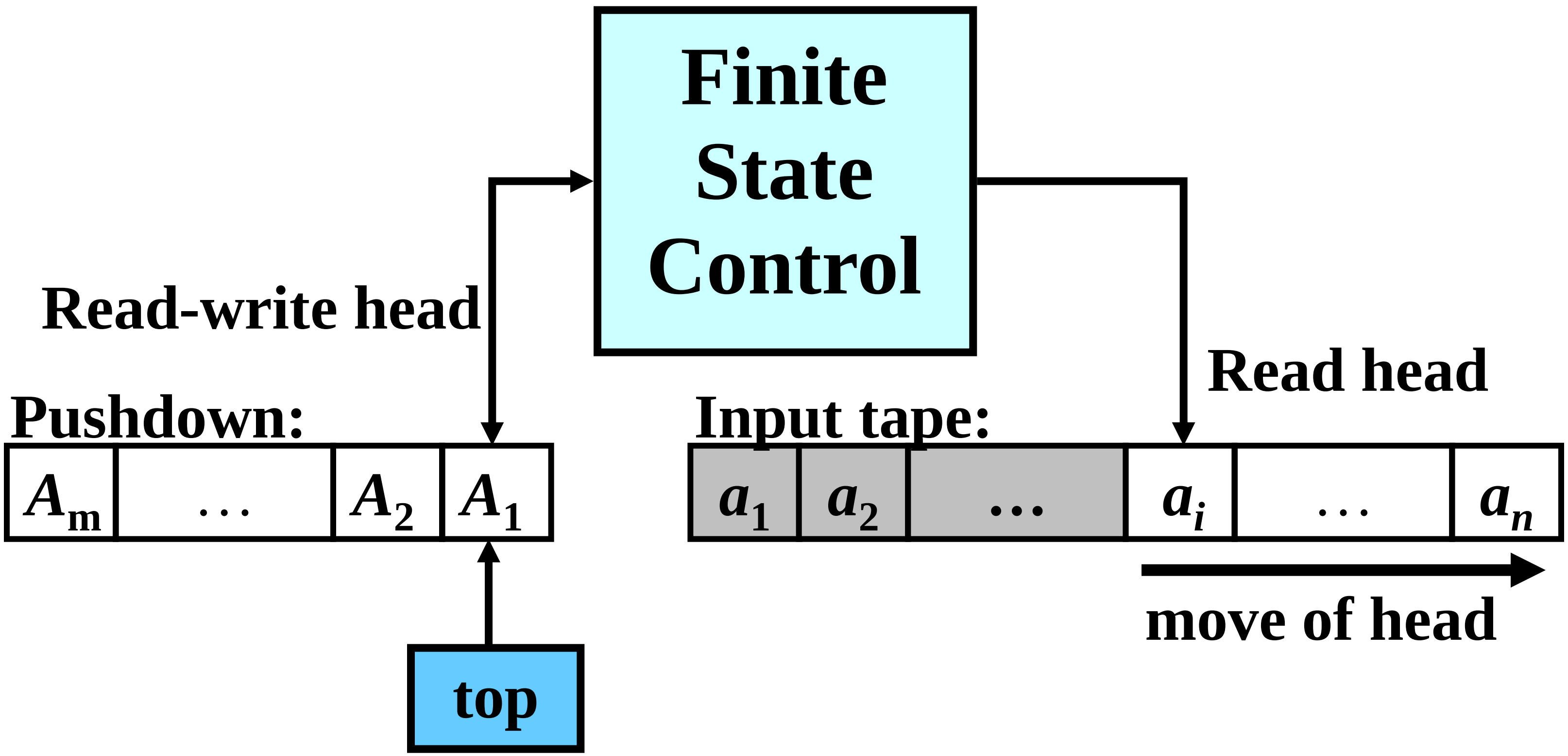
Definition: A CFL, L, is *inherently ambiguous* if L is generated by no unambiguous grammar.

Example:

- G_{expr1} is **unambiguous**, because for every $x \in L(G_{expr1})$ there exists **only one derivation tree**
- G_{expr2} is **ambiguous**, because for $i+i*i \in L(G_{expr2})$ there exist **two derivation trees**
- $L_{expr} = L(G_{expr1}) = L(G_{expr2})$ is **not inherently ambiguous** because G_{expr1} **is unambiguous**

Pushdown Automata (PDA)

Gist: An FA extended by a pushdown store.



Pushdown Automata: Definition

Definition: *A pushdown automaton* (PDA) is a 7-tuple $M = (Q, \Sigma, \Gamma, R, s, S, F)$, where

- Q is a *finite set of states*
- Σ is an *input alphabet*
- Γ is a *pushdown alphabet*
- R is a *finite set of rules* of the form: $Apa \rightarrow wq$ where $A \in \Gamma$, $p, q \in Q$, $a \in \Sigma \cup \{\varepsilon\}$, $w \in \Gamma^*$
- $s \in Q$ is the *start state*
- $S \in \Gamma$ is the *start pushdown symbol*
- $F \subseteq Q$ is a set of *final states*

Notes on PDA Rules

Mathematical note on rules:

- Strictly mathematically, R is a finite relation from $\Gamma \times Q \times (\Sigma \cup \{\varepsilon\})$ to $\Gamma^{*} \times Q$
- Instead of $(\boldsymbol{Apa}, \boldsymbol{wq}) \in R$, however, we write $\boldsymbol{Apa} \rightarrow \boldsymbol{wq} \in R$

Notes on PDA Rules

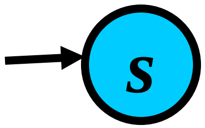
Mathematical note on rules:

- Strictly mathematically, R is a finite relation from $\Gamma \times Q \times (\Sigma \cup \{\varepsilon\})$ to $\Gamma^* \times Q$
- Instead of $(\boldsymbol{Apa}, \boldsymbol{wq}) \in R$, however, we write $\boldsymbol{Apa} \to \boldsymbol{wq} \in R$

- **Interpretation of** $\boldsymbol{Apa} \to \boldsymbol{wq}$: if the current state is $\boldsymbol{p}$, current input symbol is $\boldsymbol{a}$, and the topmost symbol on the pushdown is $\boldsymbol{A}$, then M can read $\boldsymbol{a}$, replace $\boldsymbol{A}$ with $\boldsymbol{w}$ and change state $\boldsymbol{p}$ to $\boldsymbol{q}$.
- **Note**: if $\boldsymbol{a} = \varepsilon$, no symbol is read

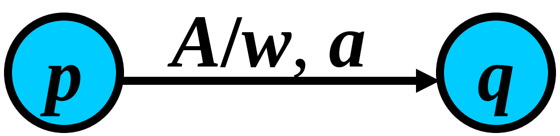
Graphical Representation

represents $q \in Q$



represents the initial state $s \in Q$

represents a final state $f \in F$



denotes $Apa \rightarrow wq \in R$

Graphical Representation: Example

$M = (Q, \Sigma, \Gamma, R, s, S, F)$

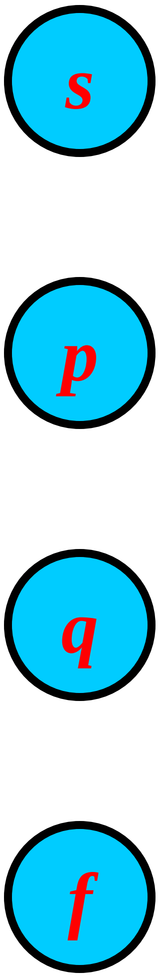
where:

Graphical Representation: Example

$M = (Q, \Sigma, \Gamma, R, s, S, F)$

where:

- $Q = \{\boldsymbol{s}, \boldsymbol{p}, \boldsymbol{q}, \boldsymbol{f}\}$;

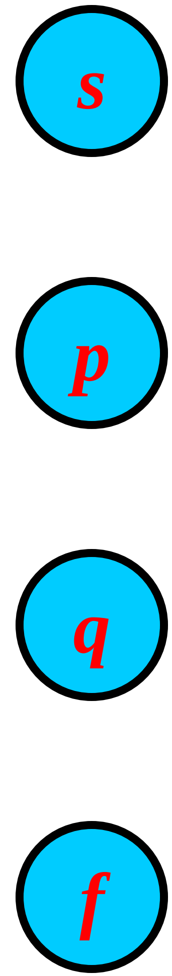


Graphical Representation: Example

$M = (Q, \Sigma, \Gamma, R, s, S, F)$

where:

- $Q = \{\mathbf{s}, \mathbf{p}, \mathbf{q}, \mathbf{f}\}$;
- $\Sigma = \{\mathbf{a}, \mathbf{b}\}$;



Graphical Representation: Example

$M = (Q, \Sigma, \Gamma, R, s, S, F)$

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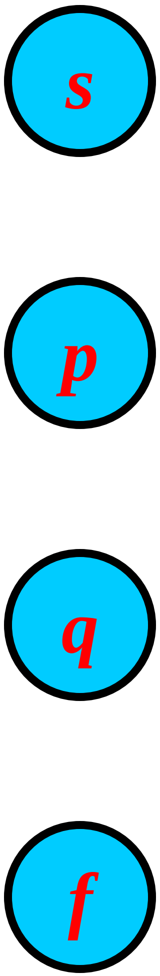
- $\Sigma = \{\boldsymbol{a}, \boldsymbol{b}\}$;
- $\Gamma = \{\boldsymbol{a}, \boldsymbol{S}\}$;

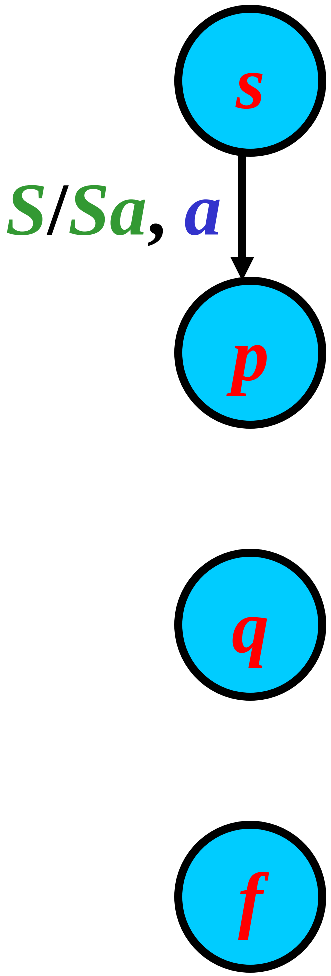


Graphical Representation: Example

$M = (Q, \Sigma, \Gamma, R, s, S, F)$

where:

- $Q = \{s, p, q, f\}$;
- $\Sigma = \{a, b\}$;
- $\Gamma = \{a, S\}$;
- $R = \{Ssa \rightarrow Sap$,



Graphical Representation: Example

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where:

- $Q = \{s, p, q, f\}$;
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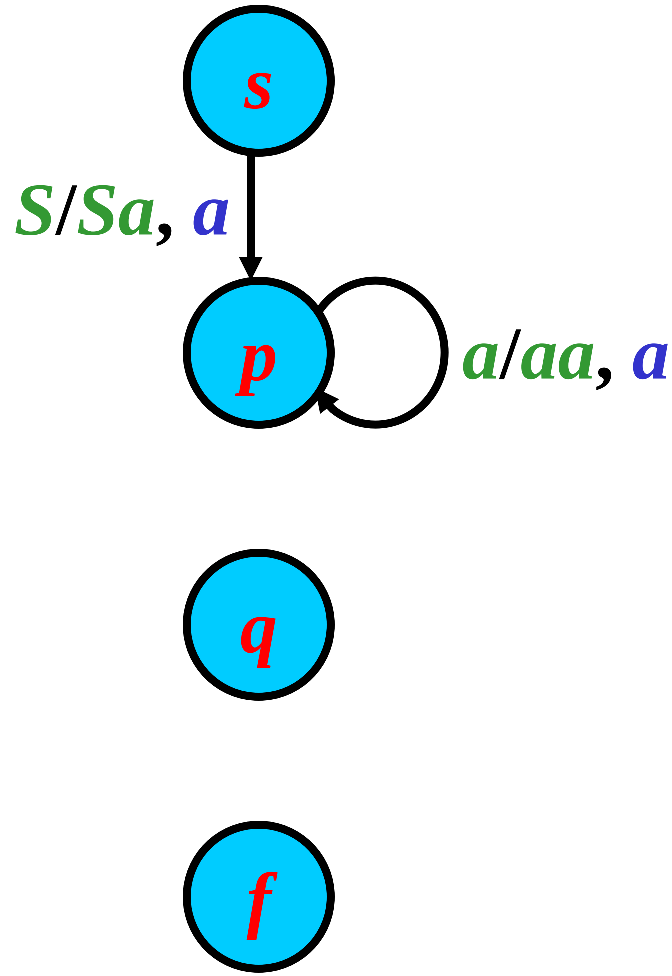
- $\Gamma = \{a, S\}$;
- $R = \{Ssa \rightarrow Sap,$
  $apa \rightarrow aap,$



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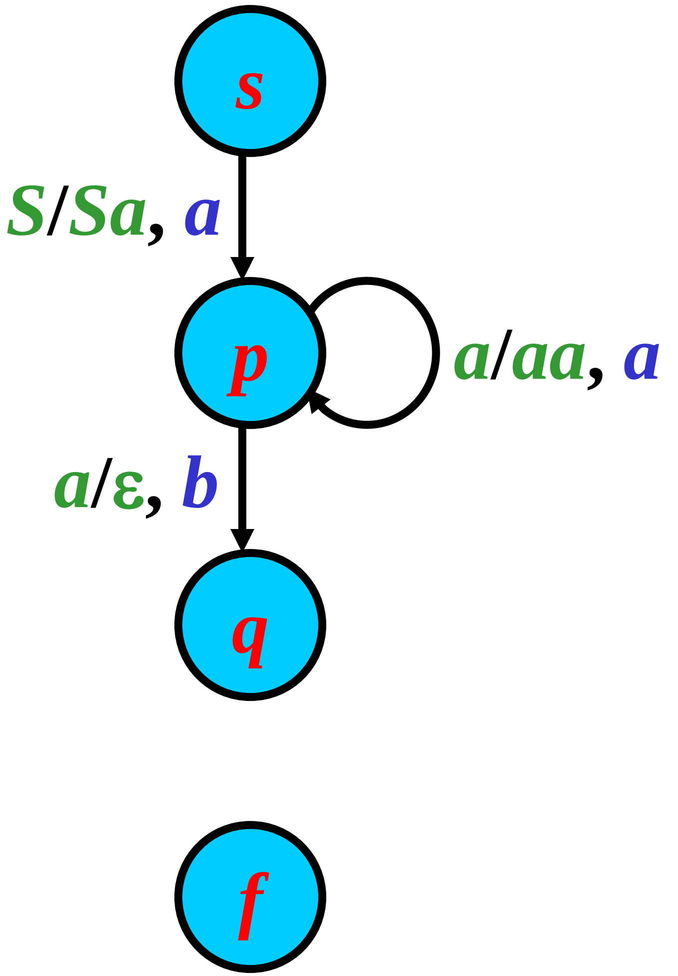
- $\Gamma = \{a, S\}$;
- $R = \{Ssa \rightarrow Sap,$
  $apa \rightarrow aap,$
  $apb \rightarrow q,$



Graphical Representation: Example

$M = (Q, \Sigma, \Gamma, R, s, S, F)$

where:

- $Q = \{s, p, q, f\}$;
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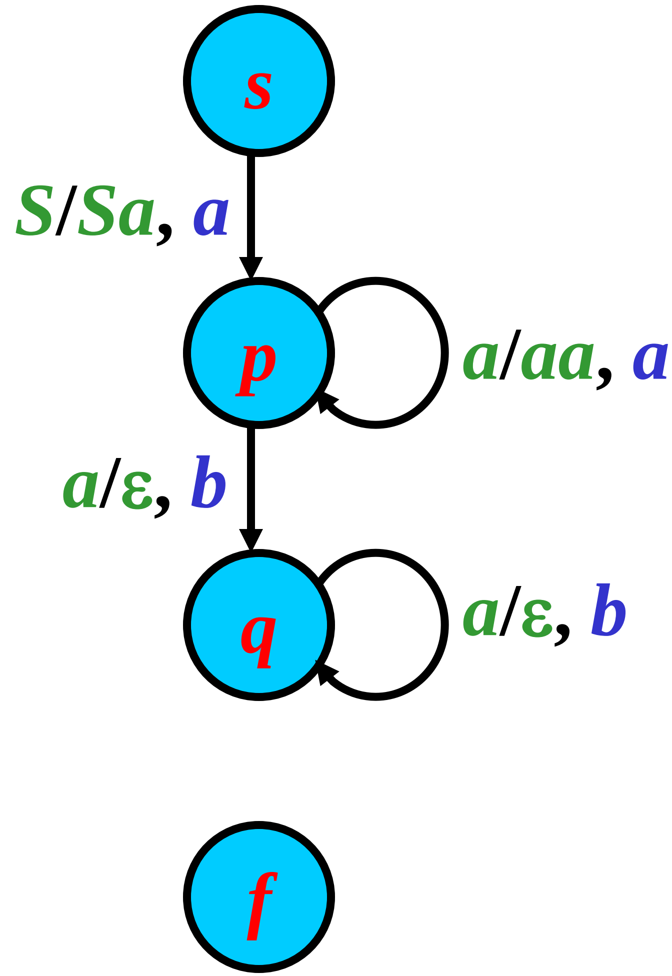
- $\Gamma = \{a, S\}$;
- $R = \{Ssa \rightarrow Sap,$
  $apa \rightarrow aap,$
  $apb \rightarrow q,$
  $aqb \rightarrow q,$



Graphical Representation: Example

$M = (Q, \Sigma, \Gamma, R, s, S, F)$

where:

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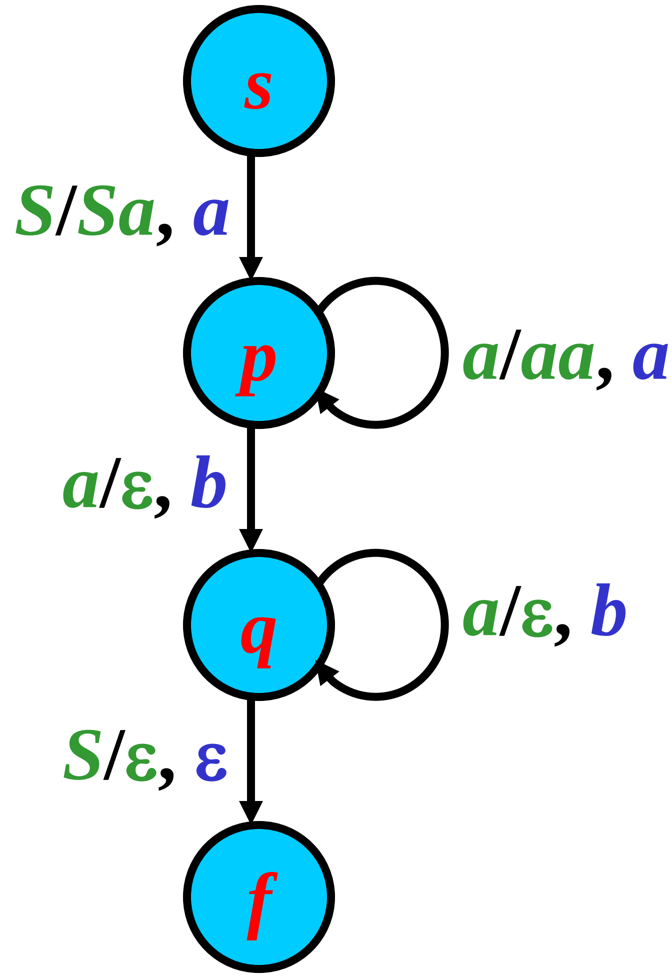

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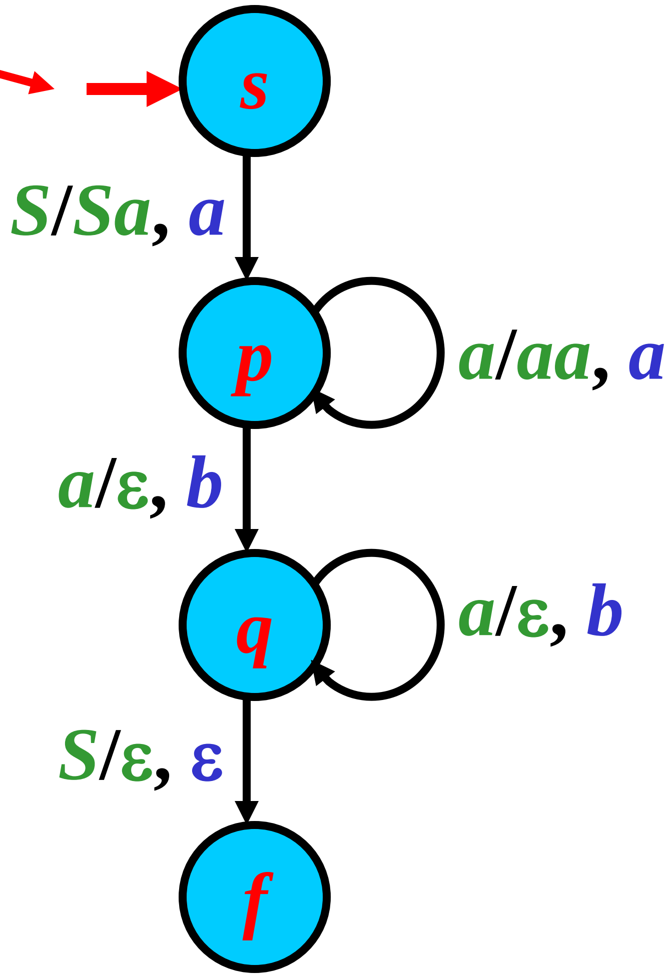

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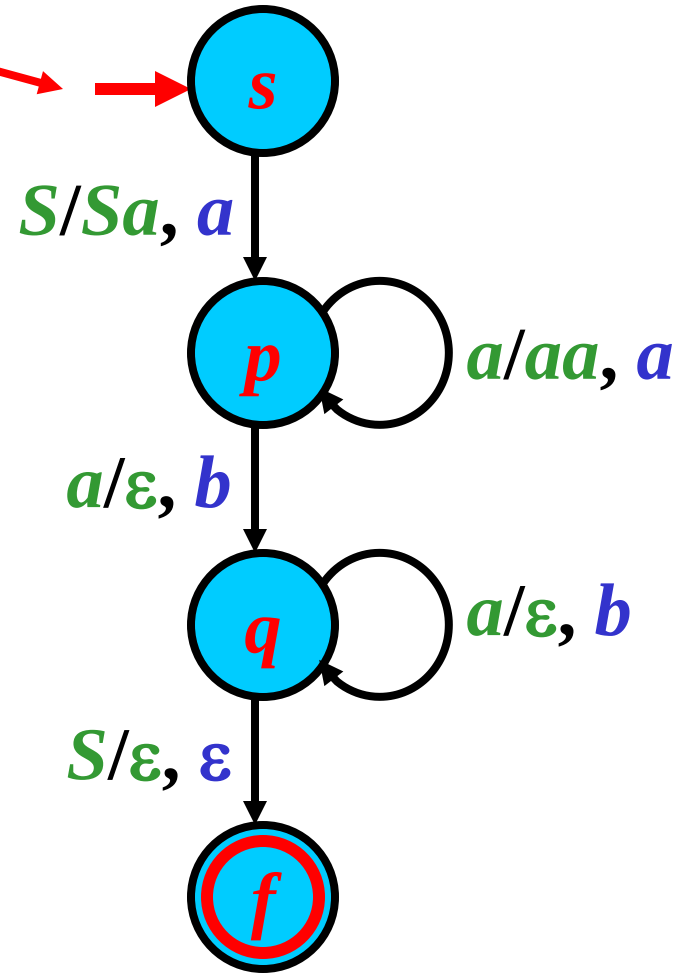
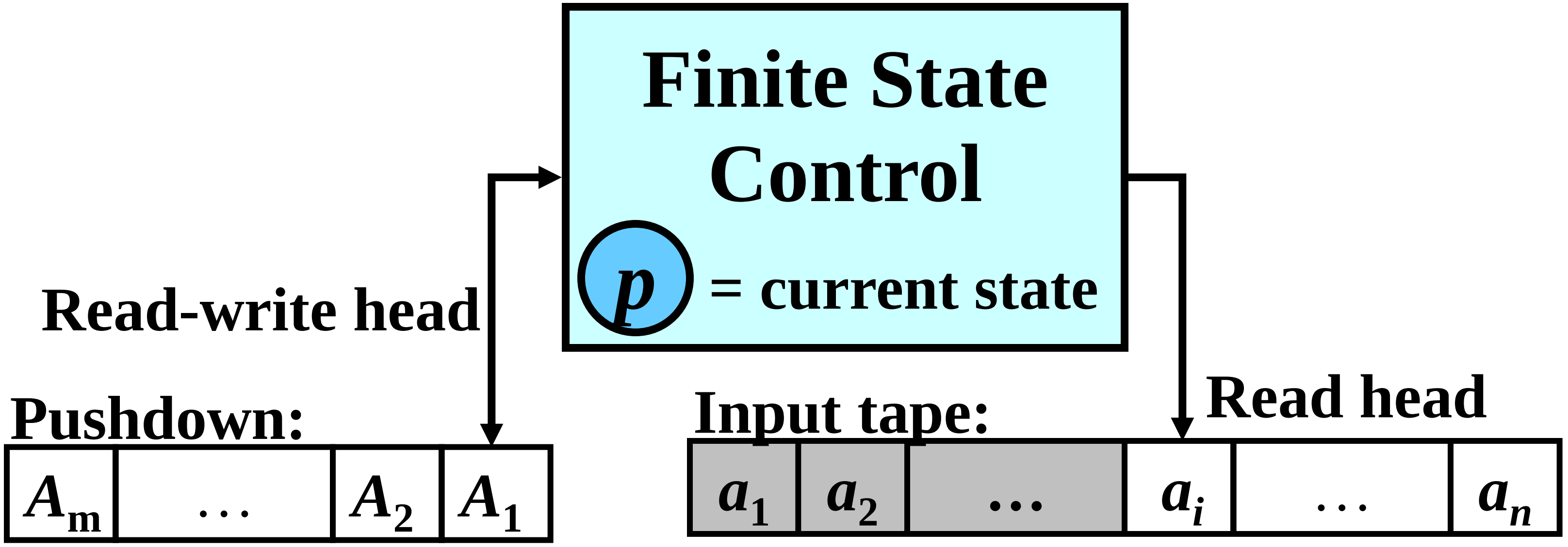


PDA Configuration

Gist: Instantaneous description of PDA

Definition: Let $M = (Q, \Sigma, \Gamma, R, s, S, F)$ be a PDA. *A configuration* of M is a string $\chi \in \Gamma^*Q\Sigma^*$



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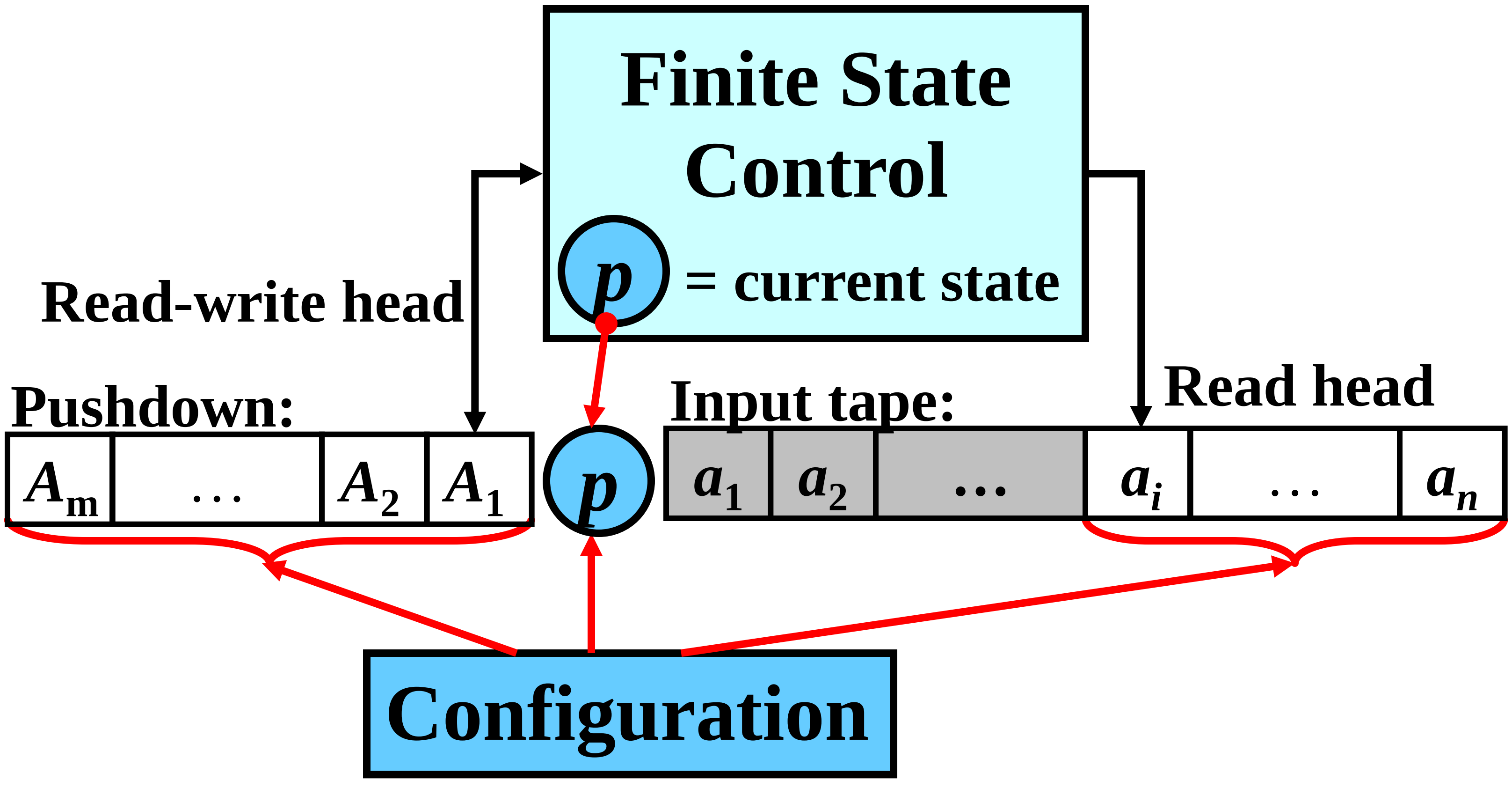

Move

Gist: A computational step made by a PDA

Definition: Let $xApay$ and $xwqy$ be two configurations of a PDA, M, where $x, w \in \Gamma^*$, $A \in \Gamma$, $p, q \in Q$, $a \in \Sigma \cup \{\varepsilon\}$, and $y \in \Sigma^*$. Let $r = Apa \rightarrow wq \in R$ be a rule. Then, M makes a *move* from $xApay$ to $xwqy$ according to r, written as $xApay \vdash xwqy$ [r] or, simply, $xApay \vdash xwqy$.

