Parallel Grammars:
Scattered Context Grammars

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Based on these Papers

- Meduna, A.: Coincidental Extention of Scattered Context Languages, *Acta Informatica* 39, 307-314, 2003
- Meduna, A. and Fernau, H.: On the Degree of Scattered Context-Sensitivity. *Theoretical Computer Science* 290, 2121-2124, 2003
- Meduna, A.: Descriptional Complexity of Scattered Rewriting and Multirewriting: An Overview. *Journal of Automata, Languages and Combinatorics*, 571-579, 2002
- Meduna, A. and Fernau, H.: A Simultaneous Reduction of Several Measures of Descriptional Complexity in Scattered Context Grammars. *Information Processing Letters*, 214-219, 2003

Classification of Parallel Grammars

I. Totally parallel grammars, such as *L* systems, rewrite **all** symbols of the sentential form during a single derivation step (not discussed in this talk).

II. Partially parallel grammars rewrite **some** symbols while leaving the other symbols unrewritten.

- Scattered Context Grammars work in a partially parallel way.
- These grammars are central to this talk.

Scattered Context Grammars (SCGs)

Essence

- semi-parallel grammars
- application of several context-free productions during a single derivation step
- stronger than CFGs

Main Topics under Discussion

- reduction of the grammatical size
- new language operations

Concept

Concept

- sequences of context-free productions
- several nonterminals are rewritten in parallel while the rest of the sentential form remains unchanged

Definition

Scattered context grammar :

- $G = (N, T, P, S)$
- N, T, and S as in a CFG
- P is a finite set of productions of the form
  $(A_1, A_2, \ldots, A_n) \rightarrow (x_1, x_2, \ldots, x_n)$
  where $A_i \in N$ and $x_i \in V^*$ with $V = N \cup T$

Direct derivation:

- $u_1A_1u_2A_2u_3 \ldots u_nA_nu_{n+1} \Rightarrow u_1x_1u_2x_2u_3 \ldots u_nx_nu_{n+1}$ if
  $(A_1, A_2, \ldots, A_n) \rightarrow (x_1, x_2, \ldots, x_n)$

Generated language:

- $L(G) = \{w\colon S \Rightarrow^* w \text{ and } w \in T^*\}$

Example

Productions:

$(S) \rightarrow (AA)$, $(A, A) \rightarrow (aA, bAc)$, $(A, A) \rightarrow (\varepsilon, \varepsilon)$

Derivation:

$S \Rightarrow AA \Rightarrow aAbAc \Rightarrow aaAbbAcc \Rightarrow aabbcc$

Generated Language:

$L(G) = \{a^i b^i c^i : i \geq 0\}$

Language Families

Language Families

- ***CS*** - *C*ontext *S*ensitive Languages
- ***RE*** - *R*ecursively *E*numerable Languages

- ***SC*** = {$L(G)$: G is a SCG}

for every $n \geq 1$,

- ***SC*(*n*)** = {$L(G)$: G is a SCG with no more than n nonterminals}

Reduction of SCGs

Reduction of SCGs

- **(A) reduction of the number of nonterminals**
- (B) reduction of the number of context (non-context-free) productions
- (C) simultaneous reduction of (A) and (B)

Reduction (A) 1/2

Reduction of the Number of Nonterminals

- **Theorem 1**: $\boldsymbol{RE} = \boldsymbol{SC}(3)$
- **Theorem 2**: $\boldsymbol{CS} \not\subset \boldsymbol{SC}(1)$

- **Proof** (Sketch): Let $L = \{a^h: h = 2^n, n \geq 1\}$. Assume that $L = L(G)$, where $G = (\{S\}, \{a\}, P, S)$ is a SCG. In G,

$$S \Rightarrow^* a^i S a^j \Rightarrow^* a^i a^k a^j$$

for some $i, j \geq 0$ such that $i + j, k \geq 1$. Thus,

$$S \Rightarrow^* a^{in} S a^{jn} \Rightarrow^* a^{in} a^k a^{jn}$$

for every $n \geq 0$. As $a^i a^k a^j \in L$, $|a^i a^k a^j| = i + k + j = 2^m$. Consider $v = a^{2i} a^k a^{2j} \in L$. Then, $2^m < |v| = 2^m + i + j < 2^{m+1}$, so $v \notin L$—a contradiction.