Note: if $a = \varepsilon$, no input symbol is read

Configuration:

Move

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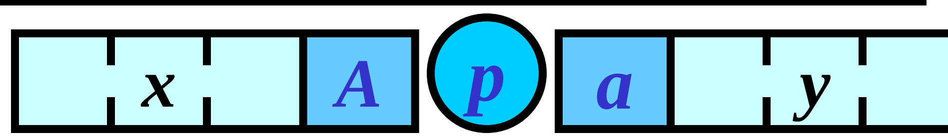

Rule: $Apa \rightarrow wq$

Move

Gist: A computational step made by a PDA

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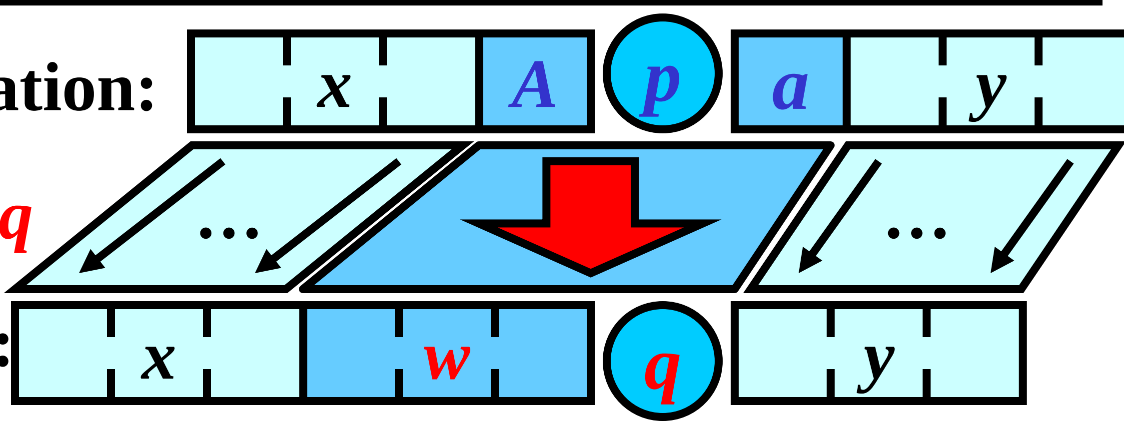
Configuration:

Rule: $Apa \rightarrow wq$

New configuration:



Sequence of Moves 1/2

Gist: Several consecutive computational steps

Definition: Let χ be a configuration. M makes *zero moves* from χ to χ; in symbols,

$$\chi \mid\!-^0 \chi\ [\varepsilon] \text{ or, simply, } \chi \mid\!-^0 \chi$$

Definition: Let $\chi_0, \chi_1, ..., \chi_n$ be a sequence of configurations, $n \geq 1$, and $\chi_{i-1} \mid\!- \chi_i\ [r_i]$, $r_i \in R$, for all $i = 1, ..., n$; that is,

$$\chi_0 \mid\!- \chi_1\ [r_1] \mid\!- \chi_2\ [r_2] \ \dots \ \mid\!- \chi_n\ [r_n]$$

Then M makes *n moves* from χ_0 to χ_n,

$$\chi_0 \mid\!-^n \chi_n\ [r_1 ... r_n] \text{ or, simply, } \chi_0 \mid\!-^n \chi_n$$

Sequence of Moves 2/2

If $\chi_0 \vdash^n \chi_n$ [ρ] for some $n \geq 1$, then

$\chi_0 \vdash^+ \chi_n$ [ρ] or, simply, $\chi_0 \vdash^+ \chi_n$

If $\chi_0 \vdash^n \chi_n$ [ρ] for some $n \geq 0$, then

$\chi_0 \vdash^* \chi_n$ [ρ] or, simply, $\chi_0 \vdash^* \chi_n$

Example: Consider

$AApabc \vdash ABqbc$ [**1**: $Apa \rightarrow Bq$], and

$ABqbc \vdash ABCrc$ [**2**: $Bqb \rightarrow BCr$].

Then, $AApabc \vdash^2 ABCrc$ [**1 2**],

$AApabc \vdash^+ ABCrc$ [**1 2**],

$AApabc \vdash^* ABCrc$ [**1 2**]

Accepted Language: Three Types

Definition: Let $M = (Q, \Sigma, \Gamma, R, s, S, F)$ be a PDA.

1) The *language that M accepts* ***by final state***, denoted by $\boldsymbol{L(M)_f}$, is defined as
$L(M)_f = \{w: w \in \Sigma^*, Ssw \vdash^* zf, z \in \Gamma^*, f \in F\}$

2) The *language that M accepts* ***by empty pushdown***, denoted by $\boldsymbol{L(M)_\varepsilon}$, is defined as
$L(M)_\varepsilon = \{w: w \in \Sigma^*, Ssw \vdash^* zf, z = \varepsilon, f \in Q\}$

3) The *language that M accepts* ***by final state and empty pushdown***, denoted by $\boldsymbol{L(M)_{f\varepsilon}}$, is defined as
$L(M)_{f\varepsilon} = \{w: w \in \Sigma^*, Ssw \vdash^* zf, z = \varepsilon, f \in F\}$

PDA: Example

$M = (Q, \Sigma, \Gamma, R, s, S, F)$

where:

- $Q = \{s, p, q, f\}$;
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  $apb \rightarrow q,$
  $aqb \rightarrow q,$
  $Sq \rightarrow f\}$
- $F = \{f\}$

Question: $aabb \in L(M)_{f\varepsilon}$**?**

S	s	a	a	b	b

$Ssaabb$

PDA: Example

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where:

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Question: $aabb \in L(M)_{f\varepsilon}$**?**

S | s | a a b b

Rule: $Ssa \rightarrow Sap$

S a | p | a b b

$Ssaabb \vdash Sapabb$

PDA: Example

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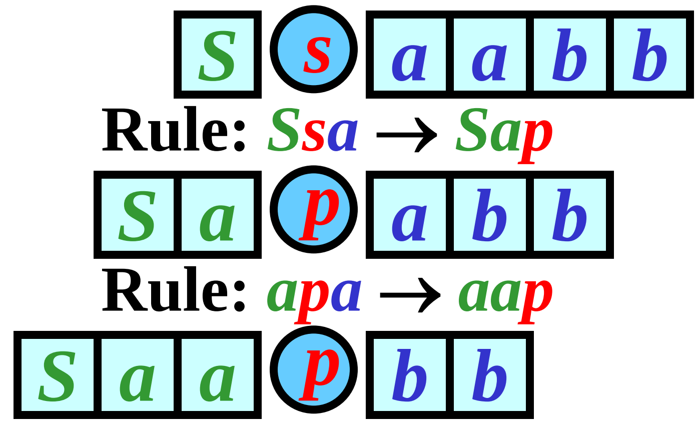

$Ssaabb \vdash Sapabb \vdash Saapbb$

PDA: Example

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S | s | $a\ a\ b\ b$

Rule: $Ssa \rightarrow Sap$

$S\ a$ | p | $a\ b\ b$

Rule: $apa \rightarrow aap$

$S\ a\ a$ | p | $b\ b$

Rule: $apb \rightarrow q$

$S\ a$ | q | b

$Ssaabb \vdash Sapabb \vdash Saapbb \vdash Saqb$

PDA: Example

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S a a | p | b b

Rule: $apb \rightarrow q$

S a | q | b

Rule: $aqb \rightarrow q$

S | q |

$Ssaabb \vdash Sapabb \vdash Saapbb \vdash Saqb \vdash Sq$

PDA: Example

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S a q b

Rule: $aqb \rightarrow q$

S q

Rule: $Sq \rightarrow f$

f

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S a a p b b

Rule: $apb \rightarrow q$

S a q b

Rule: $aqb \rightarrow q$

S q

Rule: $Sq \rightarrow f$

Empty pushdown

f

$Ssaabb \vdash Sapabb \vdash Saapbb \vdash Saqb \vdash Sq \vdash f$

PDA: Example

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S a | p | a b b

Rule: $apa \to aap$

S a a | p | b b

Rule: $apb \to q$

S a | q | b

Rule: $aqb \to q$

S | q |

Rule: $Sq \to f$

| f |

Empty pushdown

Final state

Answer: YES

$Ssaabb \vdash Sapabb \vdash Saapbb \vdash Saqb \vdash Sq \vdash f$

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$S\ a\ q\ b$

Rule: $aqb \rightarrow q$

$S\ q$

Rule: $Sq \rightarrow f$

f

Empty pushdown

Final state

Answer: YES

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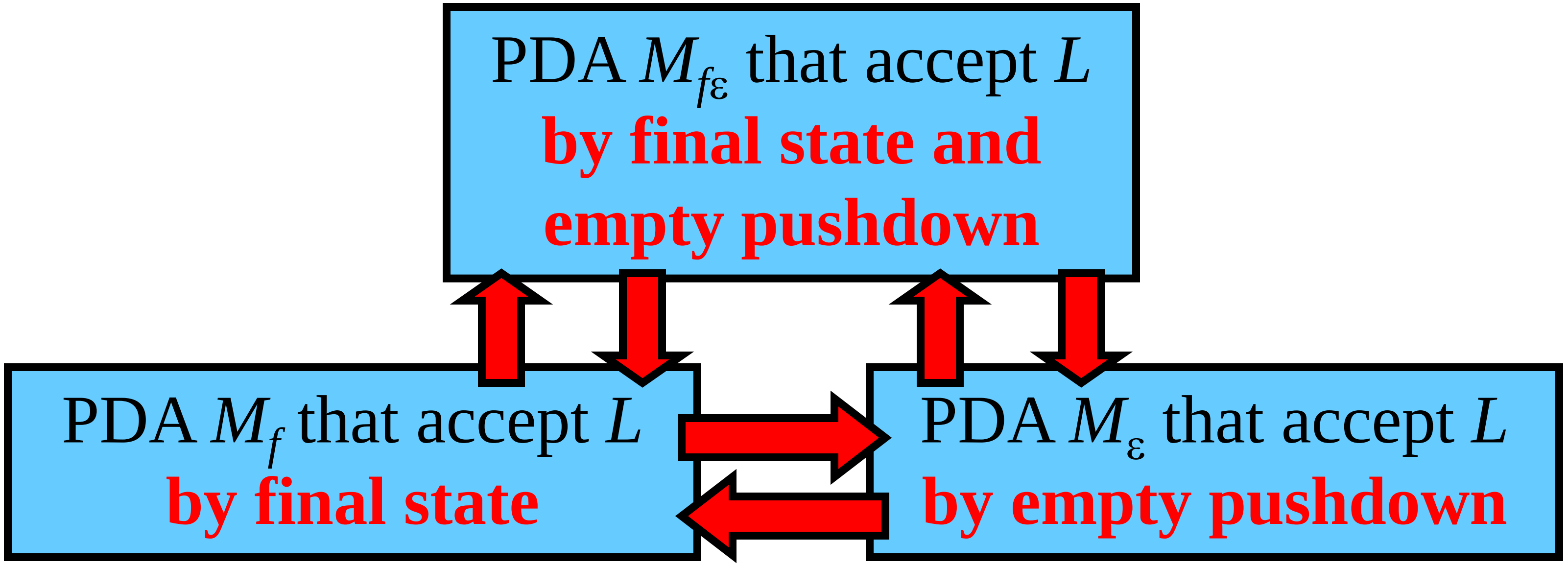
Note: $L(M)_f = L(M)_\varepsilon = L(M)_{f\varepsilon} = \{a^n b^n : n \geq 1\}$

Three Types of Acceptance: Equivalence

Theorem:

- $L = L(M_f)_f$ for a PDA $M_f \Leftrightarrow L = L(M_{f\varepsilon})_{f\varepsilon}$ for a PDA $M_{f\varepsilon}$
- $L = L(M_\varepsilon)_\varepsilon$ for a PDA $M_\varepsilon \Leftrightarrow L = L(M_{f\varepsilon})_{f\varepsilon}$ for a PDA $M_{f\varepsilon}$
- $L = L(M_f)_f$ for a PDA $M_f \Leftrightarrow L = L(M_\varepsilon)_\varepsilon$ for a PDA M_ε

Note: There exist these conversions:



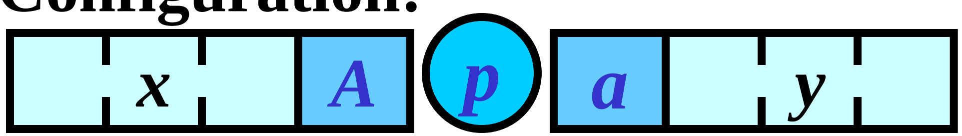
Deterministic PDA (DPDA)

Gist: Deterministic PDA makes no more than one move from any configuration.

Definition: Let $M = (Q, \Sigma, \Gamma, R, s, S, F)$ be a PDA. M is a *deterministic PDA* if for each rule $\boldsymbol{Apa} \rightarrow \boldsymbol{wq} \in R$, it holds that $R - \{\boldsymbol{Apa} \rightarrow \boldsymbol{wq}\}$ contains no rule with the left-hand side equal to $\boldsymbol{Apa}$ or $\boldsymbol{Ap}$.

Illustration:

Configuration:



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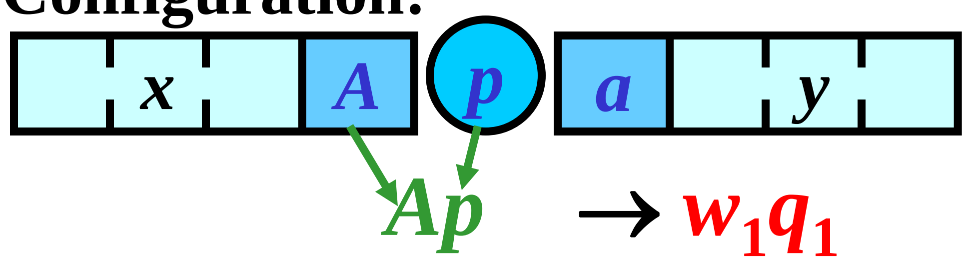

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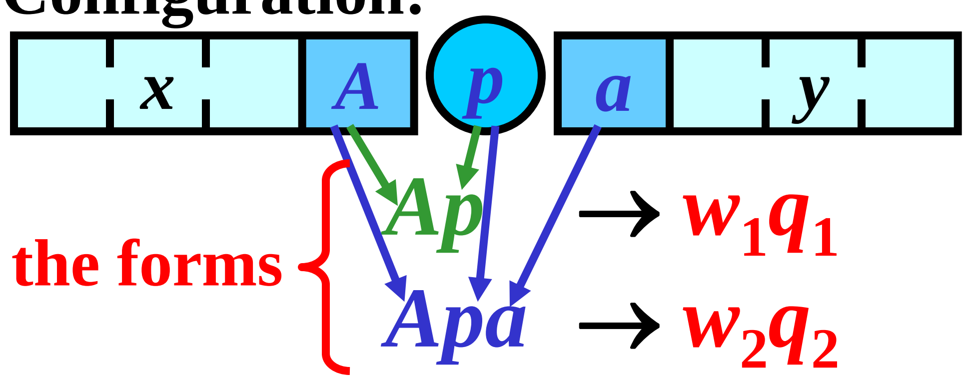

No more that one rule of the forms

PDAs are Stronger than DPDAs

Theorem: There exists no DPDA $M_{f\varepsilon}$ that accepts
$$L = \{xy: x, y \in \Sigma^*, y = reversal(x)\}$$

Proof: See page 431 in [Meduna: Automata and Languages]

Illustration:

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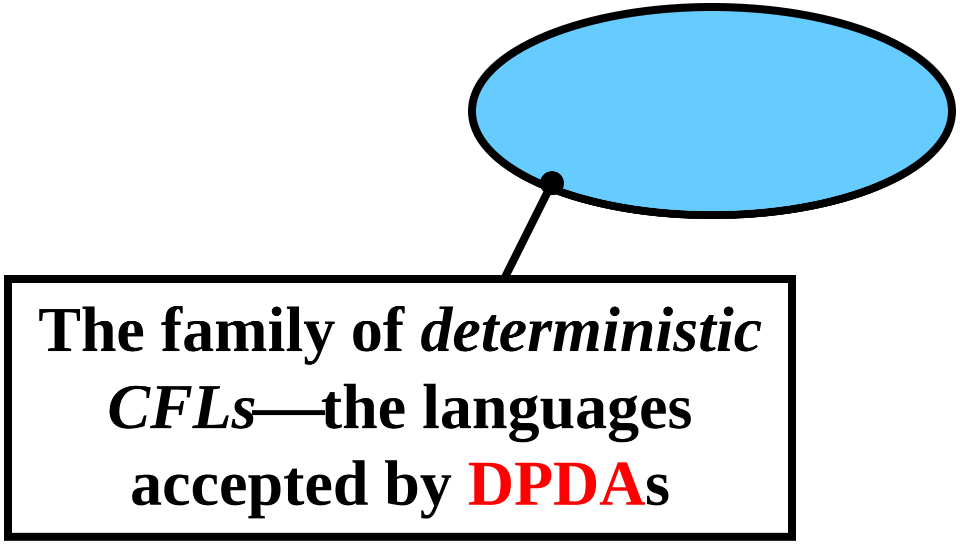

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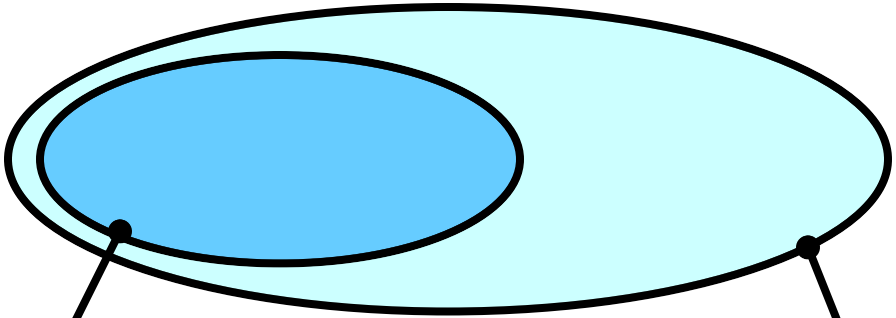

The family of *deterministic CFLs*—the languages accepted by DPDAs

The family of languages accepted by PDAs

PDAs are Stronger than DPDAs

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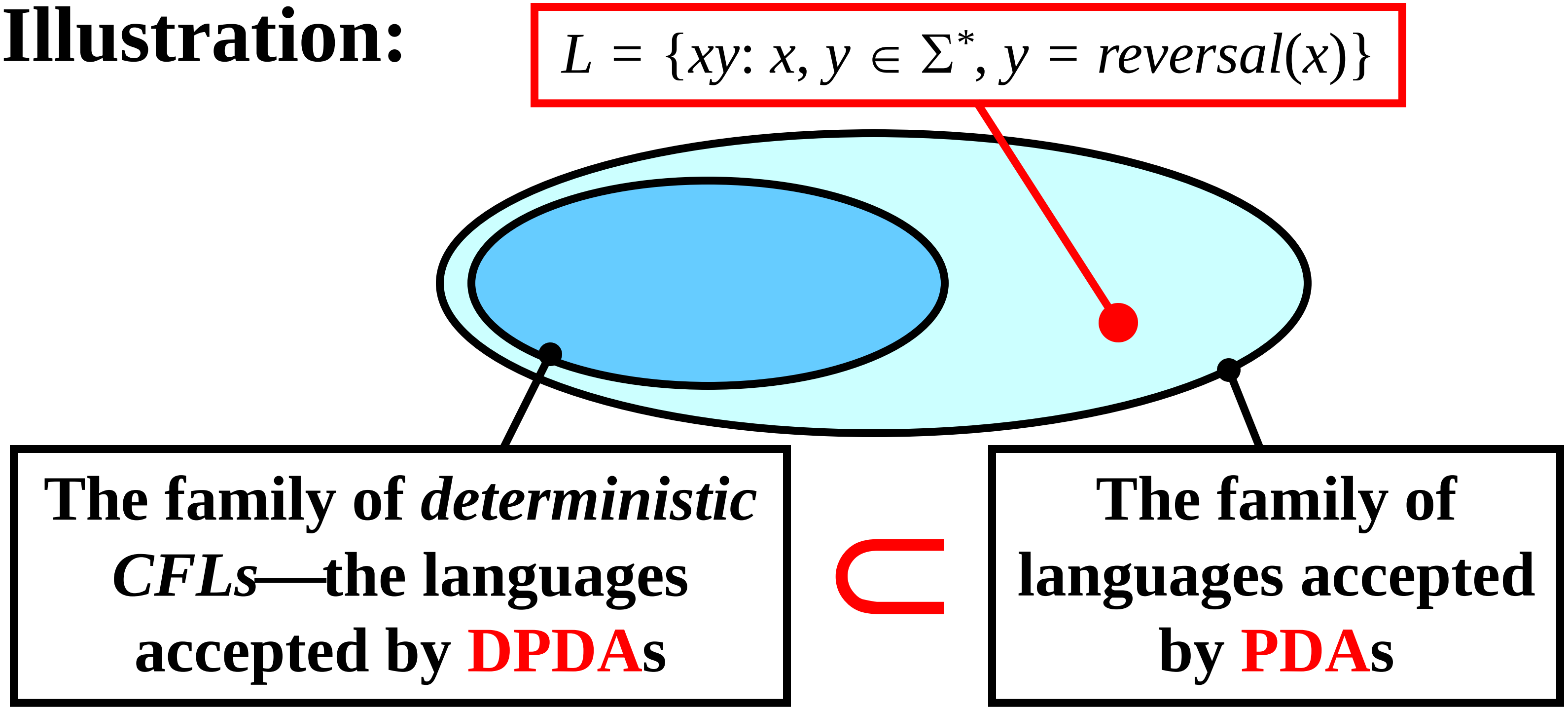


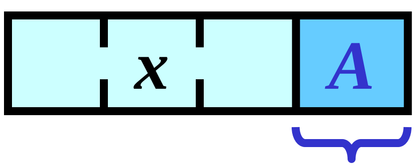
Extended PDA (EPDA)

Gist: The pushdown top of an EPDA represents a string rather than a single symbol.

Definition: *An Extended Pushdown automaton* (EPDA) is a 7-tuple $M = (Q, \Sigma, \Gamma, R, s, S, F)$, where $Q, \Sigma, \Gamma, s, S, F$ are defined as in an PDA and R is a *finite set of rules* of the form: $\boldsymbol{v}pa \rightarrow wq$, where $\boldsymbol{v}, w \in \Gamma^*$, $p, q \in Q$, $a \in \Sigma \cup \{\varepsilon\}$

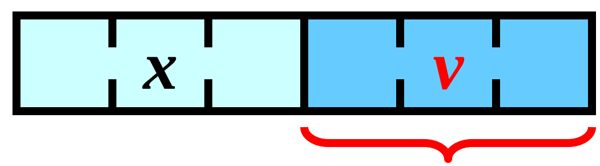
Illustration:

Pushdown of PDA:



PDA has a single symbols as the pushdown top

Pushdown of EPDA:

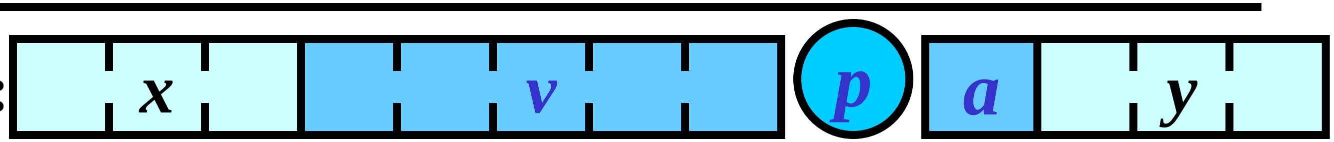


EPDA has a string as the pushdown top

Move in EPDA

Definition: Let $\boldsymbol{xvpay}$ and $\boldsymbol{xwqy}$ be two configurations of an EPDA, M, where $\boldsymbol{x}, \boldsymbol{v}, \boldsymbol{w} \in \Gamma^*$, $\boldsymbol{p}, \boldsymbol{q} \in Q$, $\boldsymbol{a} \in \Sigma \cup \{\varepsilon\}$, and $\boldsymbol{y} \in \Sigma^*$. Let $\boldsymbol{r} = \boldsymbol{vpa} \rightarrow \boldsymbol{wq} \in R$ be a rule. Then, M makes a *move* from $\boldsymbol{xvpay}$ to $\boldsymbol{xwqy}$ according to $\boldsymbol{r}$, written as $\boldsymbol{xvpay} \vdash \boldsymbol{xwqy}$ $[\boldsymbol{r}]$ or $\boldsymbol{xvpay} \vdash \boldsymbol{xwqy}$.

Configuration:



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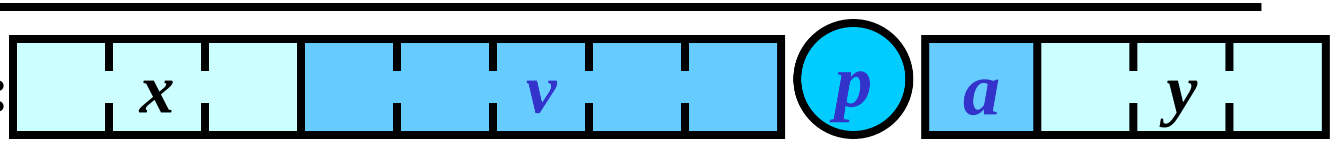
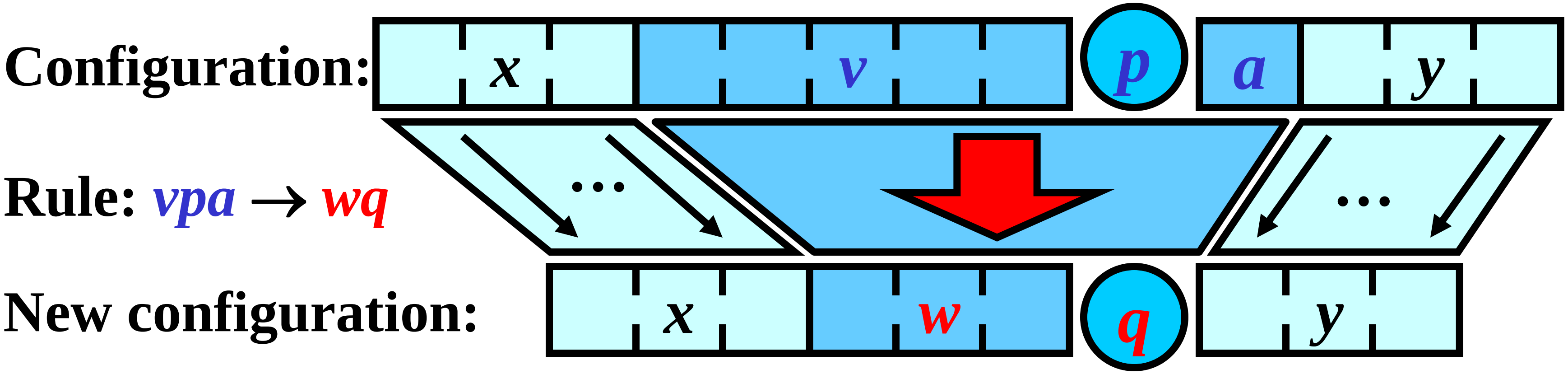


Rule: $\boldsymbol{vpa} \rightarrow \boldsymbol{wq}$

Move in EPDA

Definition: Let $xvpay$ and $xwqy$ be two configurations of an EPDA, M, where $x, v, w \in \Gamma^*$, $p, q \in Q$, $a \in \Sigma \cup \{\varepsilon\}$, and $y \in \Sigma^*$. Let $r = vpa \rightarrow wq \in R$ be a rule. Then, M makes a *move* from $xvpay$ to $xwqy$ according to r, written as $xvpay \vdash xwqy\ [r]$ or $xvpay \vdash xwqy$.



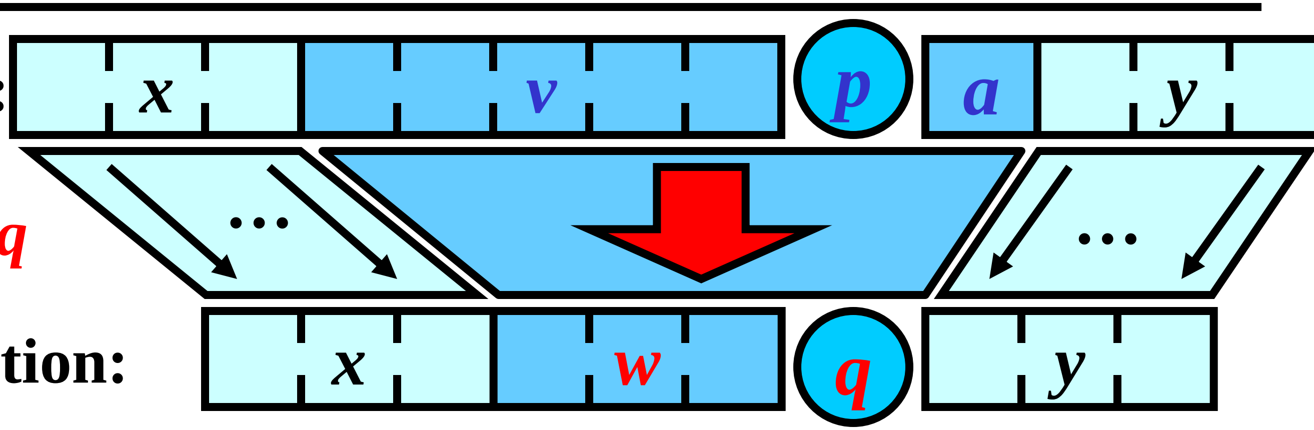
Move in EPDA

Definition: Let $xvpay$ and $xwqy$ be two configurations of an EPDA, M, where $x, v, w \in \Gamma^*$, $p, q \in Q$, $a \in \Sigma \cup \{\varepsilon\}$, and $y \in \Sigma^*$. Let $r = vpa \rightarrow wq \in R$ be a rule. Then, M makes a *move* from $xvpay$ to $xwqy$ according to r, written as $xvpay \vdash xwqy\ [r]$ or $xvpay \vdash xwqy$.

Configuration:

Rule: $vpa \rightarrow wq$

New configuration:



Note: $\vdash^n$, $\vdash^+$, $\vdash^*$, $L(M)_f$, $L(M)_\varepsilon$, and $L(M)_{f\varepsilon}$ are defined analogically to the corresponding definitions for PDA.

EPDA: Example

$M = (Q, \Sigma, \Gamma, R, s, S, F)$

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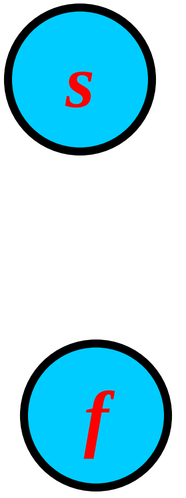

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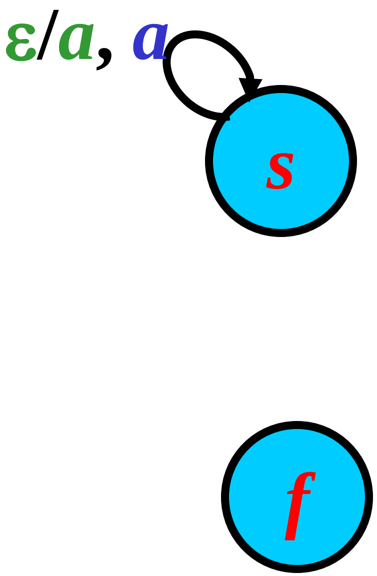

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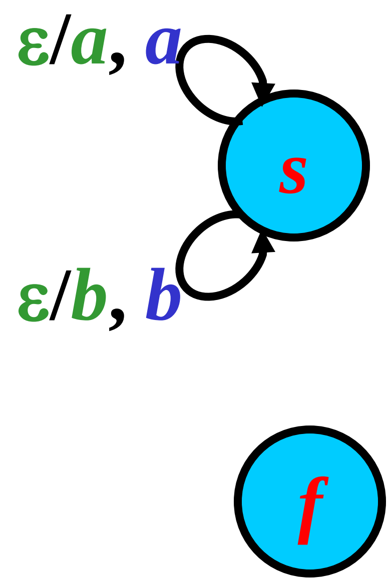

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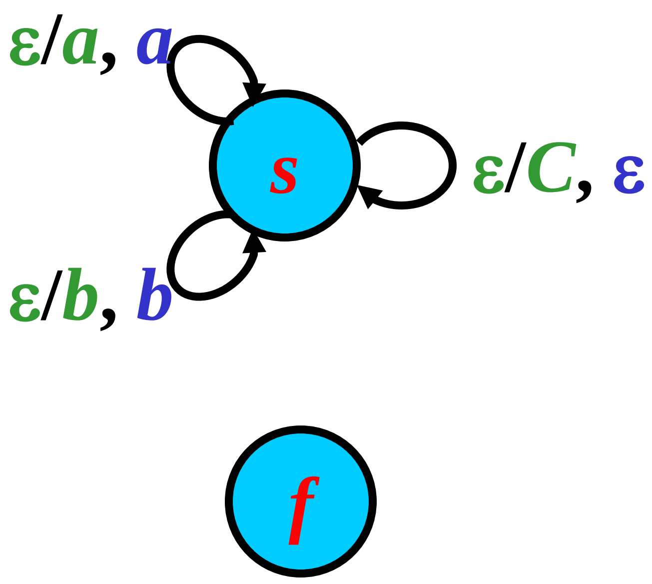

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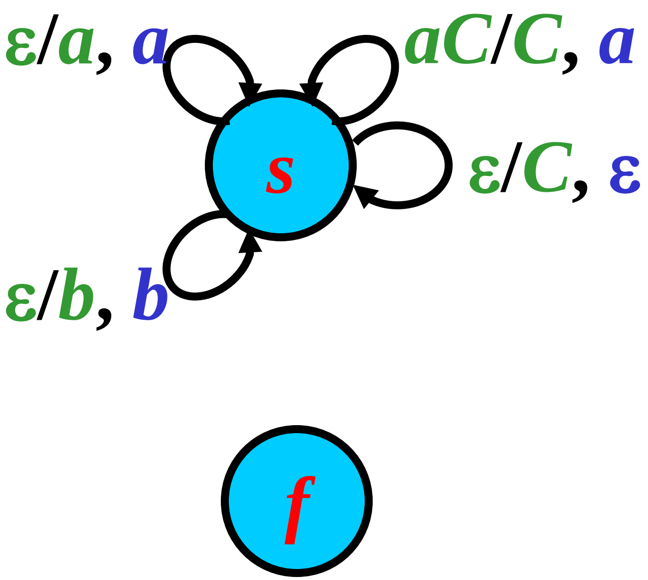

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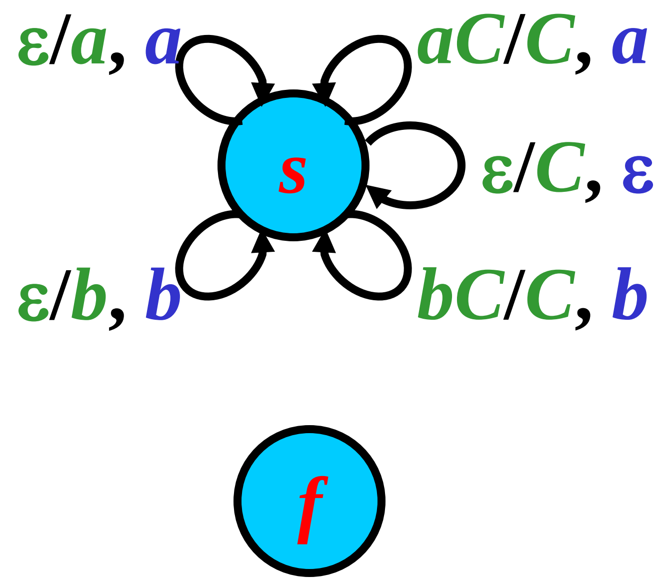

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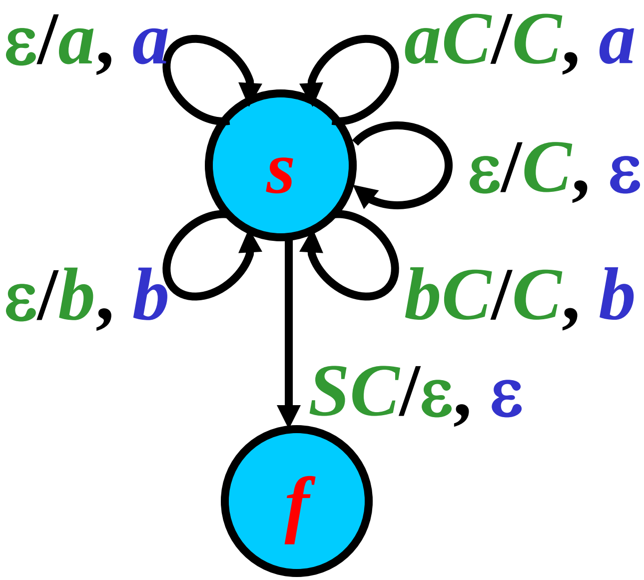

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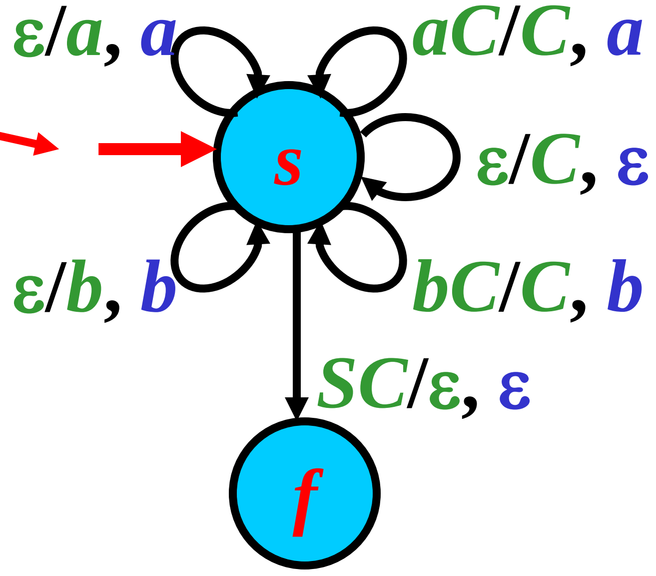

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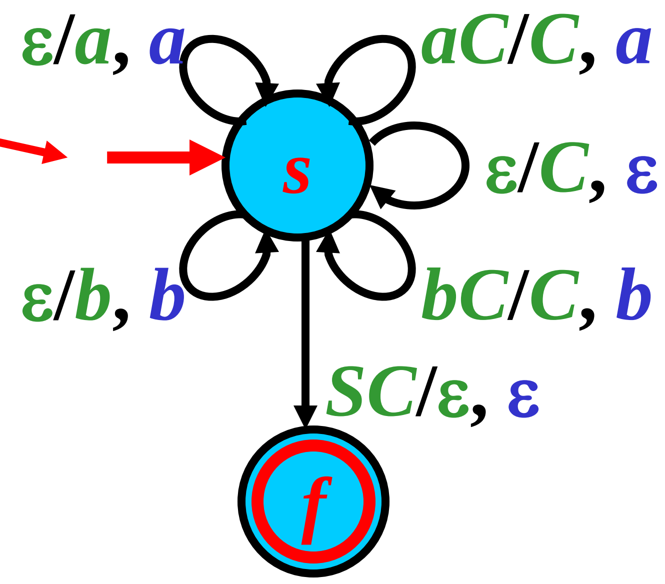

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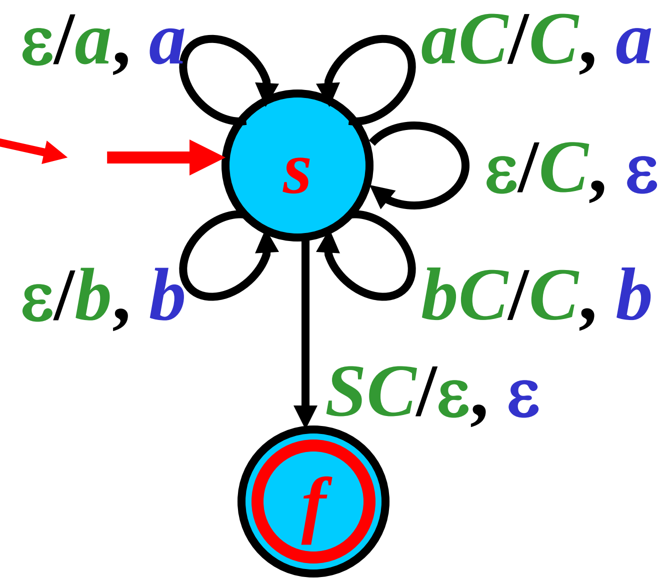

Question: $abba \in L_{f\varepsilon}(M)$?

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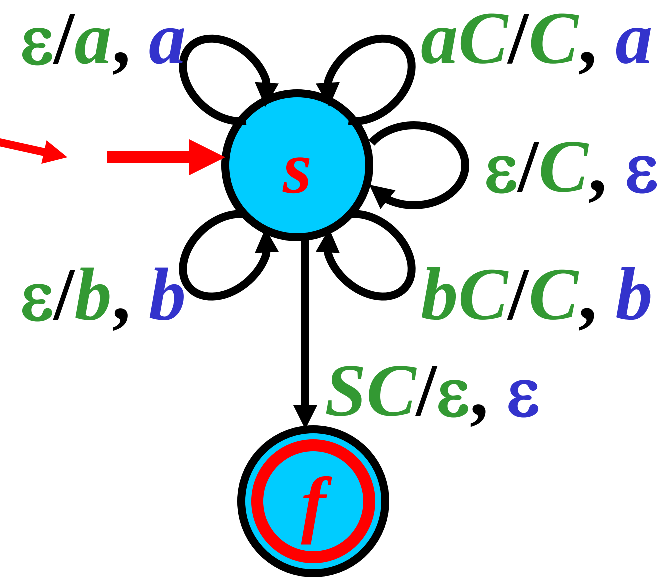

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$Ssabba$

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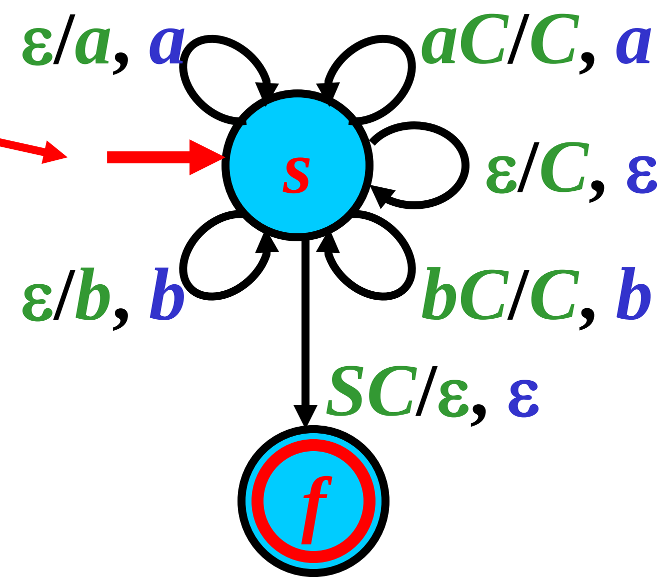

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$Ssabba \vdash Sasbba$

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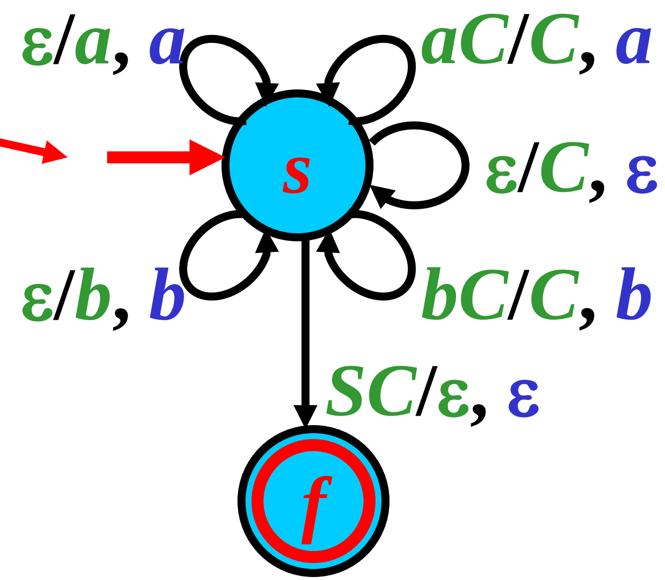

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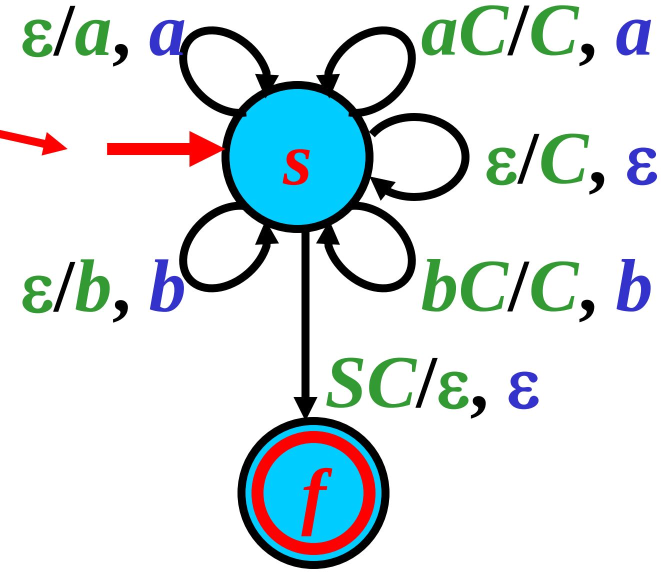

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$\vdash SabCsba$

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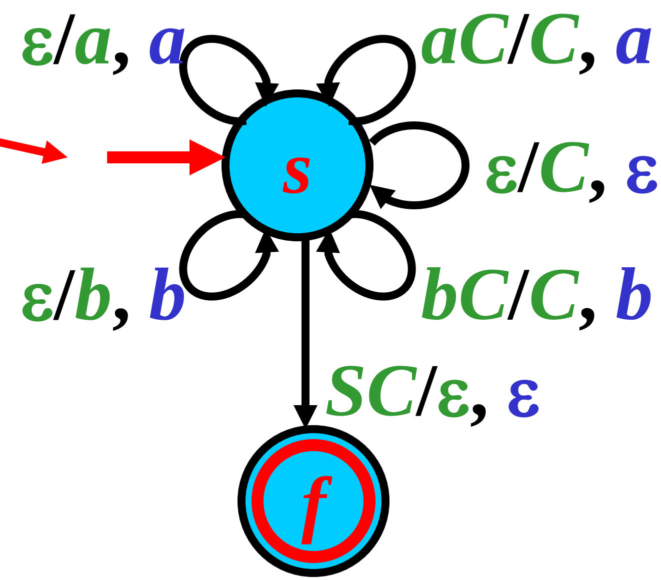

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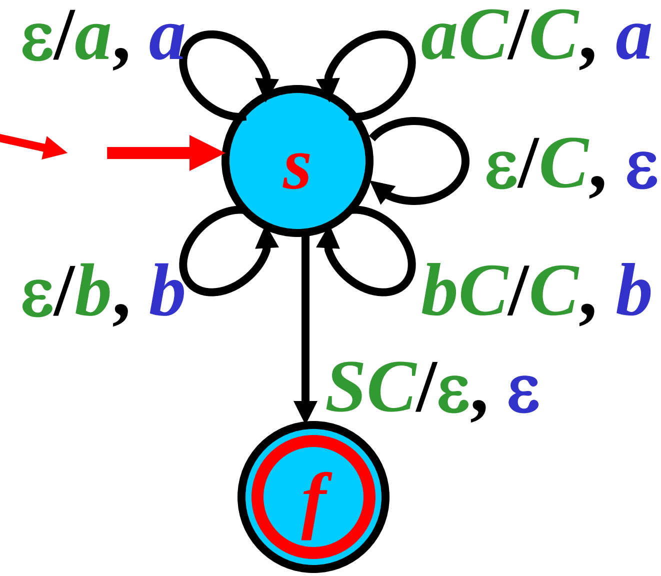

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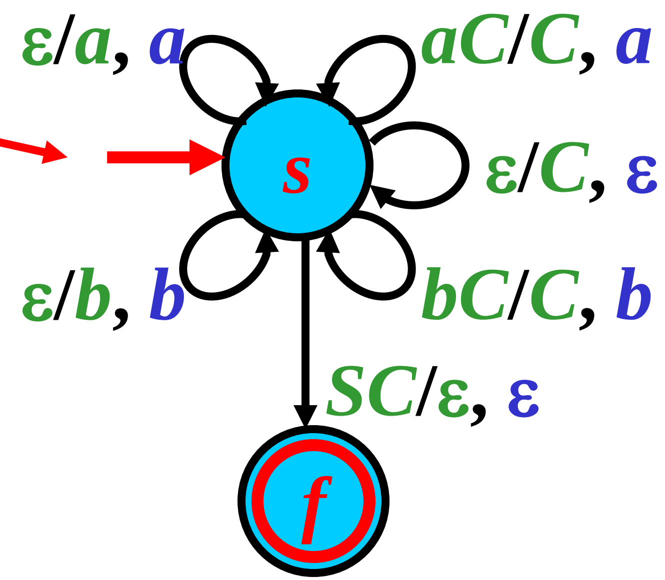

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$\vdash SabCsba \vdash SaCsa$
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Answer: YES