Reduction (A) 2/2

- **Corollary**: $\boldsymbol{SC}(1) \subset \boldsymbol{SC}(3)$ **=** $\boldsymbol{RE}$
- **Open Problem**: $\boldsymbol{RE} = \boldsymbol{SC}(2)$**?**

Reduction (B)

Reduction of SCGs

- (A) reduction of the number of nonterminals
- **(B) reduction of the number of context (non-context-free) productions**
- (C) reduction of (A) and (B)

Reduction (B) 1/5

Reduction of the Number of Context Productions

- A *context production* means a non-context-free production
$$(A_1, A_2, \ldots, A_n) \rightarrow (x_1, x_2, \ldots, x_n) \text{ with } n \geq 2$$

- **Theorem 4**: Every language in ***RE*** is generated by a scattered context grammar with only these two context productions:
$$(\$, 0, 0, \$) \rightarrow (\varepsilon, \$, \$, \varepsilon)$$
$$(\$, 1, 1, \$) \rightarrow (\varepsilon, \$, \$, \varepsilon)$$

Reduction (B) 2/5

I. Left-Extended Queue Grammar

$Q = (V, T, W, F, s, R)$

R - finite set of productions of the form (a, q, z, r). Every generation of $h \in L(Q)$ has this form

$\#a_0q_0$

$\Rightarrow a_0\#a_1x_0q_1$ $[(a_0, q_0, z_0, q_1)]$

$\Rightarrow a_0a_1\#a_2x_1q_2$ $[(a_1, q_1, z_1, q_2)]$

$\Rightarrow a_0a_1\ldots a_k\#a_{k+1}x_kq_{k+1}$

$\Rightarrow a_0a_1\ldots a_ka_{k+1}\#a_{k+2}x_{k+1}y_1q_{k+2}$ $[(a_{k+1}, q_{k+1}, y_1, q_{k+2})]$

$\Rightarrow a_0a_1\ldots a_ka_{k+1}\ldots a_{k+m-1}\# a_{k+m}y_1\ldots y_{m-1}q_{k+m}$ $[(a_{k+m-1}, q_{k+m-1}, y_{m-1}, q_{k+m})]$

$\Rightarrow a_0a_1\ldots a_ka_{k+1}\ldots a_{k+m}\#y_1\ldots y_mq_{k+m+1}$ $[(a_{k+m}, q_{k+m}, y_m, q_{k+m+1})]$

where $h = y_1\ldots y_m$ with $q_{k+m+1} \in F$

Reduction (B) 3/5

II. Substitutions

g: binary code of symbols from V

h: binary code of states from W

III. Introduction of SCG

$G = (N, T, CF \cup Context, S)$

$Context = \{ (\$, 0, 0, \$) \rightarrow (\varepsilon, \$, \$, \varepsilon),$

$(\$, 1, 1, \$) \rightarrow (\varepsilon, \$, \$, \varepsilon) \}$

IV. *CF* used to generate

$\$g(a_0 a_1 \ldots a_k a_{k+1} \ldots a_{k+m}) y_1 \ldots y_m h(q_{k+m} \ldots q_{k+1} q_k \ldots q_1 q_0)\$$

Reduction (B) 4/5

V. *Context* used to verify

$g(a_0 a_1 \ldots a_k a_{k+1} \ldots a_{k+m}) = h(q_0 q_1 \ldots q_k q_{k+1} \ldots q_{k+m})$

let $g(a_0 a_1 \ldots a_k a_{k+1} \ldots a_{k+m}) = c_0 c_1 \ldots c_{(k+m)2n}$

let $h(q_0 q_1 \ldots q_k q_{k+1} \ldots q_{k+m}) = d_0 d_1 \ldots d_{(k+m)2n}$

where each $c_i, d_i \in \{0, 1\}$

By using $(\$, 0, 0, \$) \rightarrow (\varepsilon, \$, \$, \varepsilon)$ and $(\$, 1, 1, \$) \rightarrow (\varepsilon, \$, \$, \varepsilon)$, G makes