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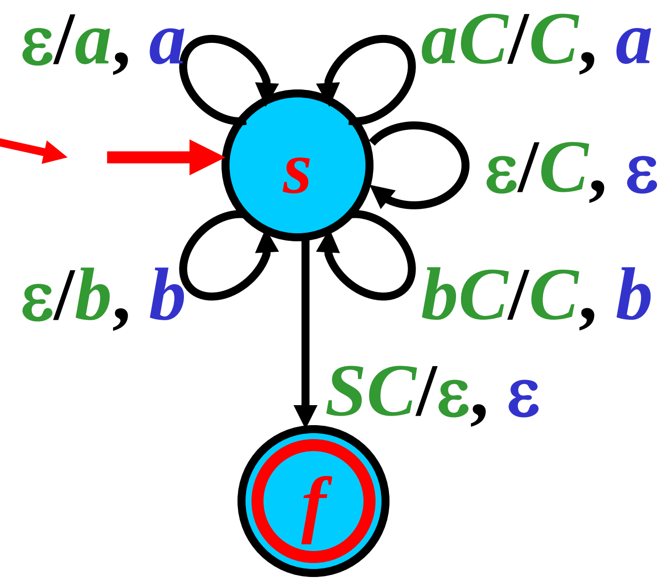

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Answer: YES

Note: $L(M)_f = L(M)_\varepsilon = L(M)_{f\varepsilon} = \{xy: x, y \in \Sigma^*, y = \text{reversal}(x)\}$

Three Types of Acceptance: Equivalence

Theorem:

- $L = L(M_f)_f$ for an EPDA $M_f \Leftrightarrow L = L(M_{f\varepsilon})_{f\varepsilon}$ for an EPDA $M_{f\varepsilon}$
- $L = L(M_\varepsilon)_\varepsilon$ for an EPDA $M_\varepsilon \Leftrightarrow L = L(M_{f\varepsilon})_{f\varepsilon}$ for an EPDA $M_{f\varepsilon}$
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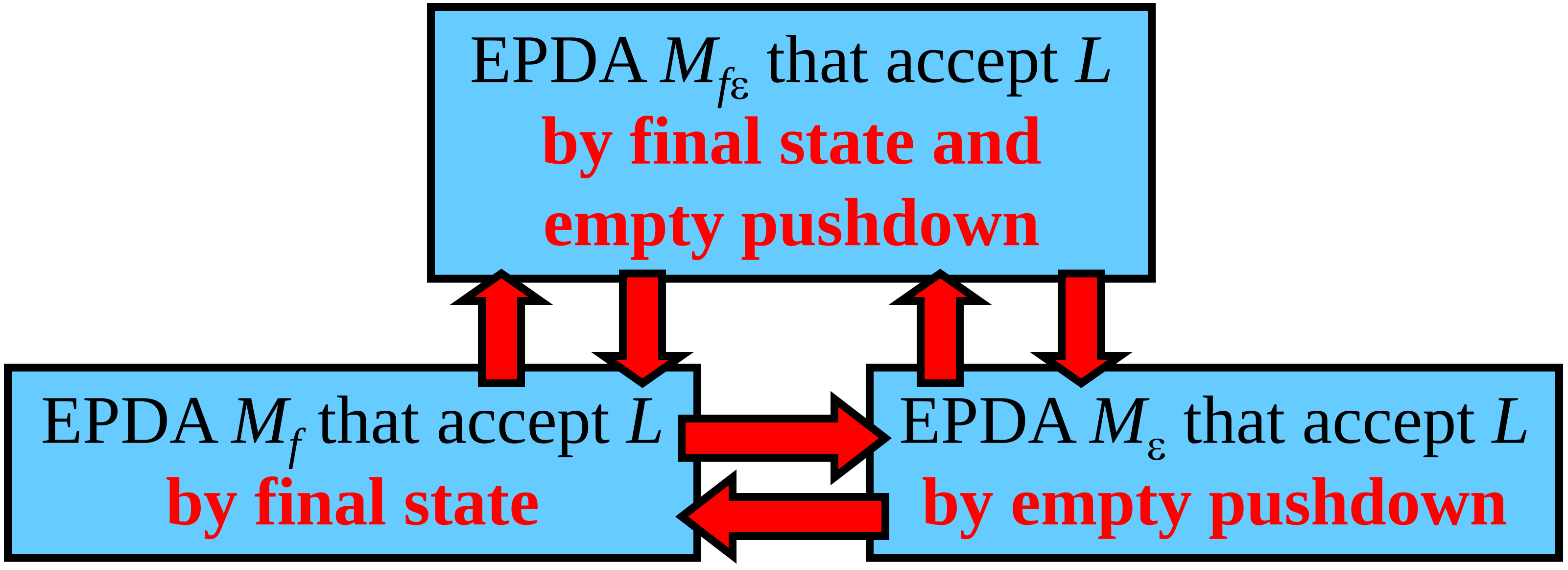
Note: There exist these conversion:



EPDAs and PDAs are Equivalent

Theorem: For every EPDA M, there is a PDA M', and $L(M)_f = L(M')_f$.

Proof: See page 419 in [Meduna: Automata and Languages]

Illustration:

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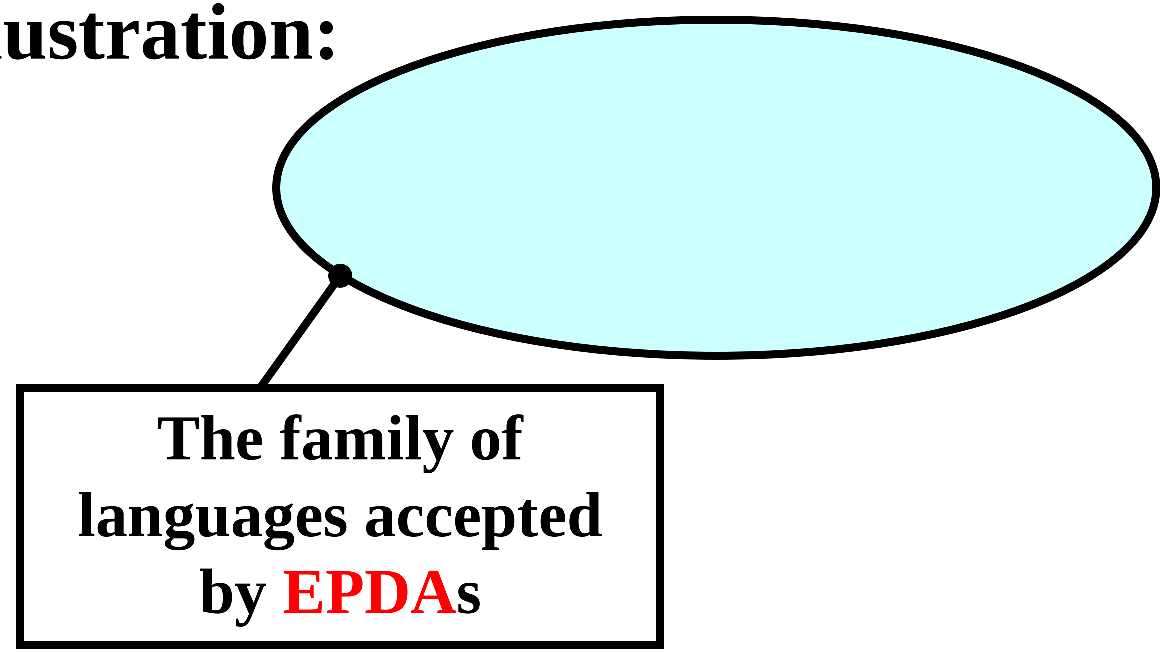
Illustration:



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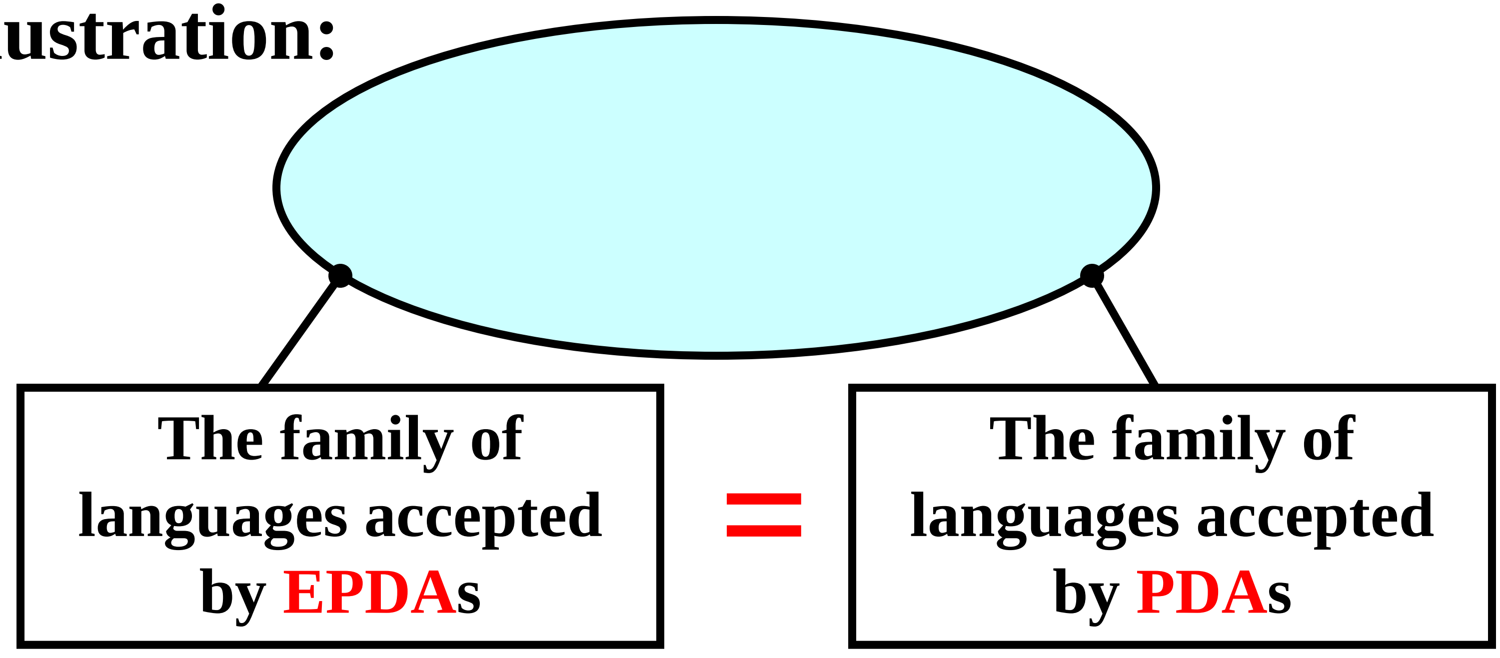
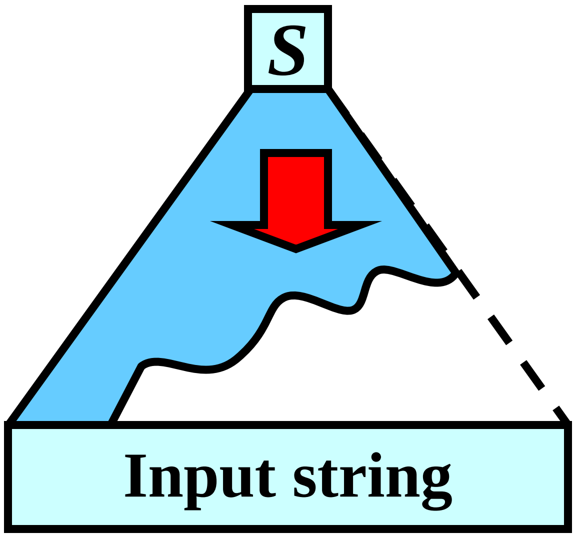


EPDAs and PDAs as Parsing Models for CFGs

Gist: **An EPDA or a PDA can simulate the construction of a derivation tree for a CFG**

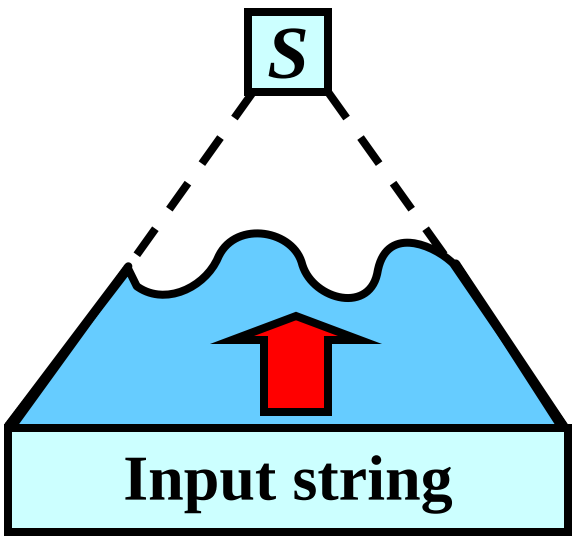
- **Two basic approaches:**

1) Top-Down Parsing



From *S* towards the input string

2) Bottom-Up Parsing

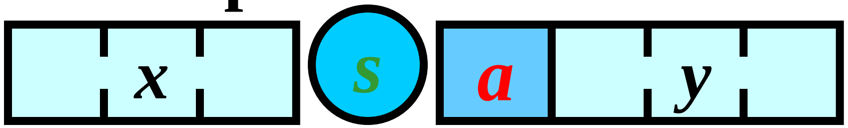


From the input string towards *S*

EPDAs as Models of Bottom-Up Parsers 1/2

Gist: An EPDA *M* underlies a bottom-up parser

1) *M* contains *shift* rules that copy the input symbols onto the pushdown:



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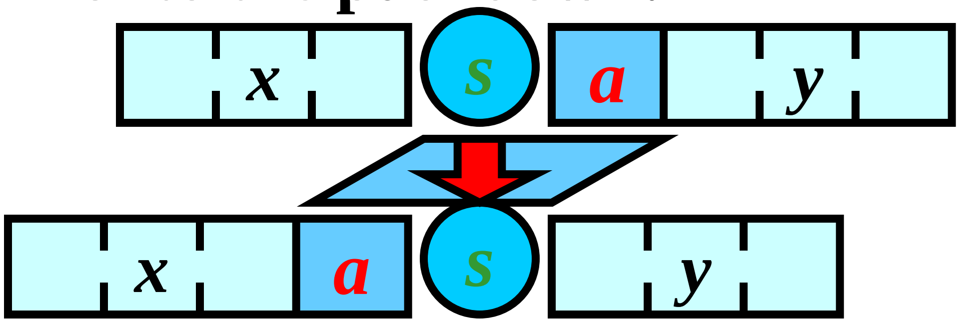

for every $a \in \Sigma$:

add $sa \rightarrow as$ to R;

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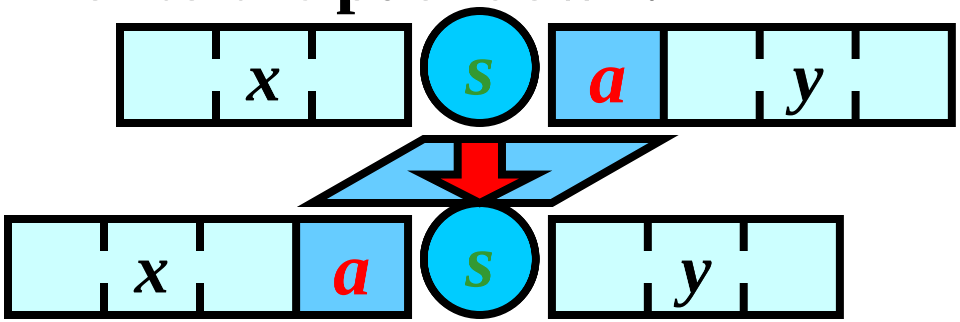
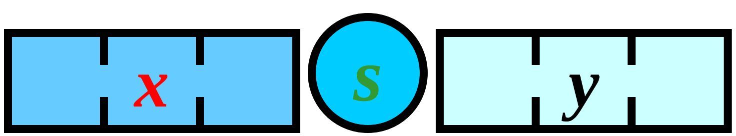


for every ***a*** $\in \Sigma$:

add ***sa*** $\rightarrow$ ***as*** to *R*;

2) *M* contains *reduction* rules that simulate the application of a grammatical rule in reverse:



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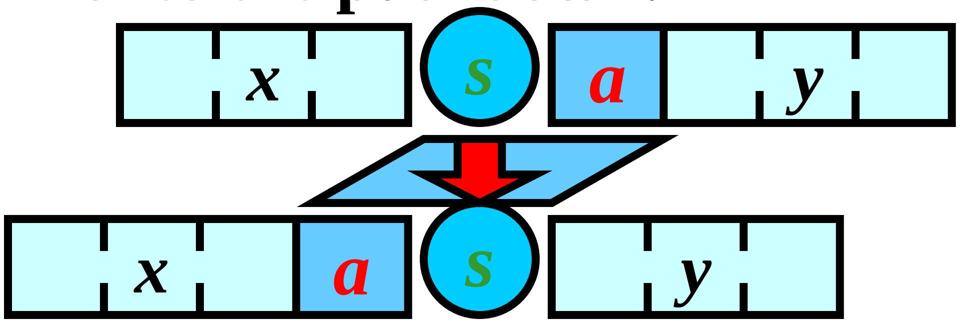

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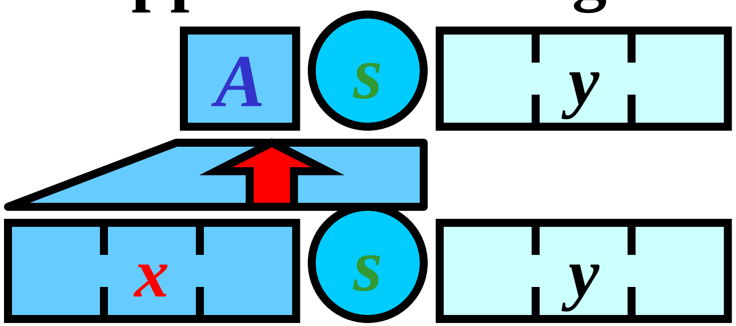

for every $A \rightarrow x \in P$ in G:
add $xs \rightarrow As$ to R;

EPDAs as Models of Bottom-Up Parsers 1/2

Gist: An EPDA *M* underlies a bottom-up parser

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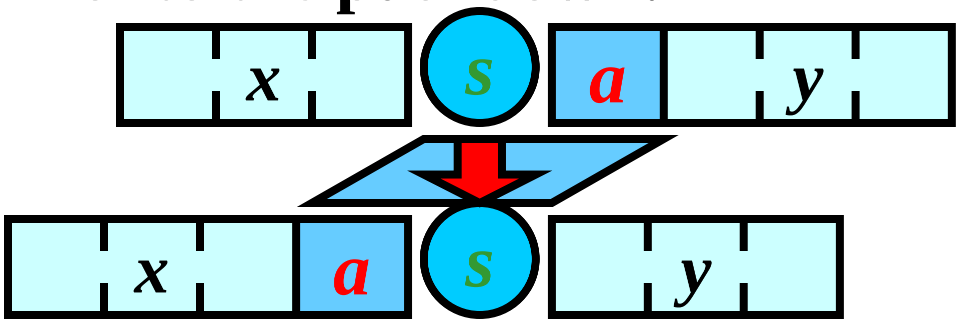

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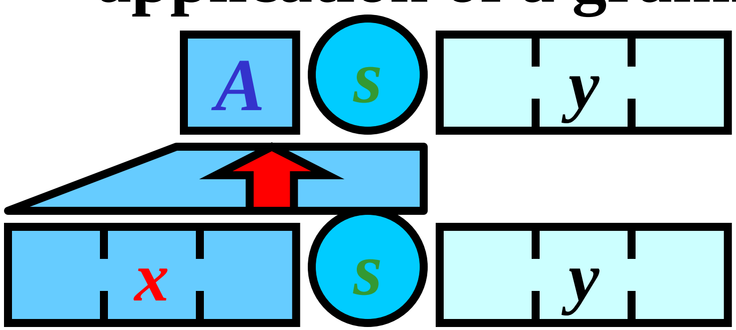


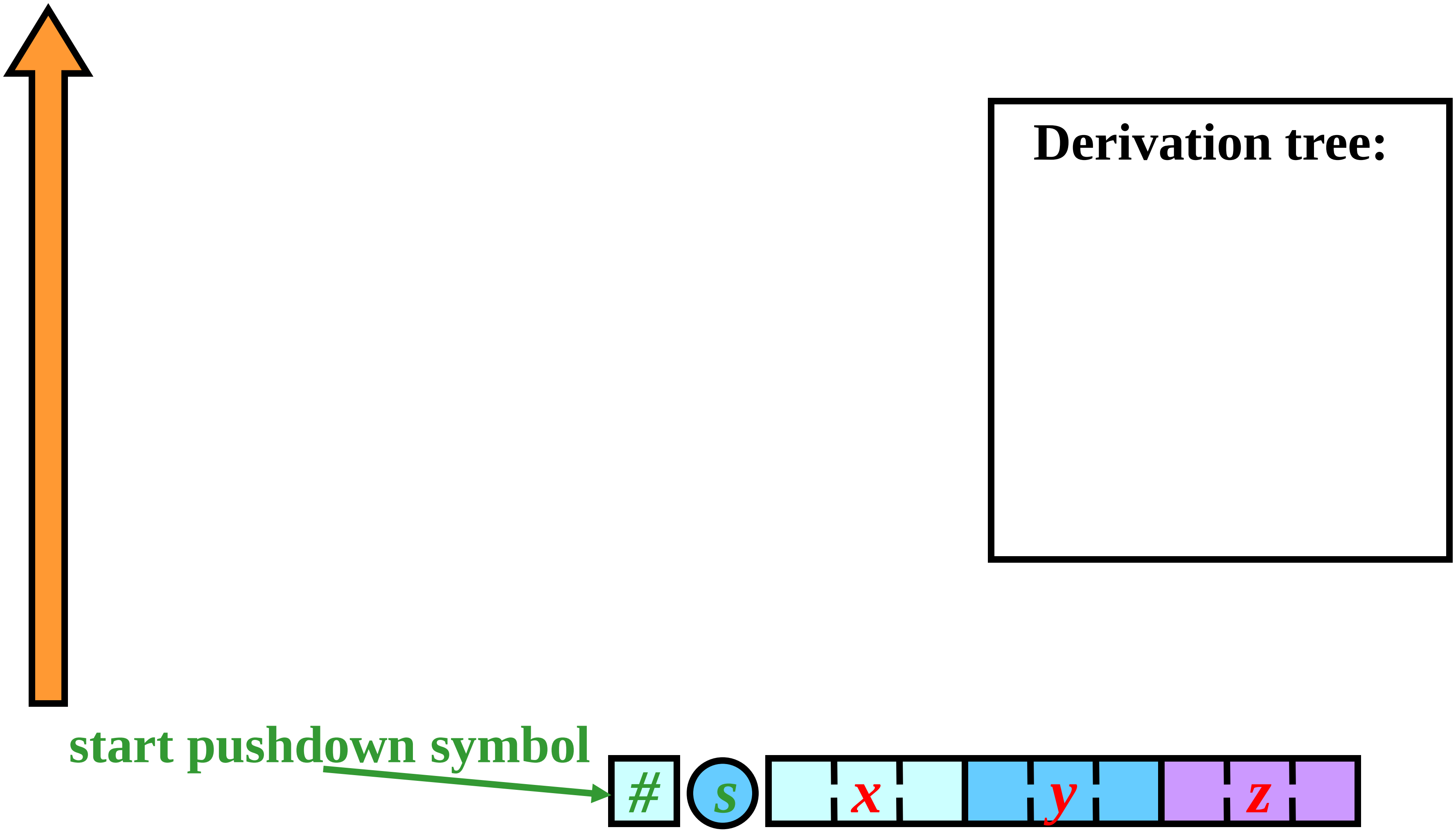
for every $A \rightarrow x \in P$ in G:

add $xs \rightarrow As$ to R;

3) *M* also contains the rule $\#Ss \rightarrow f$ that takes *M* to a final state f

EPDAs as Models of Bottom-Up Parsers 2/2

Bottom-up construction of a derivation tree:



EPDAs as Models of Bottom-Up Parsers 2/2

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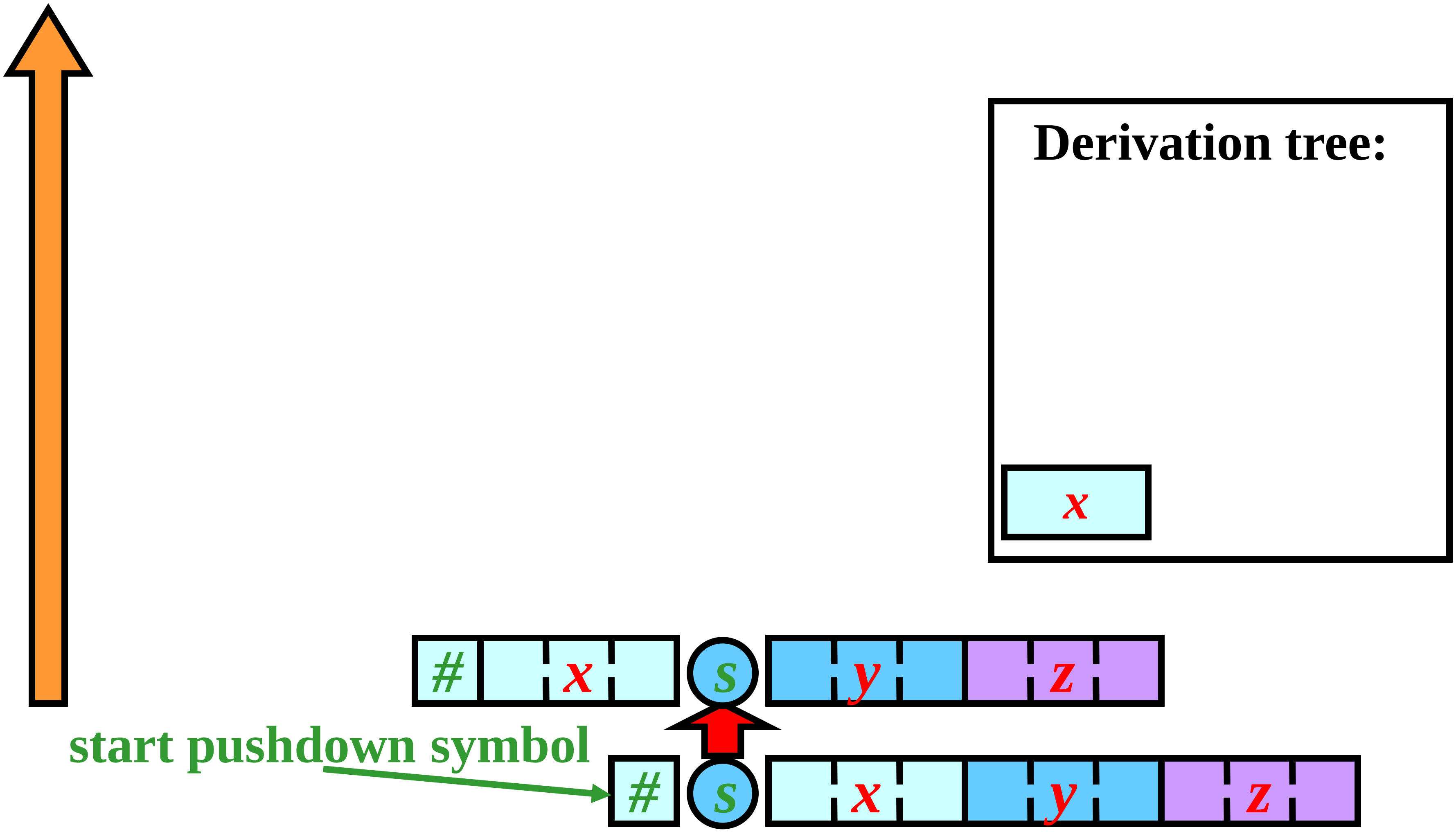

EPDAs as Models of Bottom-Up Parsers 2/2

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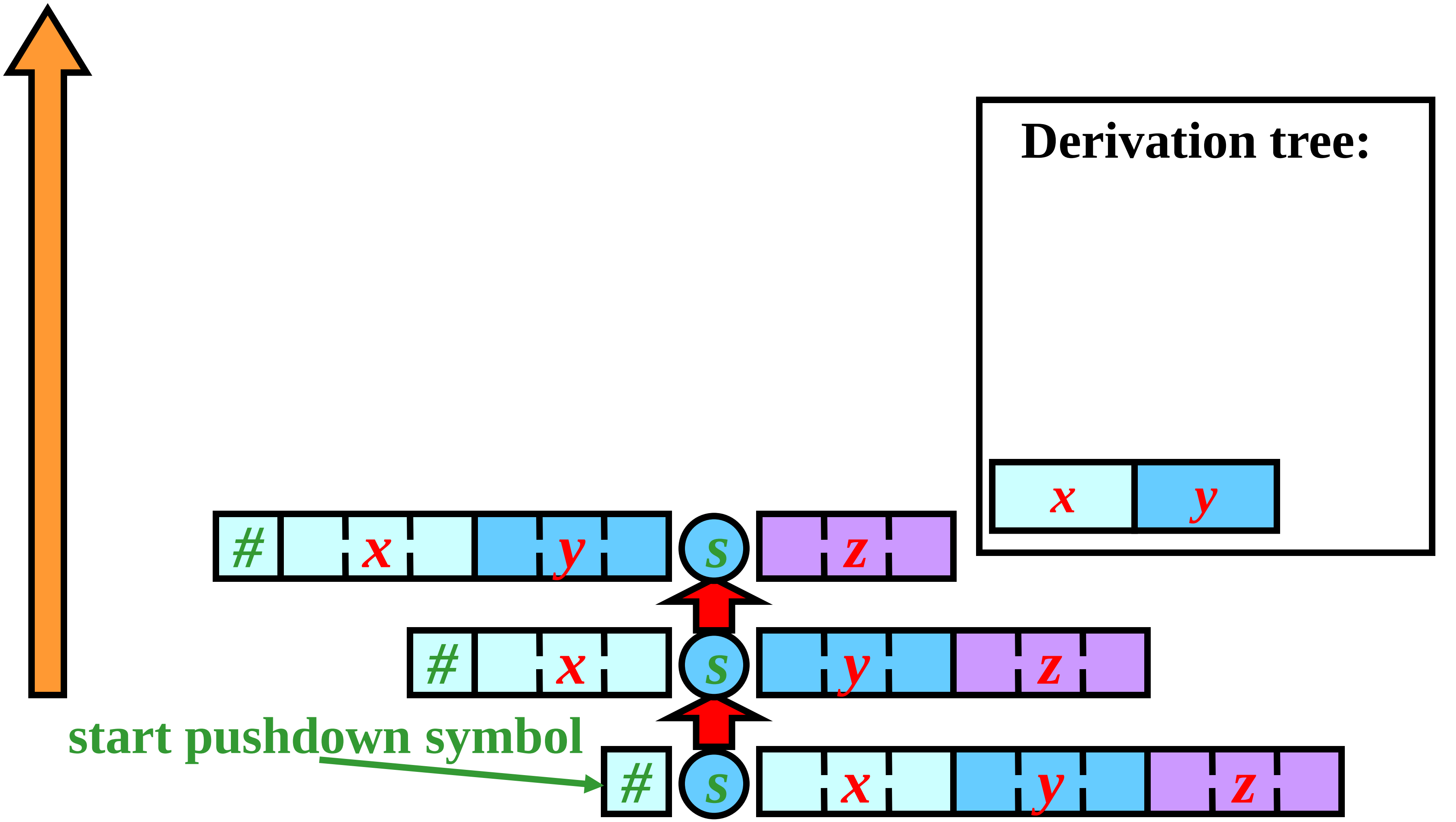

EPDAs as Models of Bottom-Up Parsers 2/2

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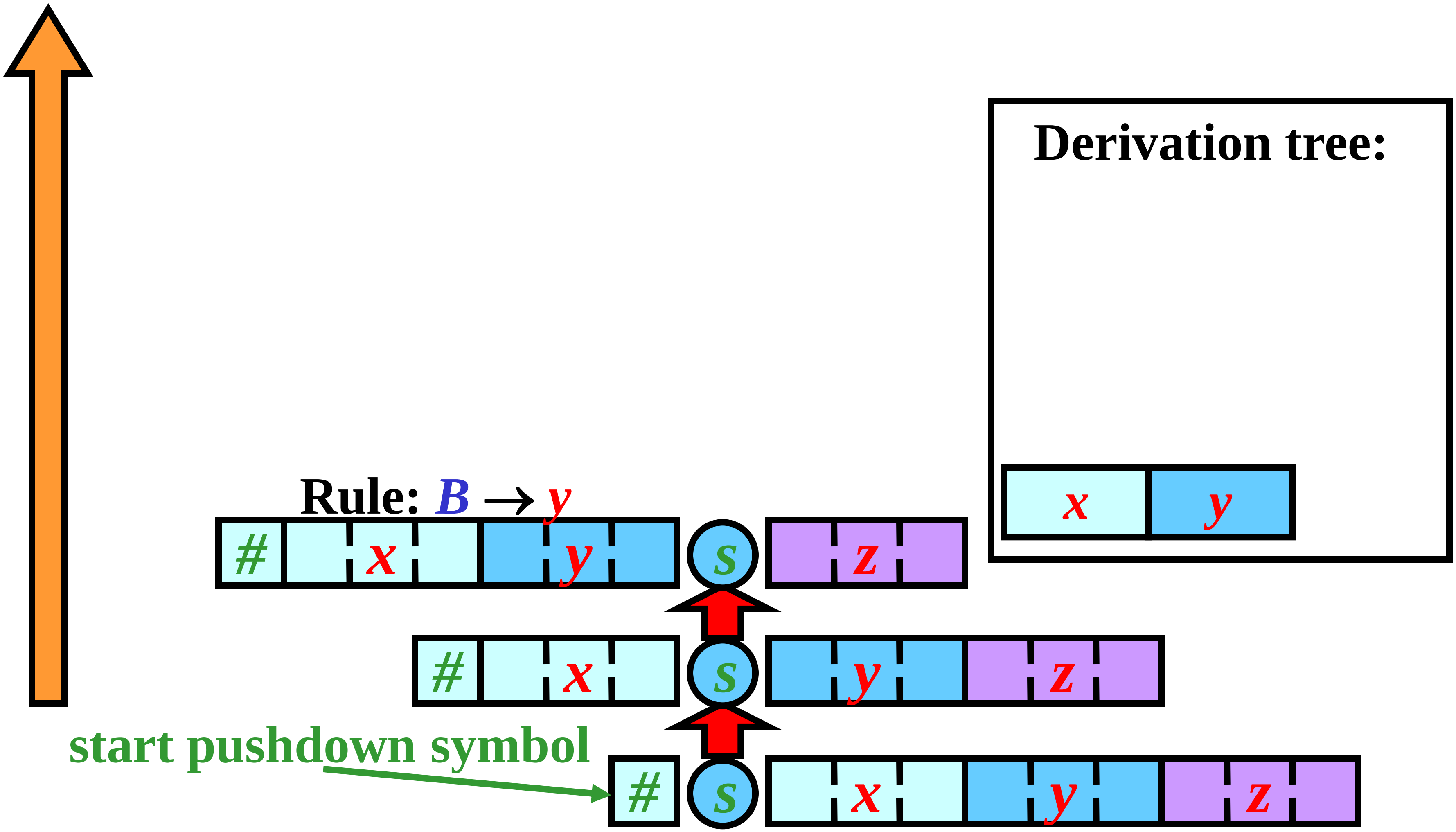

EPDAs as Models of Bottom-Up Parsers 2/2

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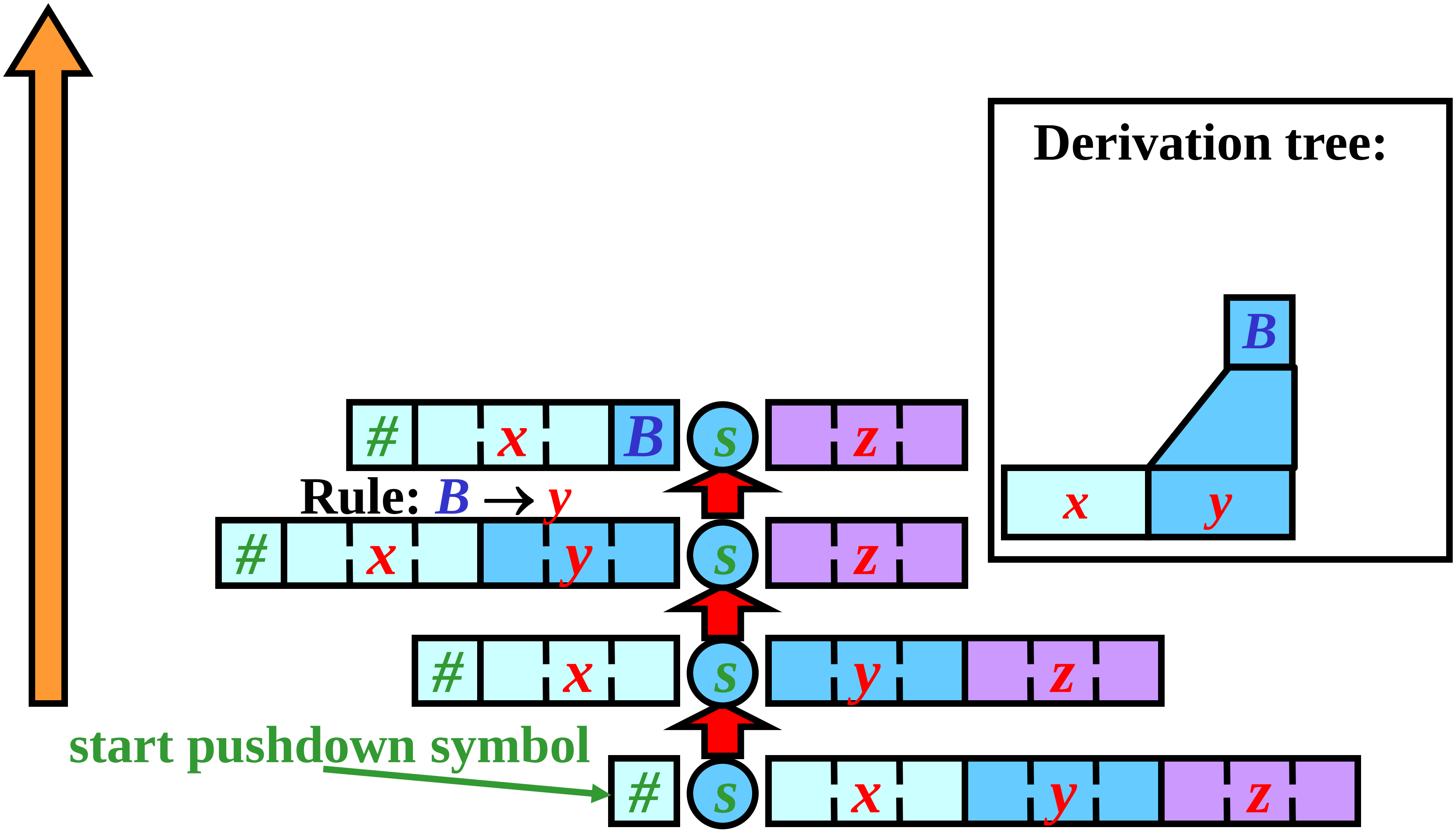
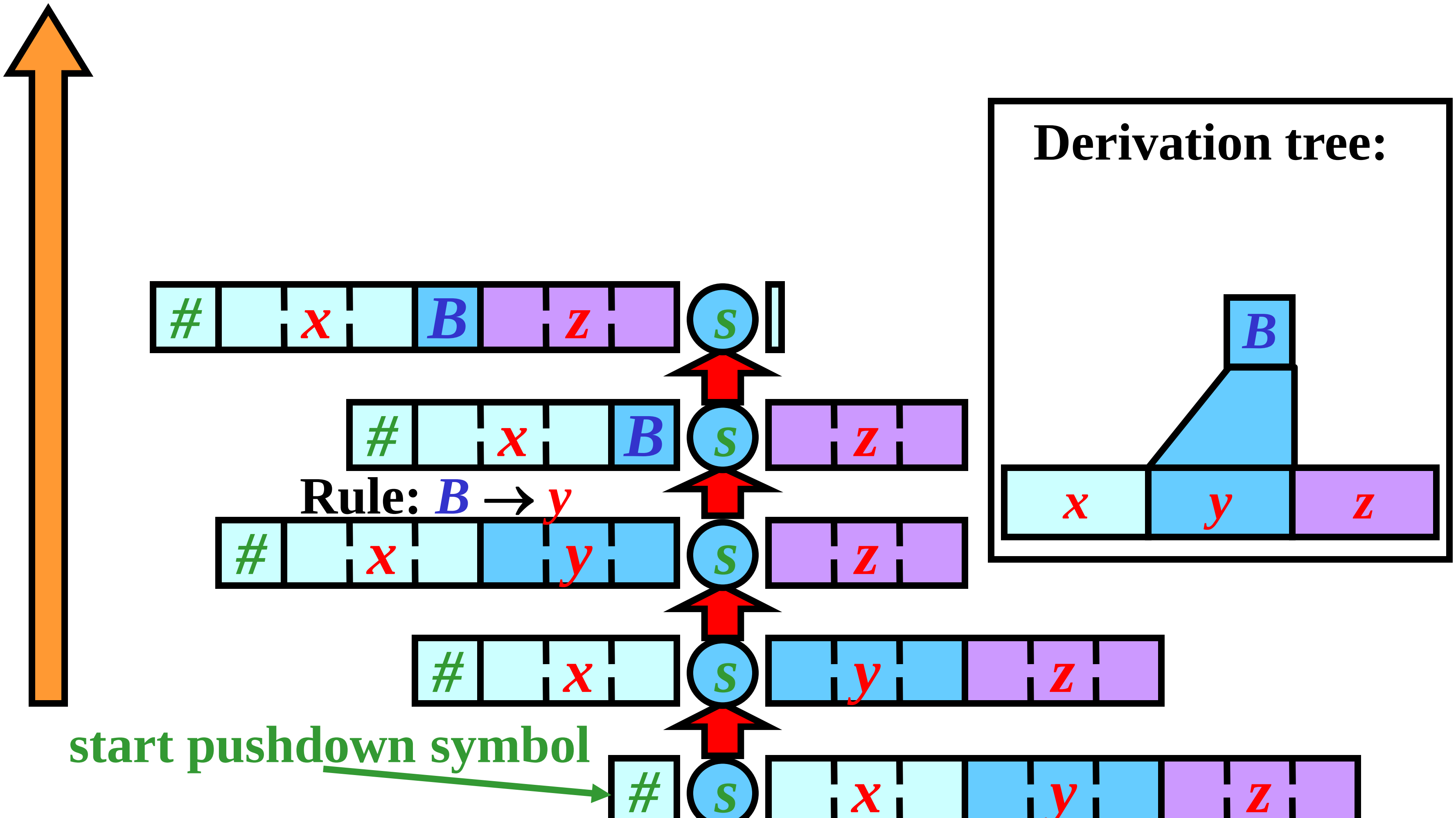


EPDAs as Models of Bottom-Up Parsers 2/2

Bottom-up construction of a derivation tree:



EPDAs as Models of Bottom-Up Parsers 2/2

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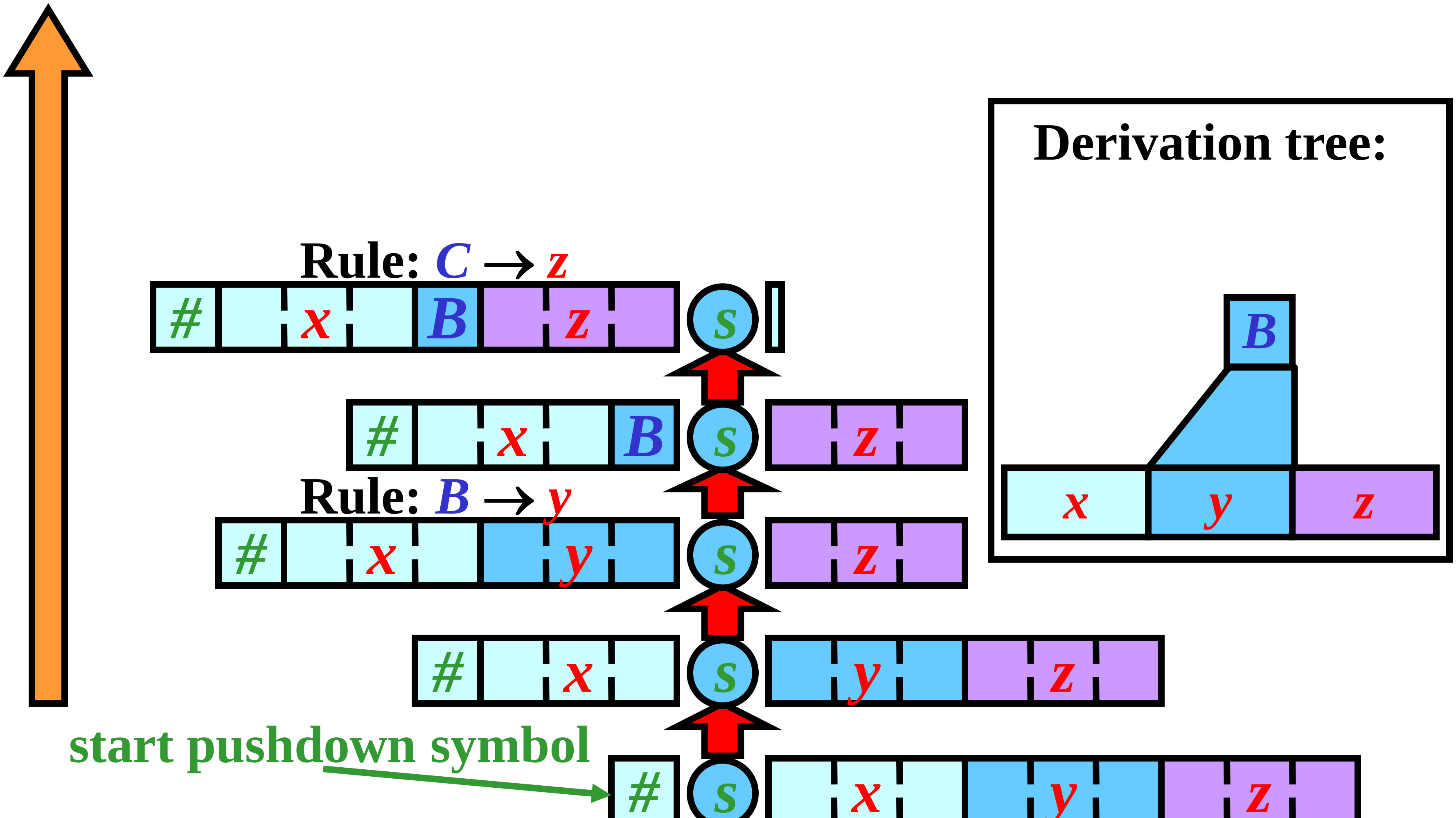

EPDAs as Models of Bottom-Up Parsers 2/2

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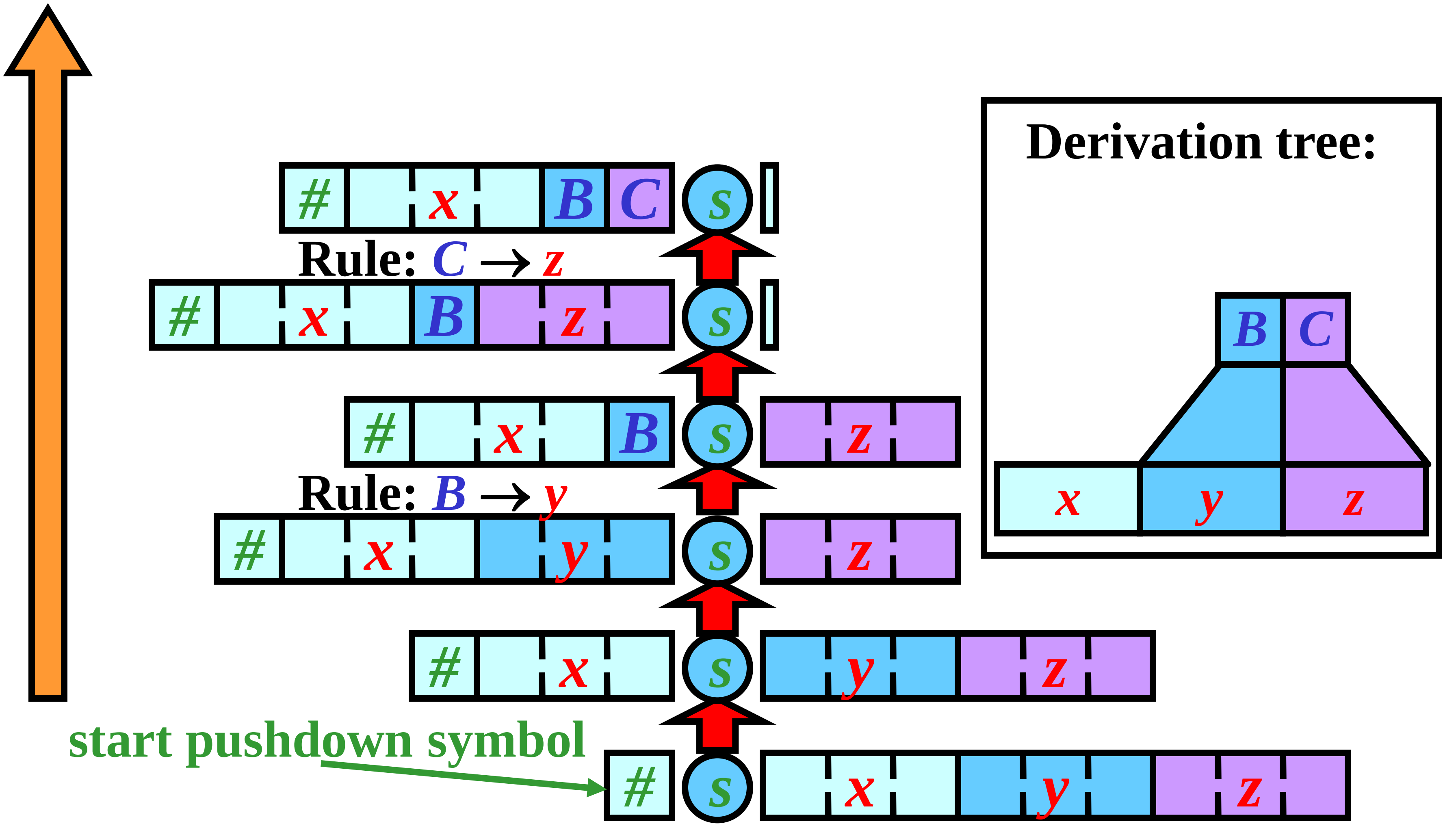

EPDAs as Models of Bottom-Up Parsers 2/2

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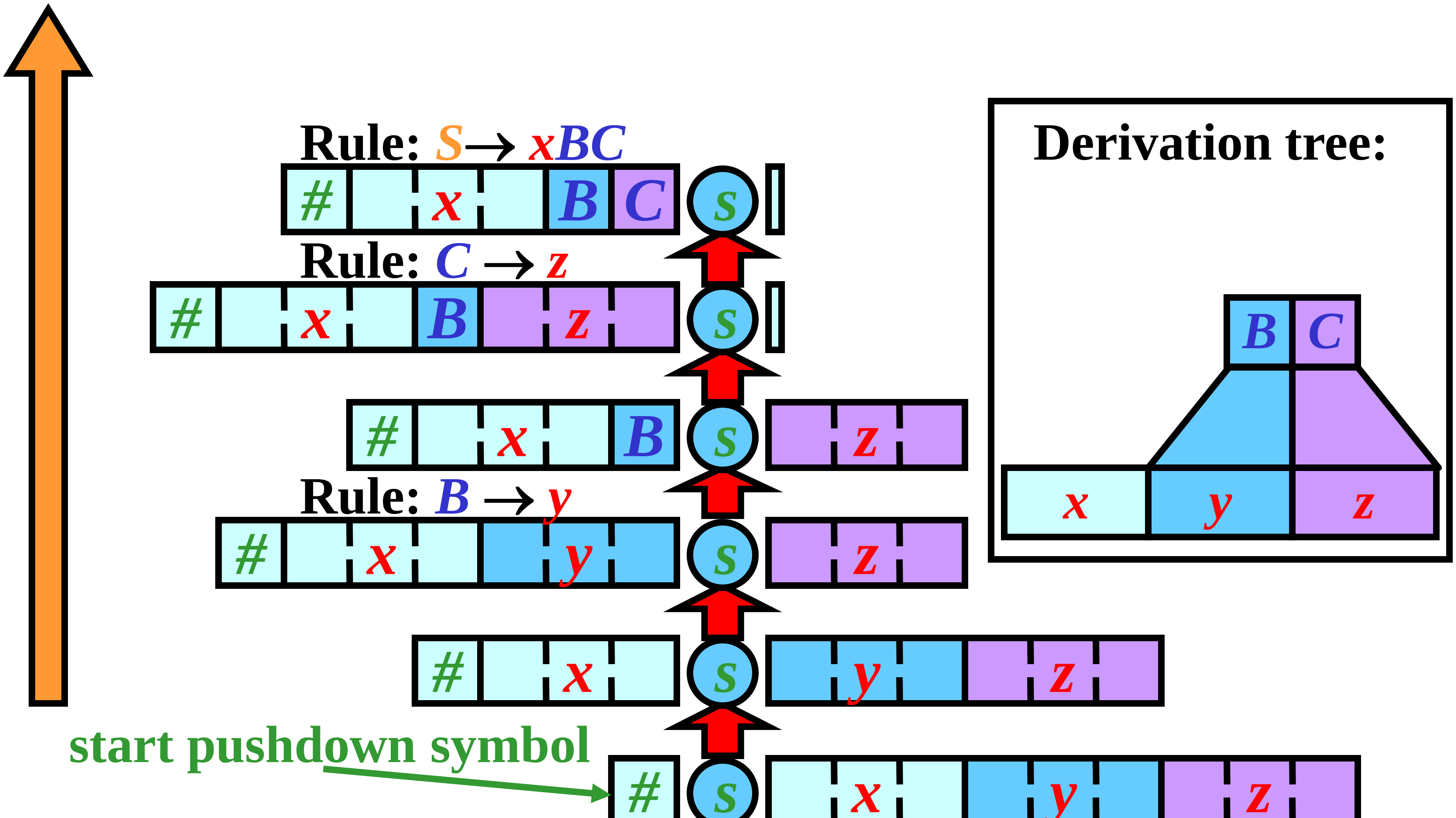

EPDAs as Models of Bottom-Up Parsers 2/2

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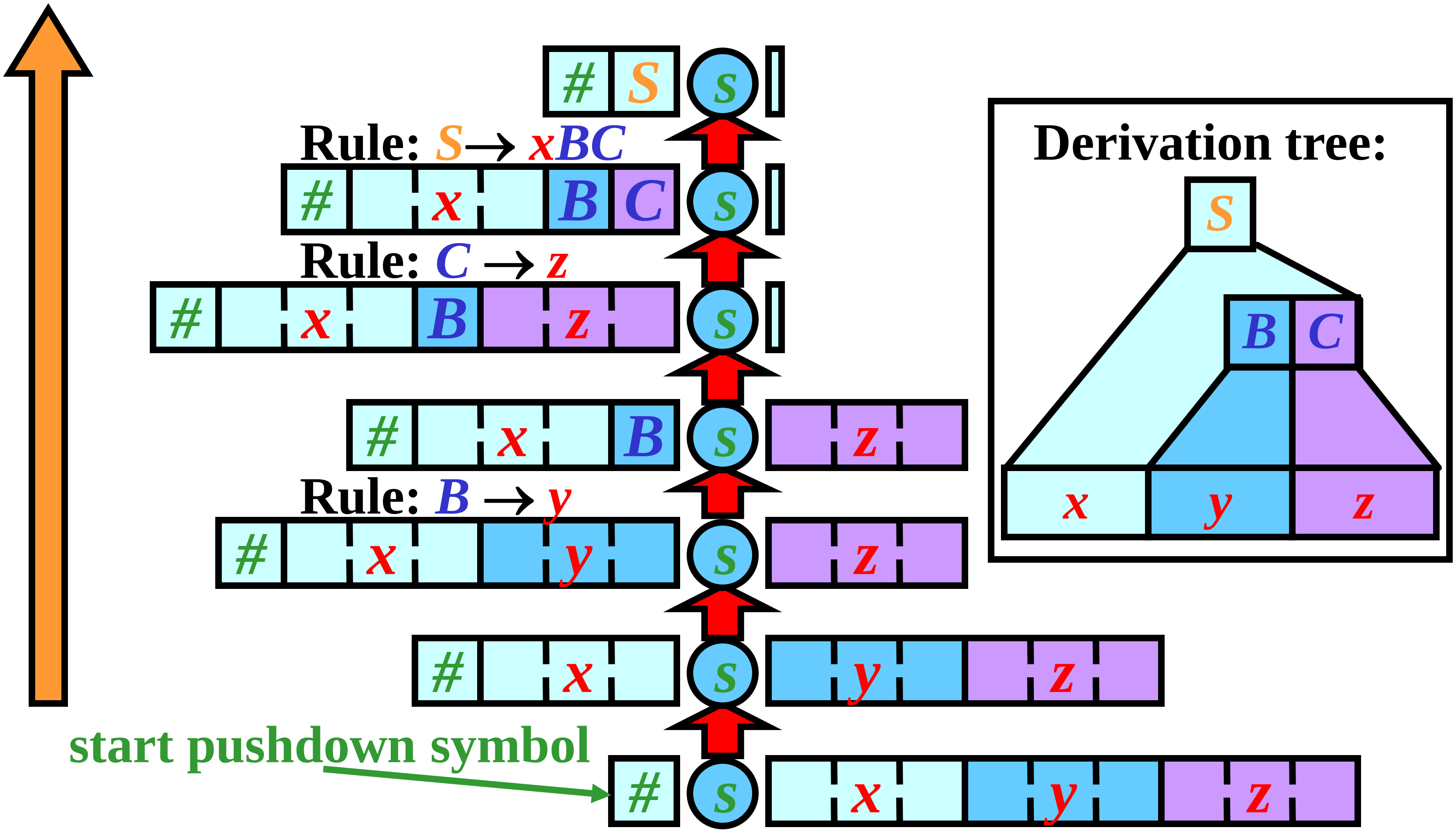
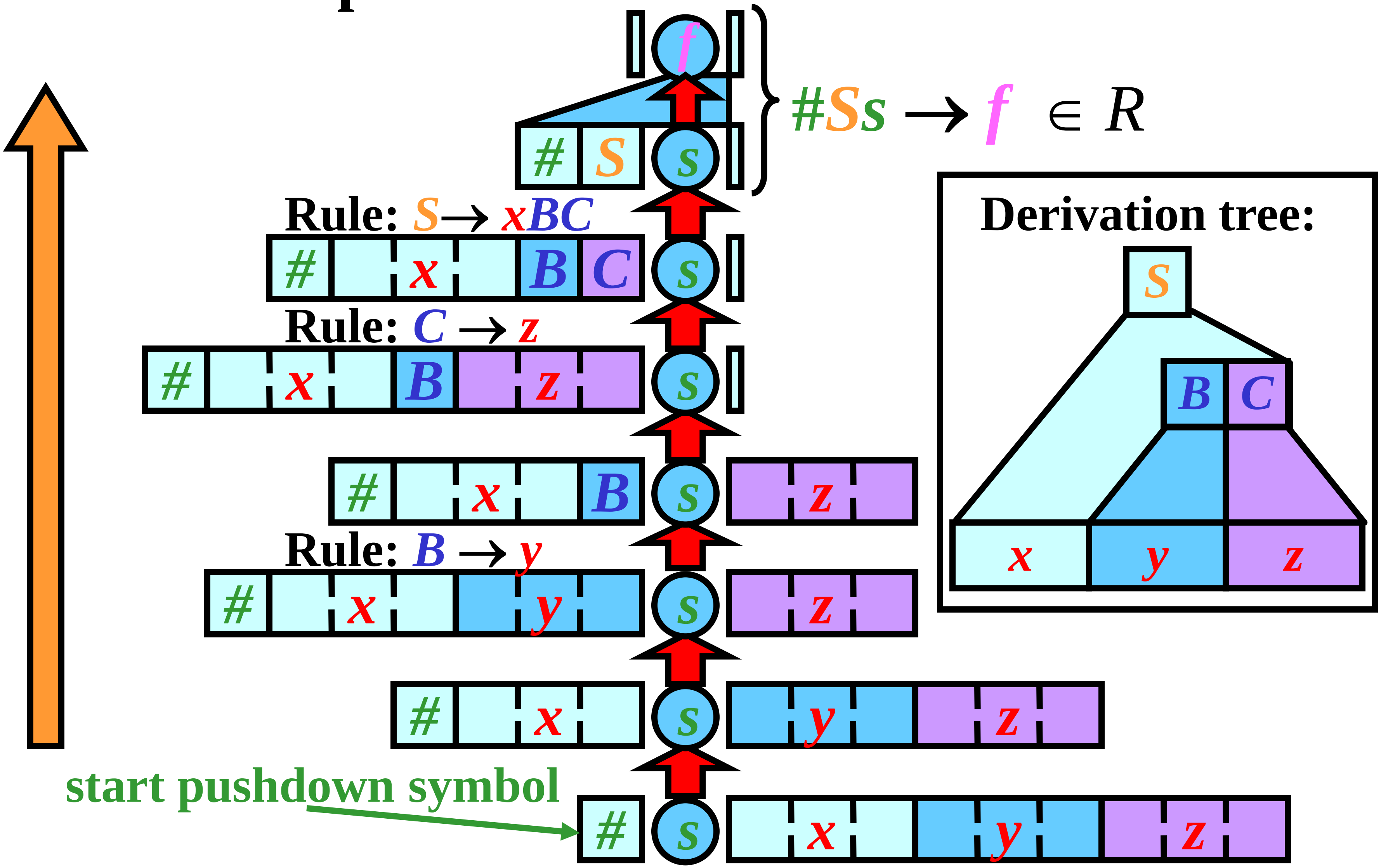


EPDAs as Models of Bottom-Up Parsers 2/2

Bottom-up construction of a derivation tree:



Algorithm: From CFG to EPDA

- **Input:** CFG $G = (N, T, P, S)$
- **Output:** EPDA $M = (Q, \Sigma, \Gamma, R, \mathbf{s}, \#, F)$; $L(G) = L(M)_f$

- **Method:**
- $Q := \{\mathbf{s}, f\}$;
- $\Sigma := T$;
- $\Gamma := N \cup T \cup \{\#\}$;
- Construction of R:
    - for every $a \in \Sigma$, add $\mathbf{s}a \rightarrow a\mathbf{s}$ to R;
    - for every $A \rightarrow x \in P$, add $x\mathbf{s} \rightarrow A\mathbf{s}$ to R;
    - add $\#S\mathbf{s} \rightarrow f$ to R;
- $F := \{f\}$;

From CFG to EPDA: Example 1/2

- $G = (N, T, P, S)$, where:

  $N = \{S\}$, $T = \{(,)\}$, $P = \{S \rightarrow (S), S \rightarrow ( )\}$

  Objective: An EPDA M such that $L(G) = L(M)_f$

$M = (Q, \Sigma, \Gamma, R, s, \#, F)$ where:

From CFG to EPDA: Example 1/2

- $G = (N, T, P, \mathbf{S})$, where:

$N = \{\mathbf{S}\}$, $T = \{\mathbf{(}, \mathbf{)}\}$, $P = \{\mathbf{S} \rightarrow \mathbf{(S)}, \mathbf{S} \rightarrow \mathbf{(\ )}\}$

Objective: An EPDA M such that $L(G) = L(M)_f$

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From CFG to EPDA: Example 1/2

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"(" $\in T$

$\Downarrow$

$R = \{s(\rightarrow (s,$

From CFG to EPDA: Example 1/2

- $G = (N, T, P, \boldsymbol{S})$, where:

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"(" $\in$ ***T*** **")"** $\in$ ***T***

$R = \{\mathbf{s(} \rightarrow \mathbf{(s}$, $\mathbf{s)} \rightarrow \mathbf{)s}$,

From CFG to EPDA: Example 1/2

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Objective: An EPDA M such that $L(G) = L(M)_f$

$M = (Q, \Sigma, \Gamma, R, s, \#, F)$ where:

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"(" $\in T$ "$)$" $\in T$ $\underbrace{S}$ $\rightarrow$ $\underbrace{(S)}$ $\in P$

$R = \{s(\rightarrow (s,\ \ s) \rightarrow)s,\ \ (S)s \rightarrow Ss,$

From CFG to EPDA: Example 1/2

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“**(**” $\in$ ***T*** “**)**” $\in$ ***T*** $\mathbf{S} \rightarrow \mathbf{(S)} \in \boldsymbol{P}$ $\mathbf{S} \rightarrow \mathbf{(\ )} \in \boldsymbol{P}$

$R = \{\mathbf{s(} \rightarrow \mathbf{(s}$, $\mathbf{s)} \rightarrow \mathbf{)s}$, $\mathbf{(S)s} \rightarrow \mathbf{Ss}$, $\mathbf{(\ )s} \rightarrow \mathbf{Ss}$,

From CFG to EPDA: Example 1/2

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"(" $\in T$ ")" $\in T$ $S \to (S) \in P$ $S \to () \in P$

$$R = \{\underbrace{s(\to (s,\ \ s) \to)s,}_{\textbf{shift rules}}\ \ \underbrace{(S)s \to Ss,\ \ ()s \to Ss,}_{\textbf{reduction rules}}\ \ \#Ss \to f\}$$

From CFG to EPDA: Example 1/2

- $G = (N, T, P, S)$, where:

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“(” $\in T$ “)” $\in T$ $S \to (S) \in P$ $S \to ( ) \in P$

$$R = \{\underbrace{s(\to (s,\quad s) \to)s,}_{\textbf{shift rules}}\quad \underbrace{(S)s \to Ss,\quad ( )s \to Ss,}_{\textbf{reduction rules}}\quad \#Ss \to f\ \}$$

$F = \{f\}$

From CFG to EPDA: Example 2/2

$M = (Q, \Sigma, \Gamma, R, s, \#, F)$, where:

$Q = \{s, f\}$, $\Sigma = T = \{(,)\}$, $\Gamma = \{(,), S, \#\}$, $F = \{f\}$

$R = \{s(\rightarrow (s,\ s) \rightarrow)s,\ (S)s \rightarrow Ss,\ ()s \rightarrow Ss,\ \#Ss \rightarrow f\}$

Question: $(()) \in L(M)_f$**?**

#	s	(	(	)	)

From CFG to EPDA: Example 2/2

$M = (Q, \Sigma, \Gamma, R, s, \#, F)$, where:

$Q = \{s, f\}$, $\Sigma = T = \{(,)\}$, $\Gamma = \{(,), S, \#\}$, $F = \{f\}$

$R = \{s(\to (s,\ s) \to)s,\ (S)s \to Ss,\ ()s \to Ss,\ \#Ss \to f\}$

Question: $(())\in L(M)_f$?

#	s	(	(	)	)

Rule: $s(\to (s$

#	(	s	(	)	)

(

From CFG to EPDA: Example 2/2

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Question: $(()) \in L(M)_f$?

\# s (( ))

Rule: $s(\rightarrow (s$

\# (s ( )) (

Rule: $s(\rightarrow (s$

\# ((s)) ((

From CFG to EPDA: Example 2/2

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s (( ))

Rule: $s(\rightarrow (s$

(s ( )) (

Rule: $s(\rightarrow (s$

((s)) ((

Rule: $s) \rightarrow)s$

(( ) s) (( )

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Question: $(())\ \in L(M)_f$?

\# **s** (( ))

Rule: $s(\rightarrow (s$

\# (**s** ( ))   (

Rule: $s(\rightarrow (s$

\# ((**s**))   ((

Rule: $s) \rightarrow)s$

\# (( ) **s**)   (( )

Rule: $( )s \rightarrow S$

\# (S **s**)   S ← (( )

From CFG to EPDA: Example 2/2

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Question: $(( )) \in L(M)_f$?

#	s	(	(	)	)

Rule: $s(\rightarrow (s$

#	(	s	(	)	)

(

Rule: $s(\rightarrow (s$

#	(	(	s	)	)

((

Rule: $s) \rightarrow)s$

#	(	(	)	s	)

(( )

Rule: $( )s \rightarrow Ss$

#	(	S	s	)

S

(( )

Rule: $s) \rightarrow)s$

#	(	S	)	s

S

(( ))

From CFG to EPDA: Example 2/2

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Question: $(()) \in L(M)_f$?

\# s (( ))

Rule: $s(\rightarrow (s$

\# (s ( ))   (

Rule: $s(\rightarrow (s$

\# ((s))   ((

Rule: $s) \rightarrow)s$

\# (( ) s)   (( )

Rule: $()s \rightarrow Ss$

\# (S s)

S
(( )

Rule: $s) \rightarrow)s$

\# (S) s

S
(( ))

Rule: $(S)s \rightarrow Ss$

\# S s

S
S
(( ))

From CFG to EPDA: Example 2/2

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Question: $(())\in L(M)_f$?

$\# \; s \; (())$

Rule: $s(\rightarrow (s$

$\#(\; s \; ())$ $\qquad ($

Rule: $s(\rightarrow (s$

$\#((\; s \;))$ $\qquad (($

Rule: $s) \rightarrow)s$

$\#(() \; s \;)$ $\qquad (()$

Rule: $()s \rightarrow S$

$\#(S \; s \;)$

Rule: $s) \rightarrow)s$

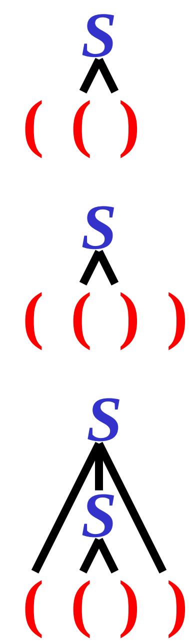
$\#(S) \; s$

Rule: $(S)s \rightarrow S$

$\#S \; s$

Rule: $\#Ss \rightarrow f$

f



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$R = \{s(\rightarrow (s,\ s) \rightarrow)s,\ (S)s \rightarrow Ss,\ ()s \rightarrow Ss,\ \#Ss \rightarrow f\ \}$

Question: $(())\ \in L(M)_f$?

$\#\ s\ (\ (\ )\ )$

Rule: $s(\rightarrow (s$

$\#\ (\ s\ (\ )\ )$

(

Rule: $s(\rightarrow (s$

$\#\ (\ (\ s\ )\ )$

((

Rule: $s) \rightarrow)s$

$\#\ (\ (\ )\ s\ )$

(( )

Rule: $()s \rightarrow S$

$\#\ (\ S\ s\ )$

S

(( )

Rule: $s) \rightarrow)s$

$\#\ (\ S\ )\ s$

S

(( ))

Rule: $(S) \rightarrow S$

$\#\ S\ s$

S

S

(( ))

Rule: $\#Ss \rightarrow f$

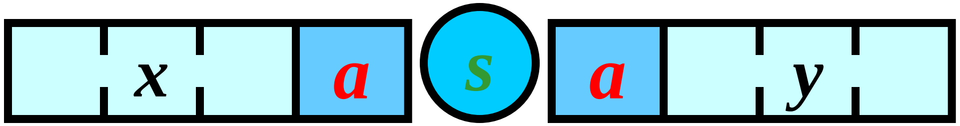
Final state f

Answer: YES

PDAs as Models of Top-Down Parsers 1/2

Gist: An PDA *M* underlies a top-down parser

1) *M* contains *popping* rules that pops the top symbol from the pushdown and reads the input symbol if both coincide:



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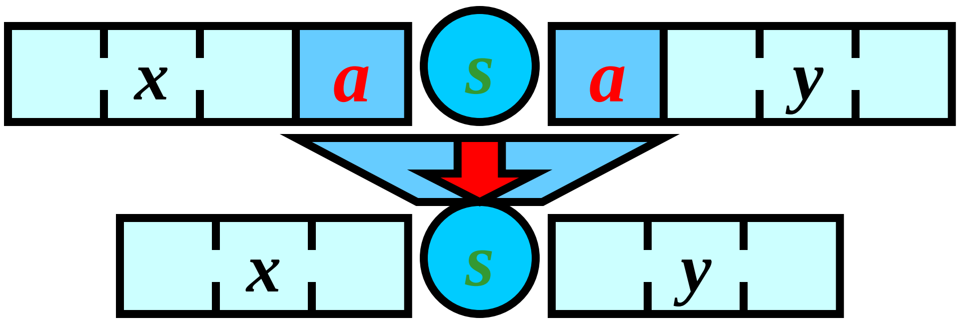

for every $a \in \Sigma$:

add $asa \rightarrow s$ to R;

PDAs as Models of Top-Down Parsers 1/2

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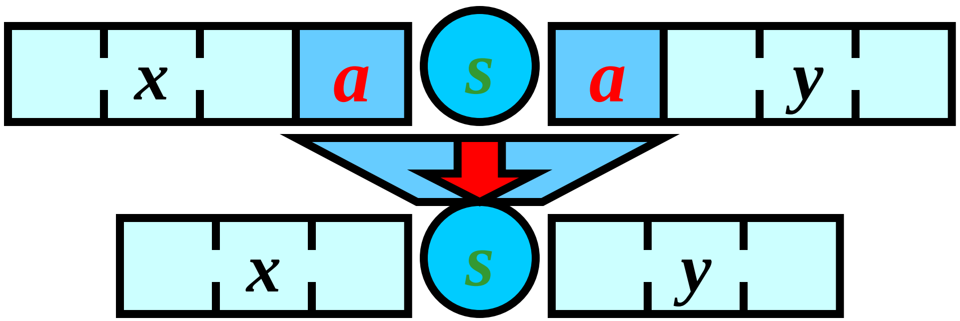
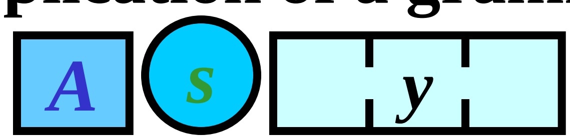


for every $a \in \Sigma$:

add $asa \rightarrow s$ to R;

2) *M* contains *expansion* rules that simulate the application of a grammatical rule:



PDAs as Models of Top-Down Parsers 1/2

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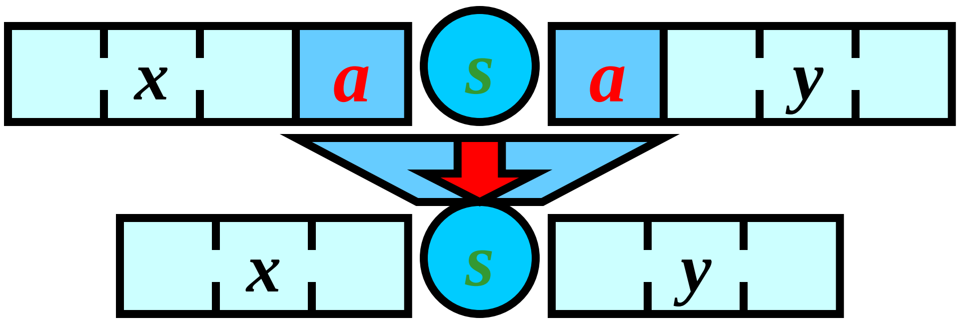

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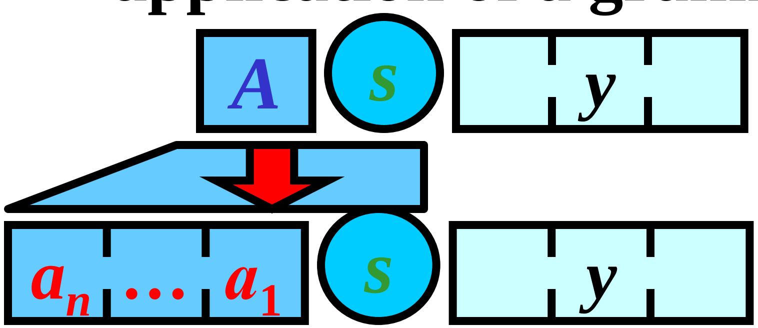
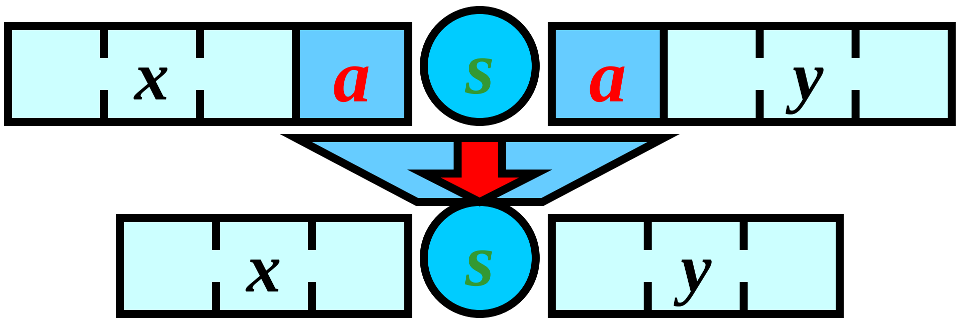


PDAs as Models of Top-Down Parsers 1/2

Gist: An PDA *M* underlies a <u>top-down</u> parser

---

1) *M* contains *popping* rules that pops the top symbol from the pushdown and reads the input symbol if both coincide:



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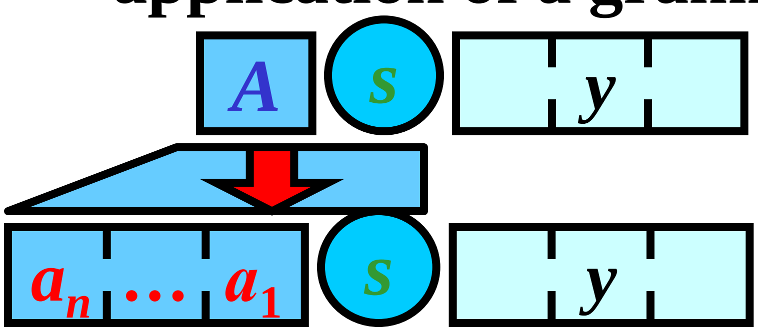


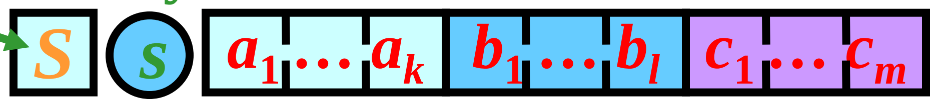
for every $A \rightarrow a_1 \ldots a_n \in P$ in G,

add $As \rightarrow \underbrace{a_n \ldots a_1}_{= \textbf{reversal}(a_1 \ldots a_n)} s$ to R;

PDAs as Models of Top-Down Parsers 2/2

Top-down construction of a derivation tree:

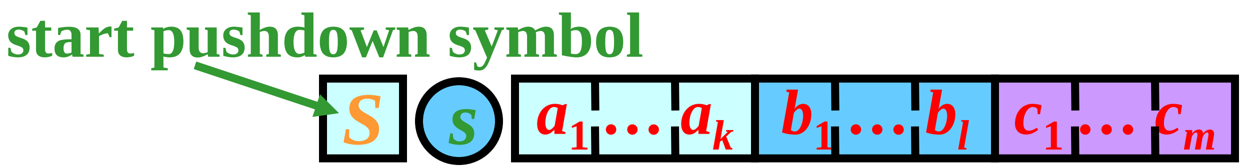
start pushdown symbol



Derivation tree:

PDAs as Models of Top-Down Parsers 2/2

Top-down construction of a derivation tree:

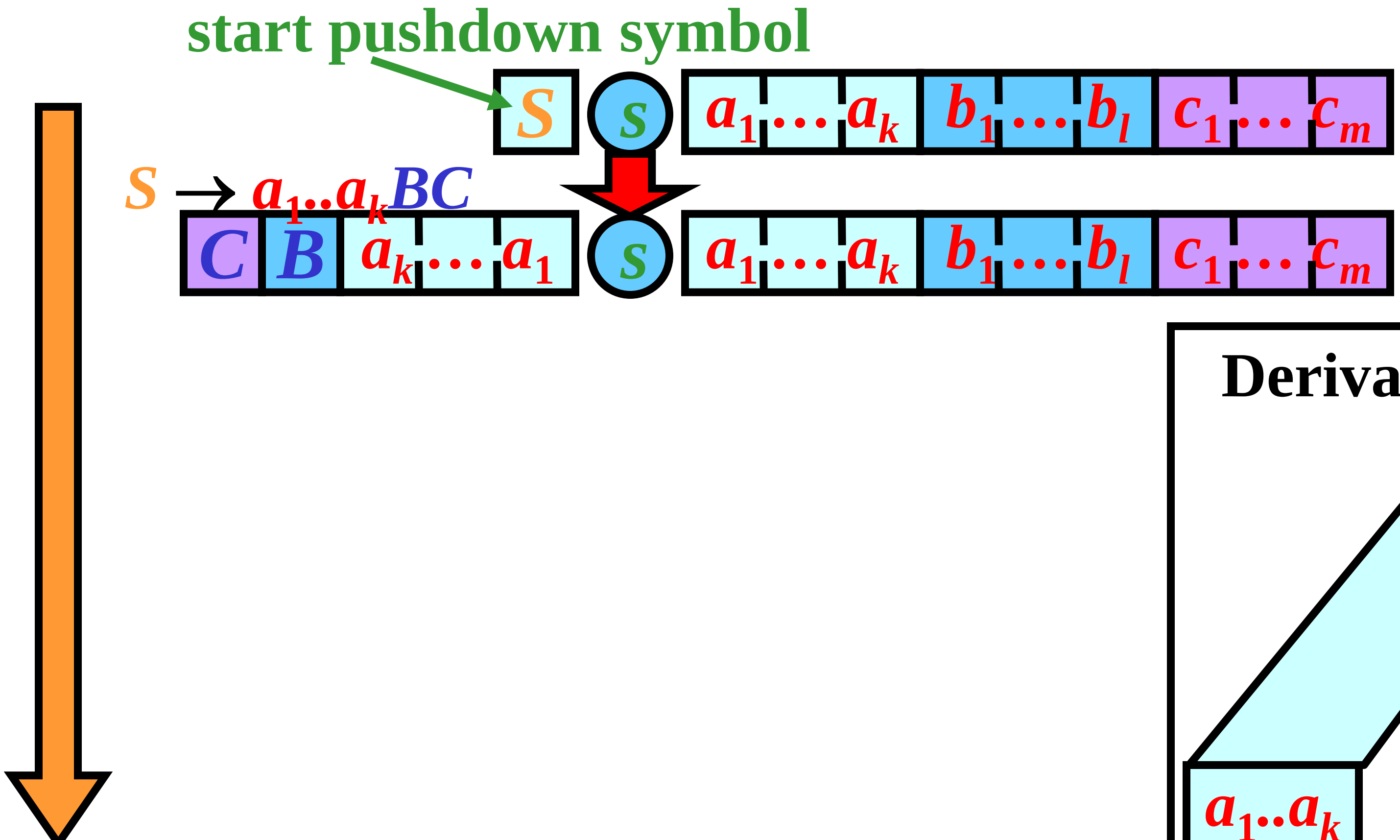


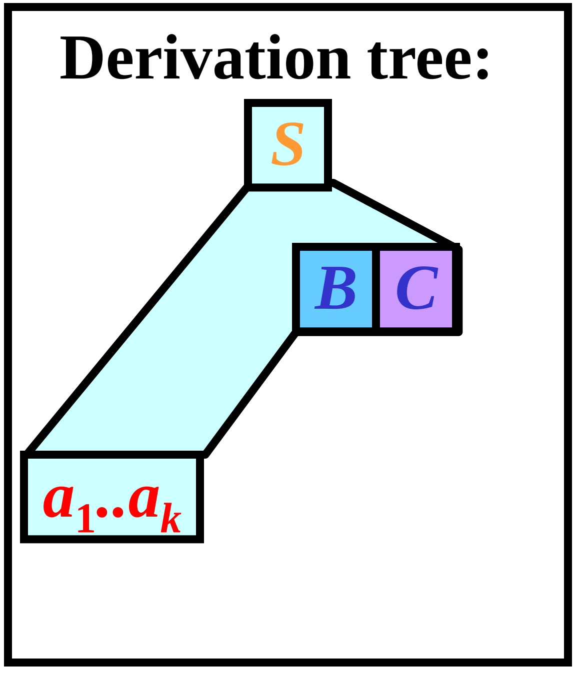
$S \rightarrow a_1..a_kBC$

Derivation tree:

PDAs as Models of Top-Down Parsers 2/2

Top-down construction of a derivation tree:





PDAs as Models of Top-Down Parsers 2/2

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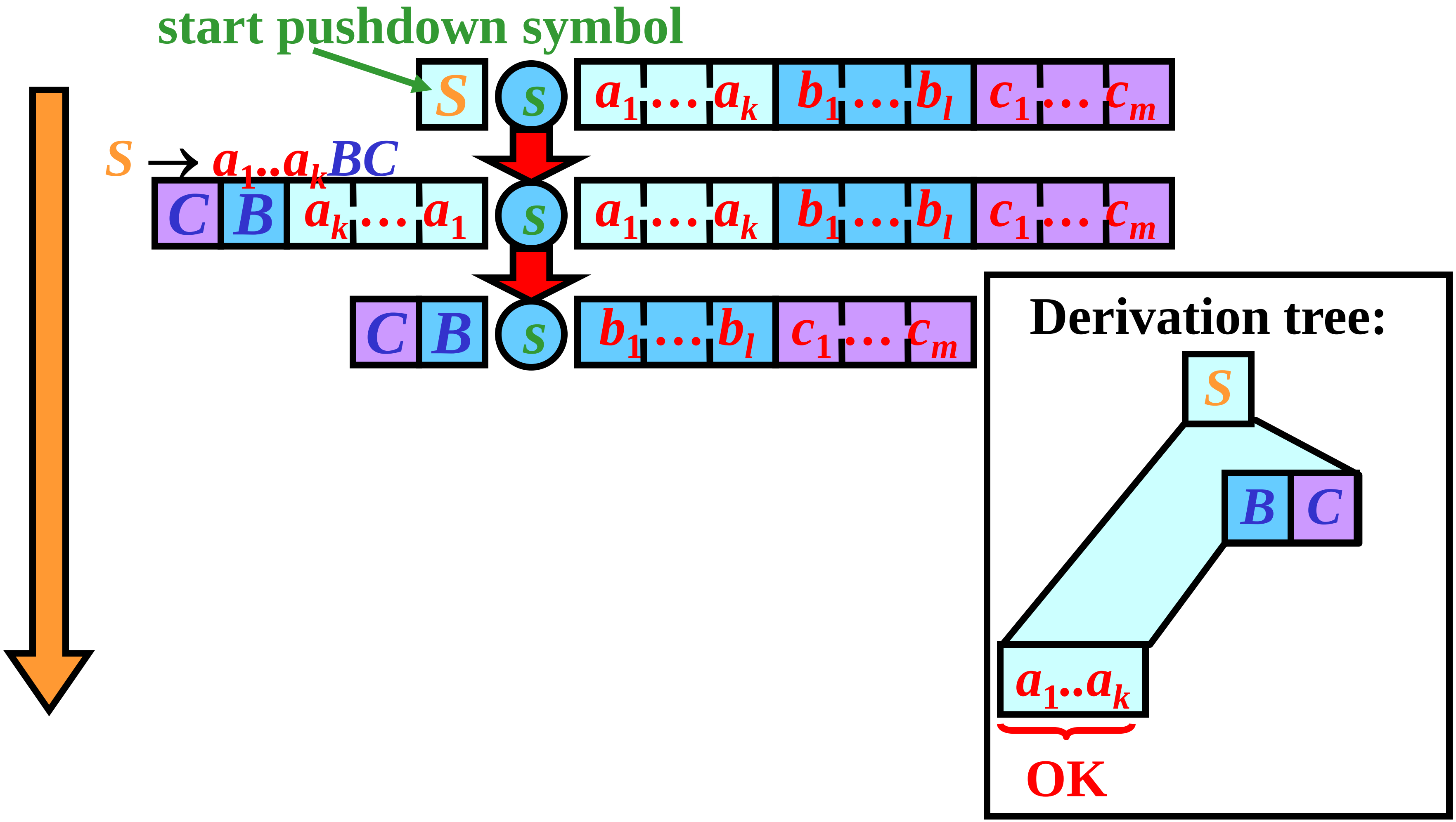

PDAs as Models of Top-Down Parsers 2/2

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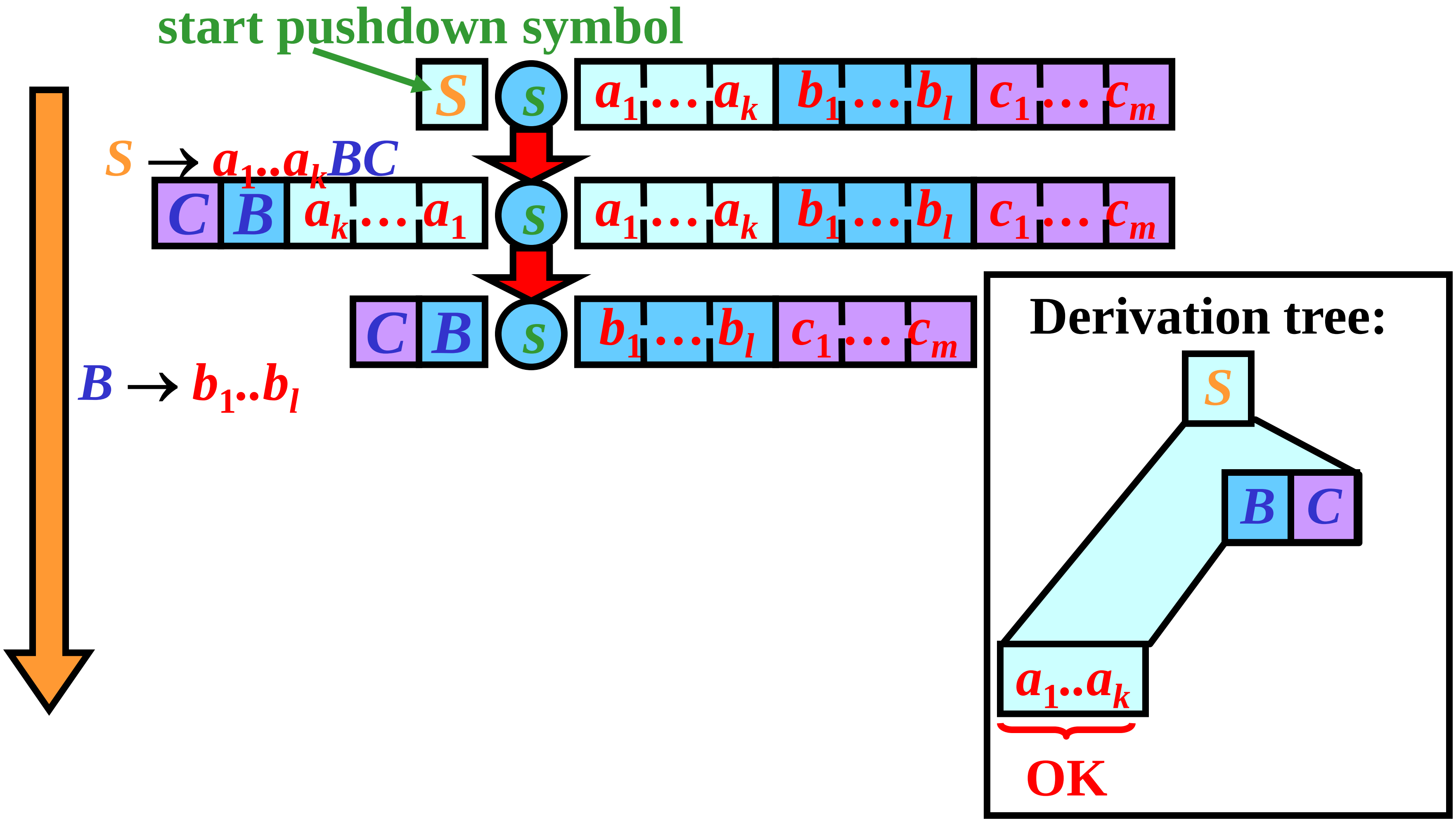
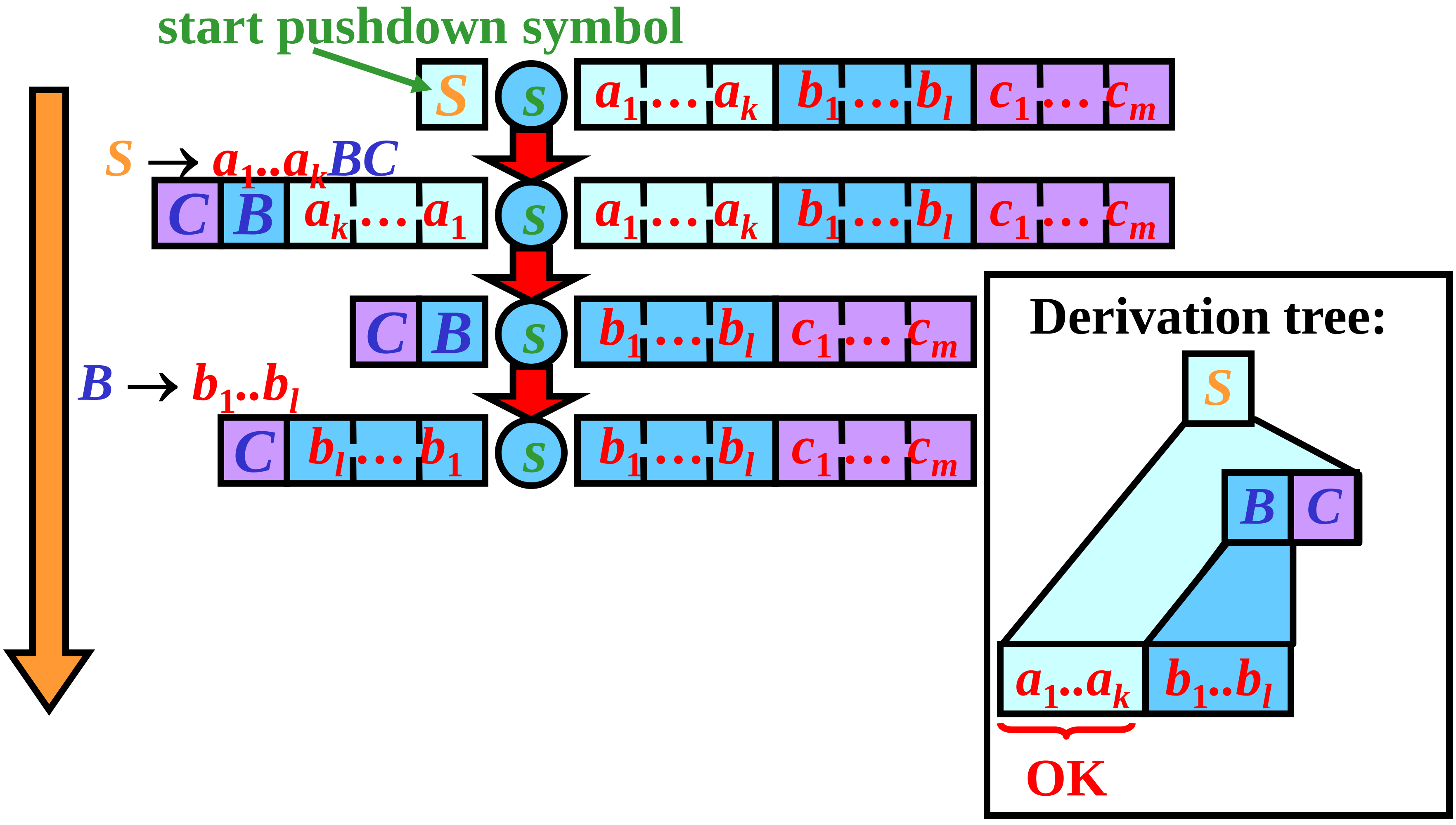


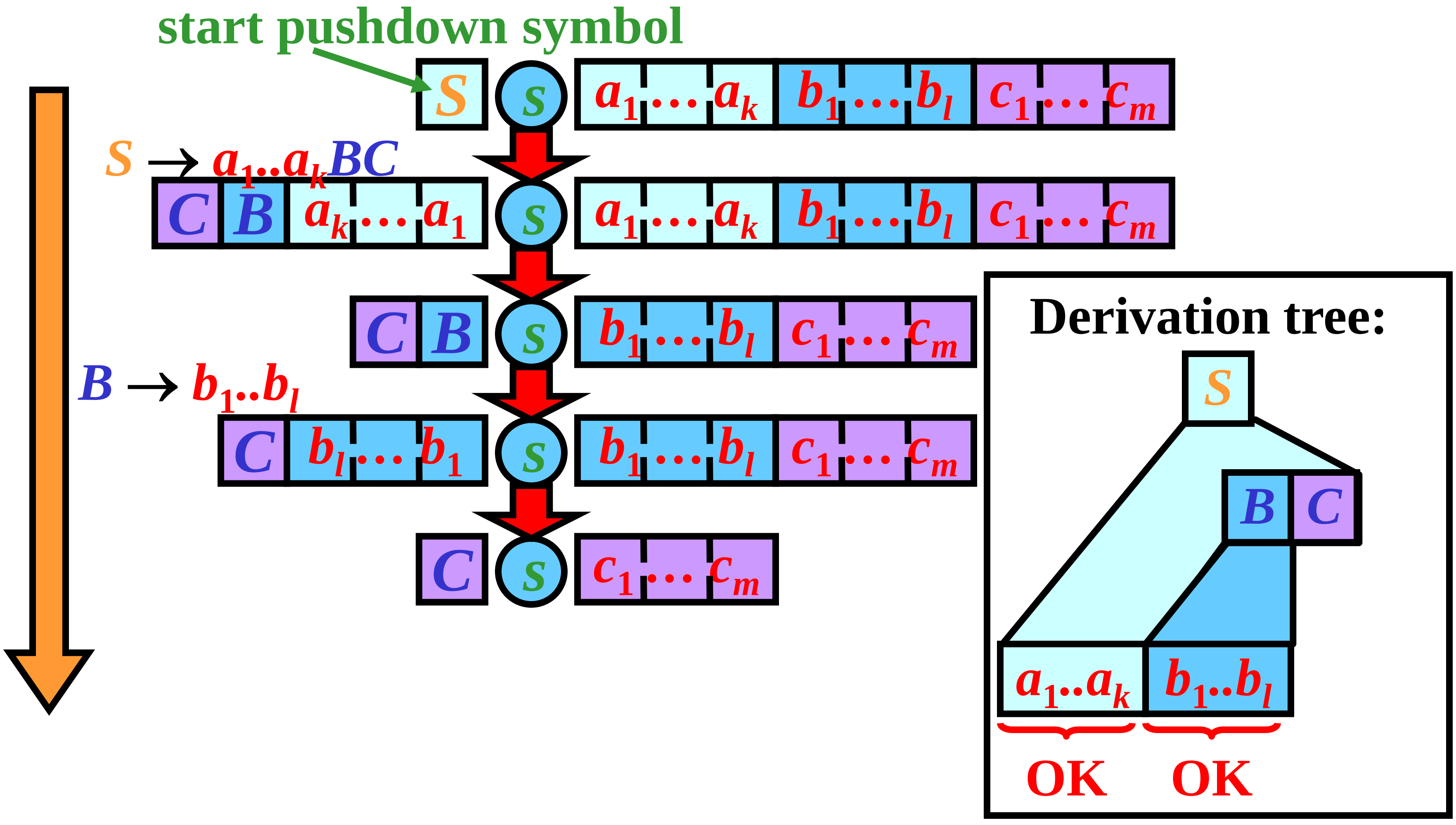
PDAs as Models of Top-Down Parsers 2/2

Top-down construction of a derivation tree:



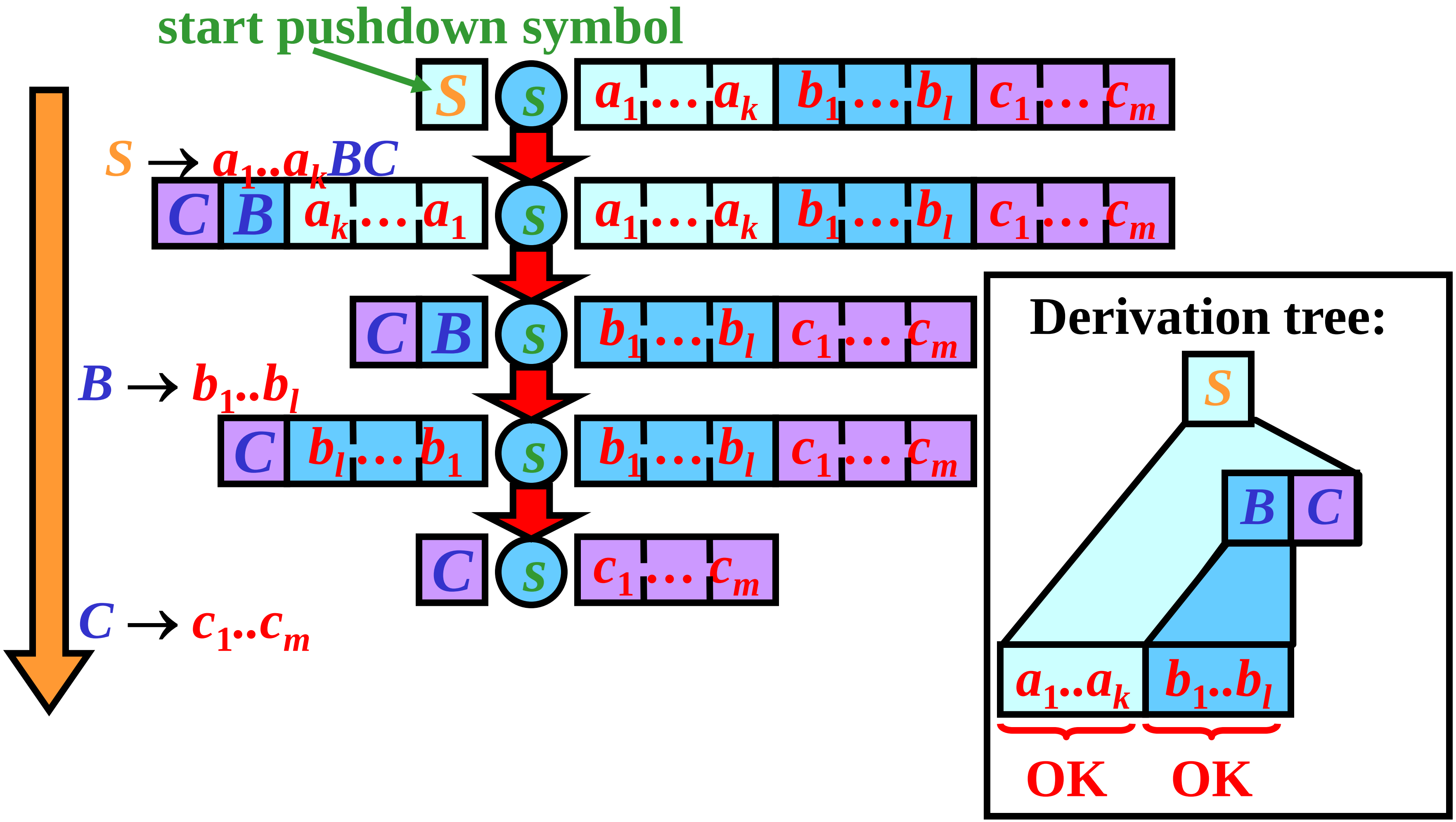
PDAs as Models of Top-Down Parsers 2/2

Top-down construction of a derivation tree:



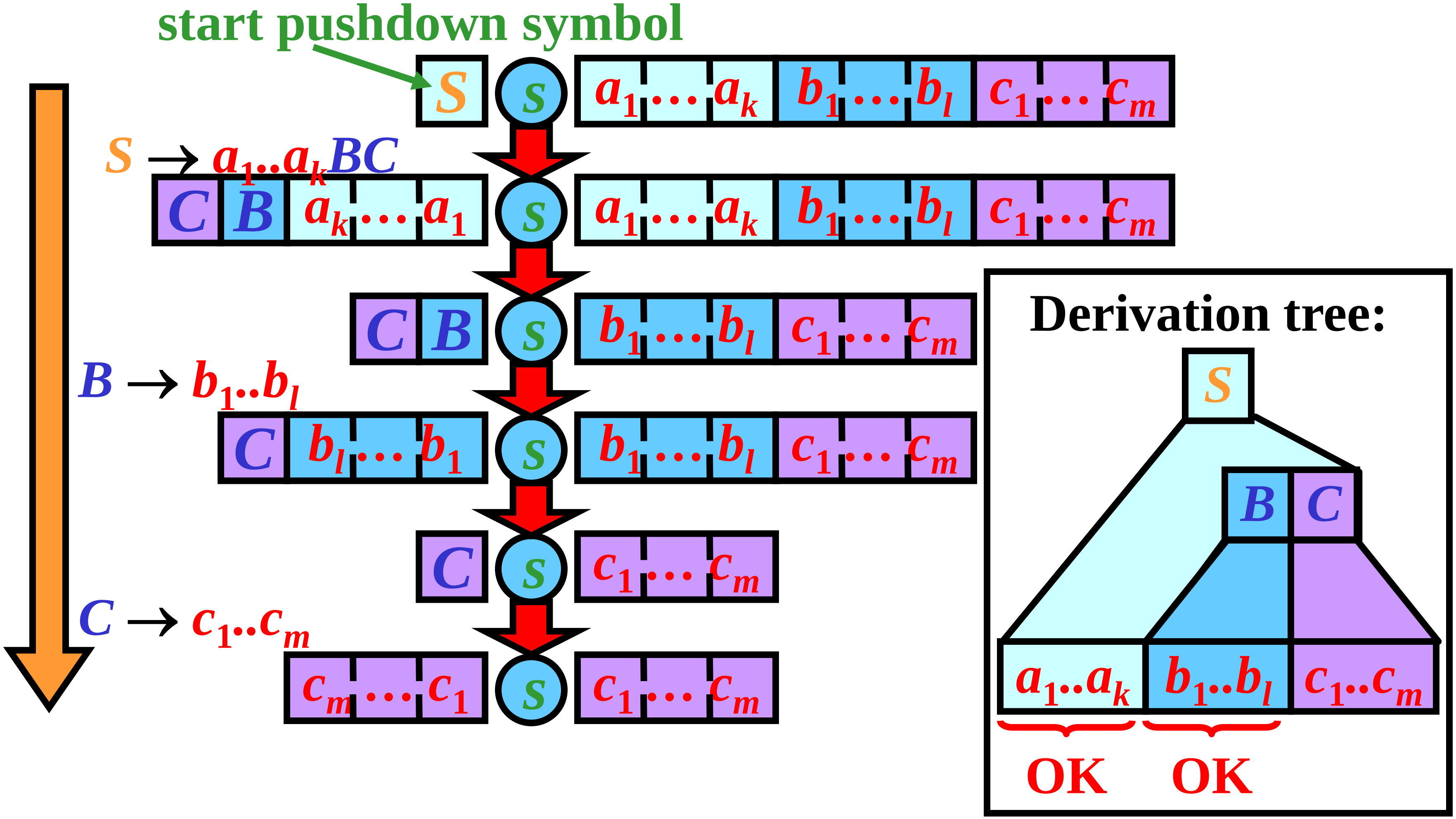
PDAs as Models of Top-Down Parsers 2/2

Top-down construction of a derivation tree:



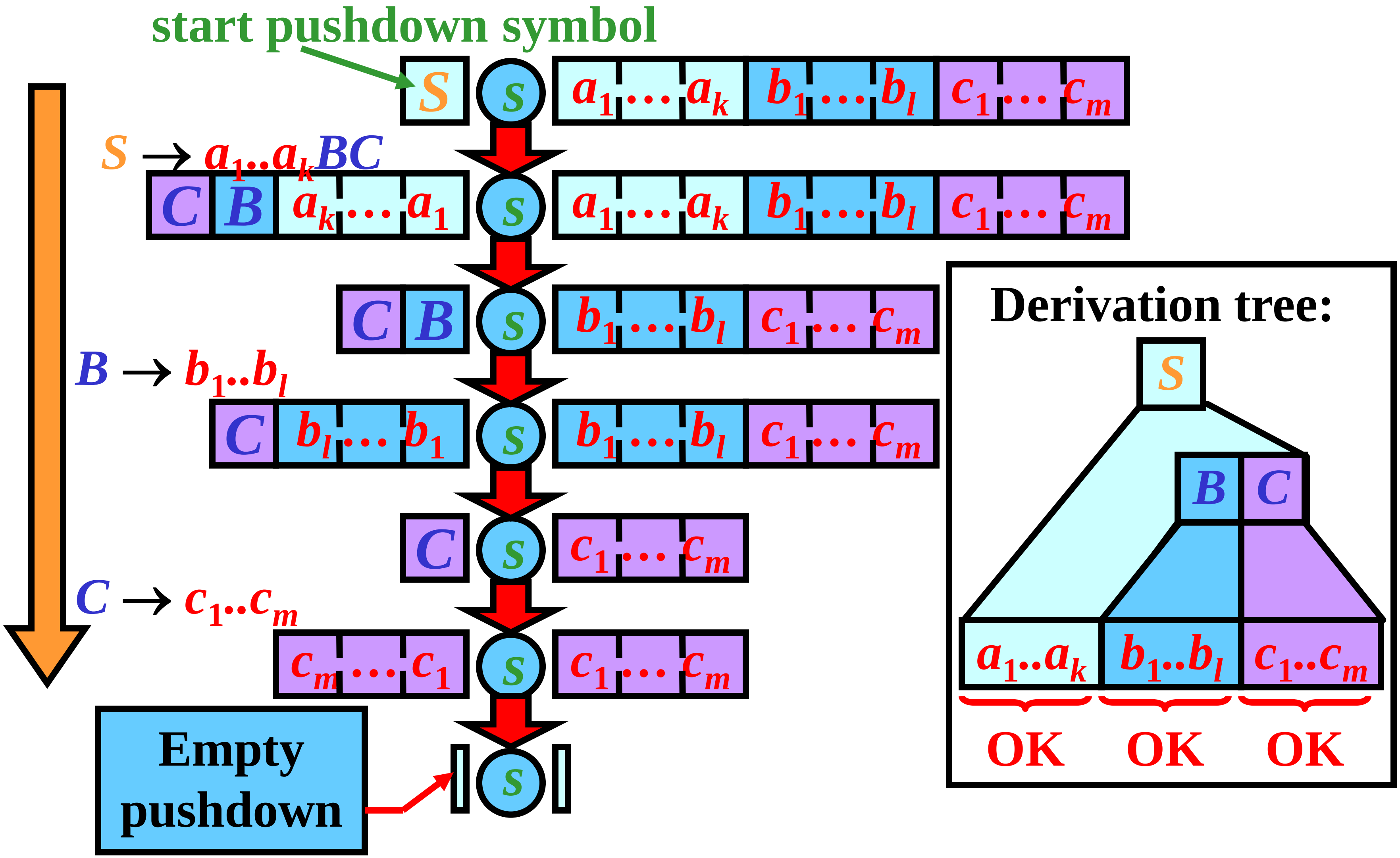
PDAs as Models of Top-Down Parsers 2/2

Top-down construction of a derivation tree:



PDAs as Models of Top-Down Parsers 2/2

Top-down construction of a derivation tree:



Algorithm: From CFG to PDA

- **Input:** CFG $G = (N, T, P, S)$
- **Output:** PDA $M = (Q, \Sigma, \Gamma, R, s, S, F)$; $L(G) = L(M)_\varepsilon$

- **Method:**
- $Q := \{s\}$;
- $\Sigma := T$;
- $\Gamma := N \cup T$;
- Construction of R:
  - for every $a \in \Sigma$, add $asa \rightarrow s$ to R;
  - for every $A \rightarrow x \in P$, add $As \rightarrow ys$ to R, where y = reversal(x);
- $F := \emptyset$;

From CFG to PDA: Example 1/2

- $G = (N, T, P, \mathbf{S})$, where:

$N = \{\mathbf{S}\}$, $T = \{\mathbf{(}, \mathbf{)}\}$, $P = \{\mathbf{S} \to \mathbf{(S)}, \mathbf{S} \to \mathbf{(\,)}\}$

Objective: An PDA M such that $L(G) = L(M)_\varepsilon$

$M = (Q, \Sigma, \Gamma, R, \mathbf{s}, \mathbf{S}, F)$ where:

From CFG to PDA: Example 1/2

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"(" $\in T$

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"(" $\in T$ ")" $\in T$

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"(" $\in T$ ")" $\in T$ $\underbrace{S} \rightarrow \underbrace{(S)}_{\text{rev}} \in P$

$R = \{(s(\rightarrow s,\ )s) \rightarrow s,\ Ss \rightarrow)S(s,$

From CFG to PDA: Example 1/2

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"(" $\in T$ ⇓ ; ")" $\in T$ ⇓ ; $\underbrace{S}$ $\to$ $\underbrace{(S)}_{\text{rev}}$ $\in P$; $\underbrace{S}$ $\to$ $\underbrace{()}_{\text{rev}}$ $\in P$

$R = \{(s(\to s,\ \ )s) \to s,\ \ Ss \to)S(s,\ \ Ss \to)(s\ \}$

From CFG to PDA: Example 1/2

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rev   **rev**

$R = \{\underbrace{(s(\to s,\quad)s) \to s,}_{\textbf{popping rules}}\quad \underbrace{Ss \to)S(s,\quad Ss \to)(s}_{\textbf{expansion rules}}\}$

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“**(**” $\in \boldsymbol{T}$ “**)**” $\in \boldsymbol{T}$ $\mathbf{S} \rightarrow \mathbf{(S)} \in \boldsymbol{P}$ $\mathbf{S} \rightarrow \mathbf{(\ )} \in \boldsymbol{P}$

rev **rev**

$$R = \{\underbrace{\mathbf{(s(} \rightarrow \mathbf{s},\ \ \mathbf{)s)} \rightarrow \mathbf{s},}_{\textbf{popping rules}}\ \underbrace{\mathbf{Ss} \rightarrow \mathbf{)S(s},\ \ \mathbf{Ss} \rightarrow \mathbf{)(s}}_{\textbf{expansion rules}}\ \}$$

$F = \varnothing$

From CFG to PDA: Example 2/2

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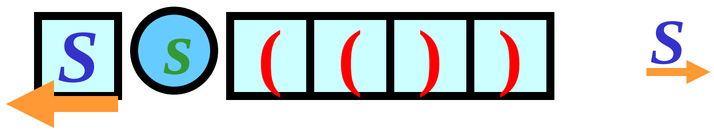
$P = \{(s(\to s, \quad)s) \to s, \quad Ss \to)S(s, \quad Ss \to)(s \}$

Question: $(())$ $\in L(M)_\varepsilon$**?**



From CFG to PDA: Example 2/2

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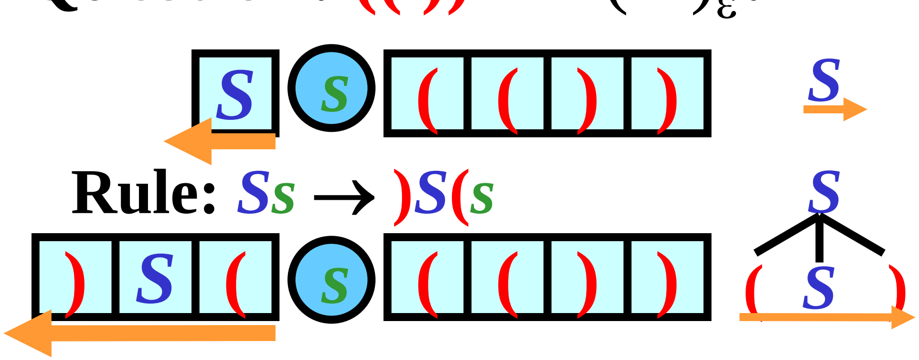

From CFG to PDA: Example 2/2

$M = (Q, \Sigma, \Gamma, R, s, S, F)$, where:

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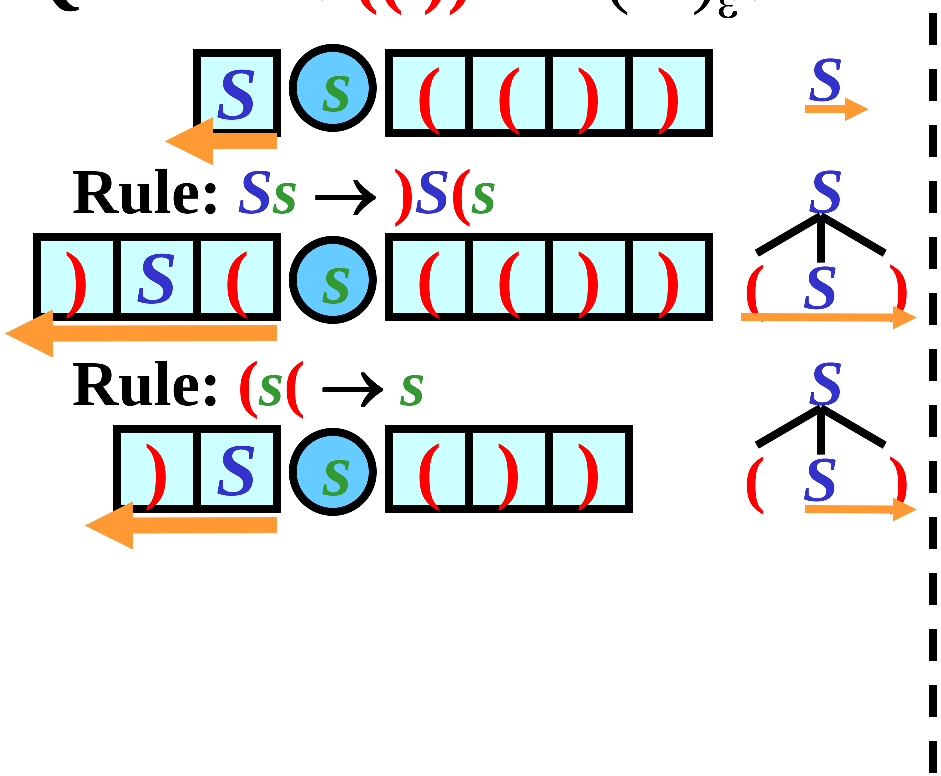

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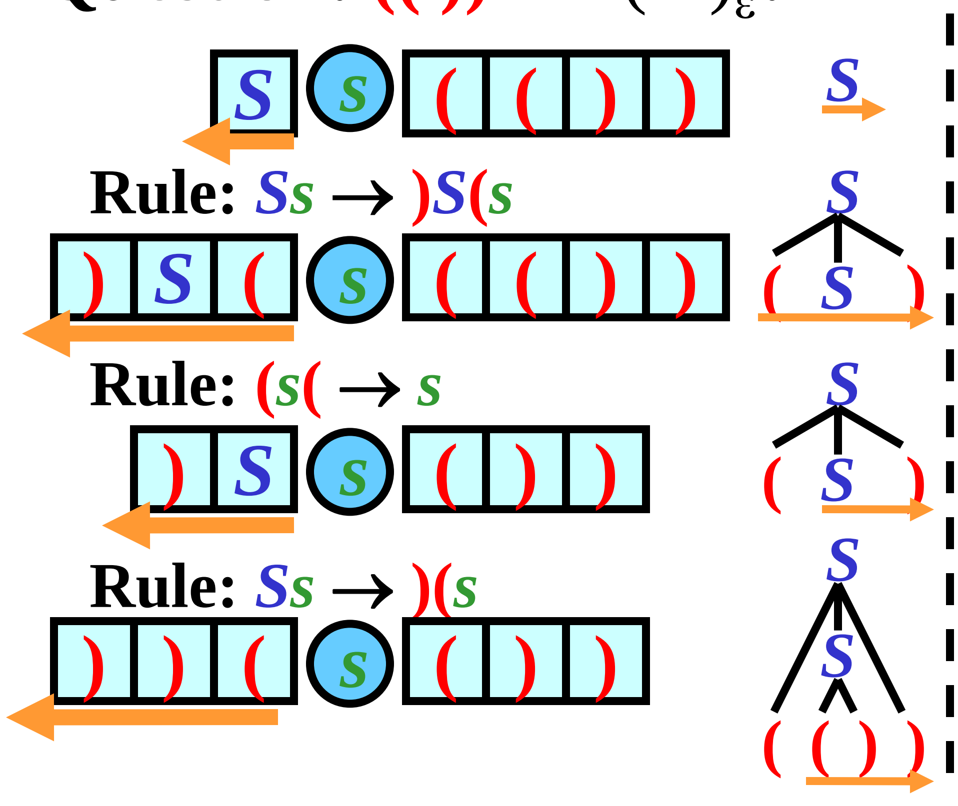

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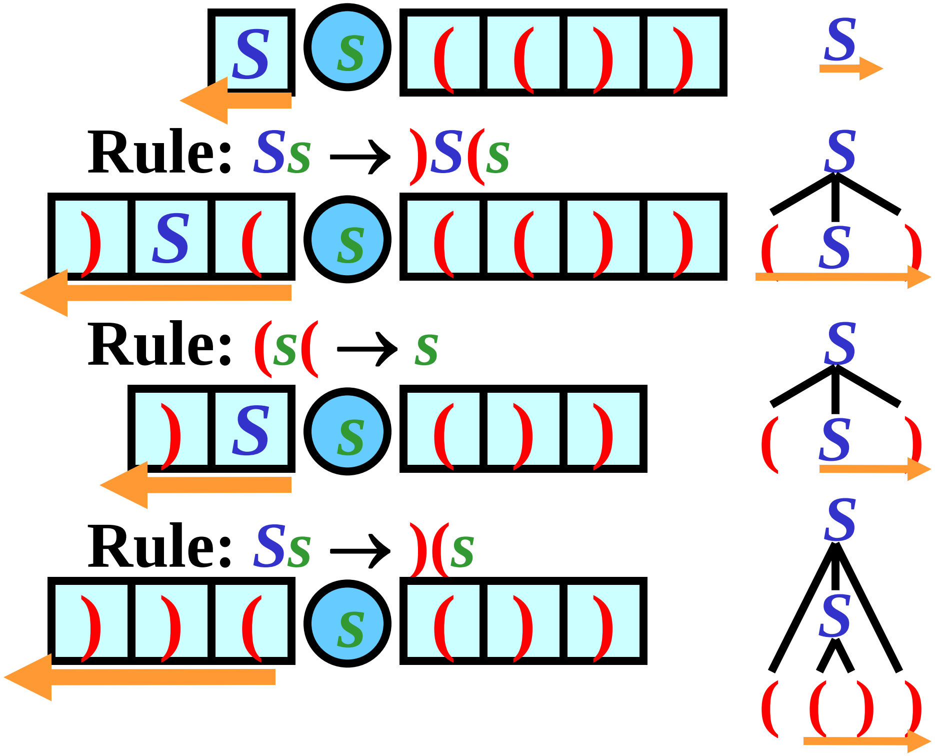

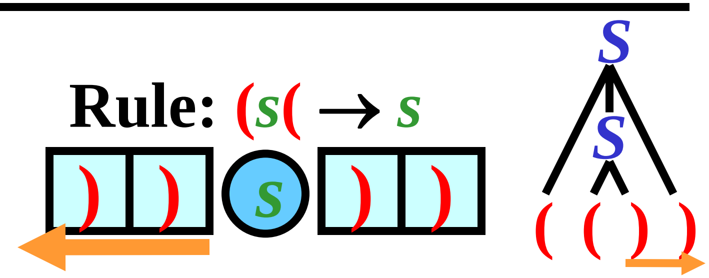

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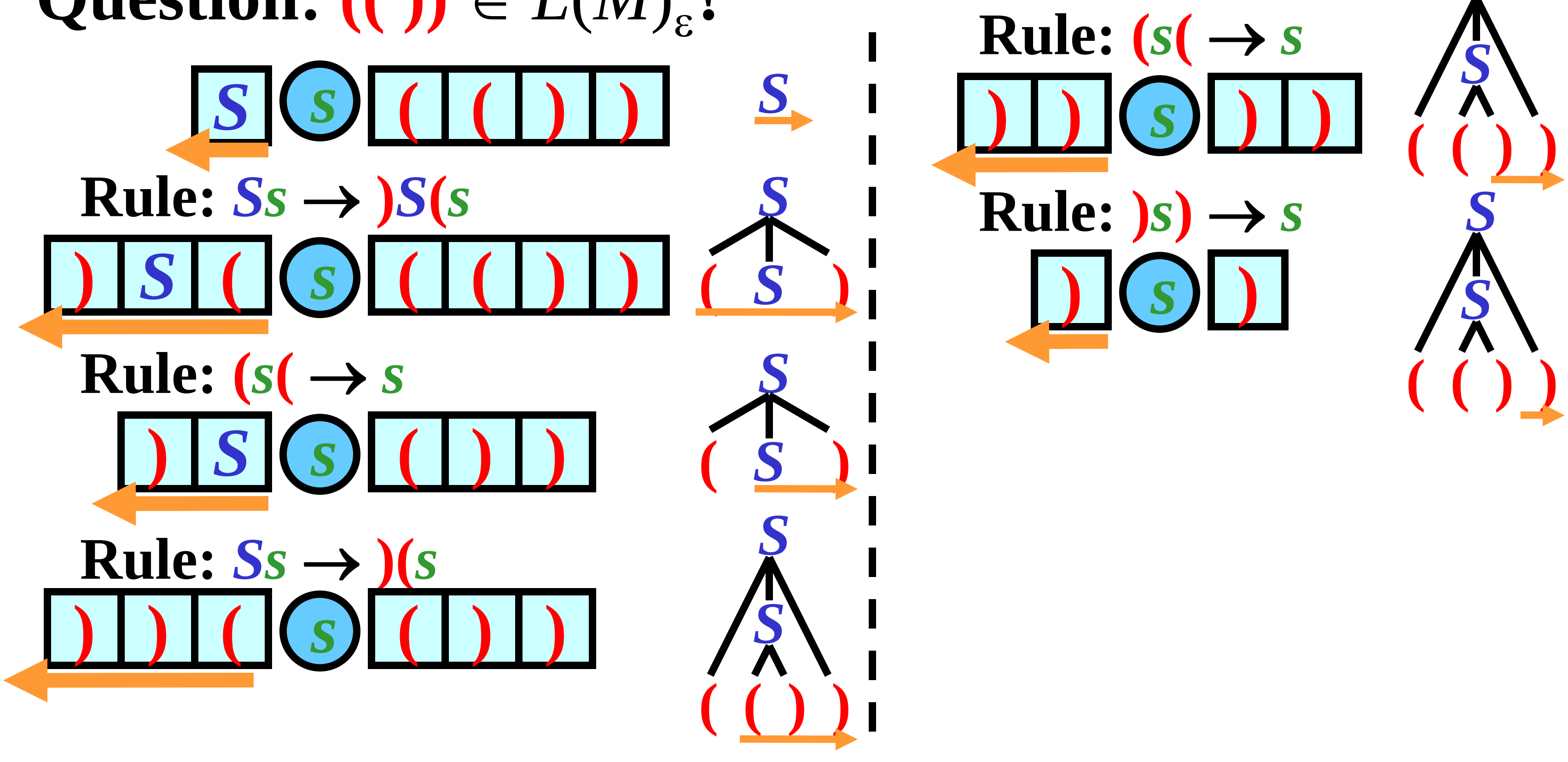

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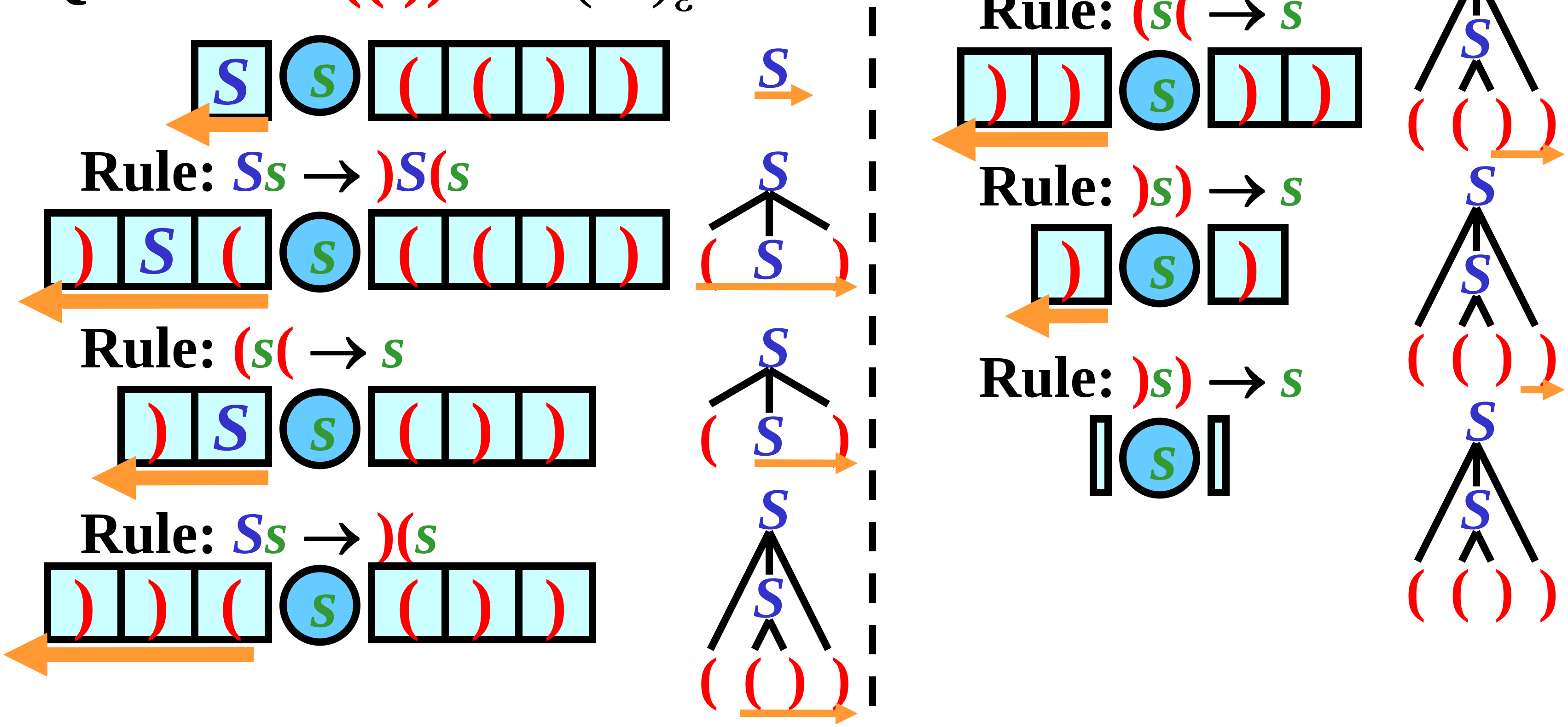

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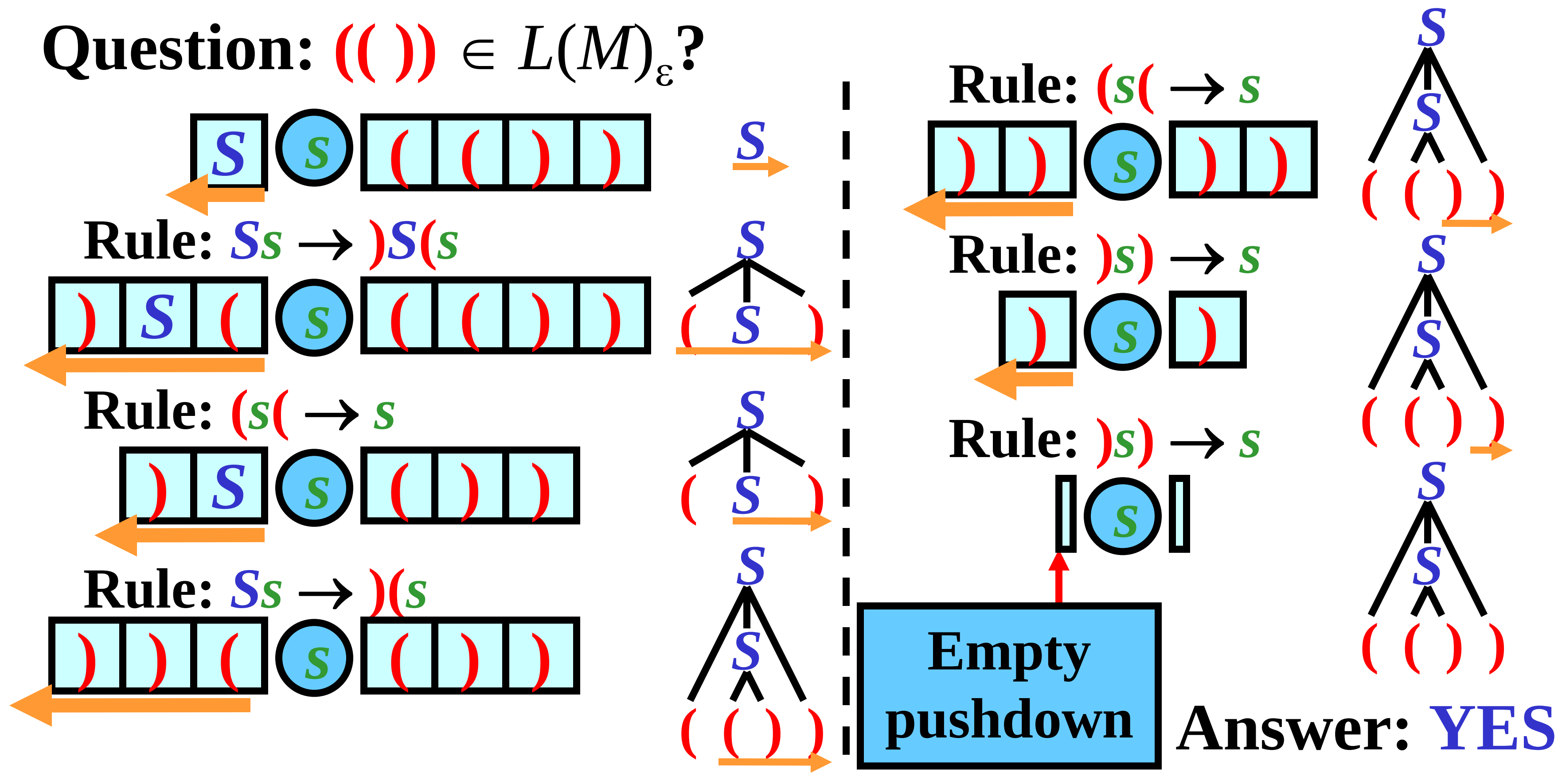

Answer: YES

Models for Context-free Languages

Theorem: For every CFG G, there is an PDA M such that $L(G) = L(M)_\varepsilon$.

Proof: See the previous algorithm.

Theorem: For every PDA M, there is a CFG G such that $L(M)_\varepsilon = L(G)$.

Proof: See page 486 in [Meduna: Automata and Languages]

Conclusion: The fundamental models for context-free languages are

1) Context-free grammars 2) Pushdown automata