$\$ c_0 c_1 c_2 \ldots c_{(k+m)2n} y_1 \ldots y_m\ d_{(k+m)2n} \ldots d_2 d_1 d_0 \$$

$\$ c_1 c_2 \ldots c_{(k+m)2n} y_1 \ldots y_m\ d_{(k+m)2n} \ldots d_2 d_1 \$$

$\$ c_2 \ldots c_{(k+m)2n} y_1 \ldots y_m\ d_{(k+m)2n} \ldots d_2 \$$

$\$ y_1 \ldots y_m \$$

$y_1 \ldots y_m$

Reduction (B) 5/5

- **Corollary 5**: The *SCGs* with two context productions characterize ***RE***.
- **Open Problem**: What is the power of the *SCGs* with a single context production?

Reduction of SCGs

Reduction of SCGs

- (A) reduction of the number of nonterminals
- (B) reduction of the number of context (non-context-free) productions
- **(C) reduction of (A) and (B)**

Simultaneous Reduction (A) & (B)

Simultaneous Reduction of the Number of Nonterminals and the Number of Context Productions

- **Note**: Next two theorems were proved in cooperation with H. Fernau (Germany).
- **Theorem**: Every type-0 language is generated by a SCG with no more than seven context productions and no more than five nonterminals
- **Theorem**: Every type-0 language is generated by a SCG with no more than six context productions and no more than six nonterminals
- **Open Problem**: Can we improve the above theorems?

New Operations

ε-free SCGs

- ε-free SCG: each production $(A_1, \ldots, A_n) \rightarrow (x_1, \ldots, x_n)$ satisfies $x_i \neq \varepsilon$
- ***ε-free SC*** = $\{L(G)$: G is an ε-free SCG $\}$
- ***ε-free SC*** $\subseteq$ ***CS*** $\subset$ ***SC*** = ***RE***

- Objective: Increase of ***ε-free SC*** to ***RE*** by a simple language operation over ***ε-free SC***

Coincidental Extension 1/6

Coincidental Extension

- For a symbol, *#*, and a string, $x = a_1a_2\ldots a_{n-1}a_n$, any string of the form $\#^i a_1 \#^i a_2 \#^i \ldots \#^i a_{n-1} \#^i a_n \#^i$, where $i \geq 0$, is a *coincidental #-extension* of *x*.

- A language, *K*, is a coincidental #-extension of *L* if every string of *K* represents a coincidental extension of a string in *L* and the deletion of all #s in *K* results in L, symbolically written as $L\ {}_{\#}\blacktriangleleft\ K$

- If $L\ {}_{\#}\blacktriangleleft\ K$ and there are an infinitely many coincidental extensions of *x* in *K* for every $x \in L$, we write $L\ {}_{\#}\blacktriangleleft_{\infty}\ K$

Coincidental Extension 2/6

Examples:

For $X = \{\#^i a \#^i b \#^i : i \geq 5\} \cup \{\#^i c^n \#^i d^n \#^i : n, i \geq 0\}$ and

$Y = \{ab\} \cup \{c^n d^n : n \geq 0\}$,

$$Y \;{}_{\#}\!\blacktriangleleft_{\infty} X, \text{ so } Y \;{}_{\#}\!\blacktriangleleft X.$$

For $A = \{\#a\#b\#\} \cup \{\#^i c^n \#^i d^n \#^i : n, i \geq 0\}$,

$$Y \;{}_{\#}\!\blacktriangleleft A \text{ holds, but } Y \;{}_{\#}\!\blacktriangleleft_{\infty} A \text{ does not hold.}$$

$B = \{\#^i a \#^i b \#^i : \mathrm{i} \geq 5\} \cup \{\#^i c^n \#^i d^n \#^{i+1} : n, i \geq 0\}$ is not the coincidental #-extension of any language.

Coincidental Extension 3/6

- **Theorem**: Let $K \in RE$. Then, there exists a ε-free SCG, G, such that $K \ {}_{\#}\!\blacktriangleleft_{\infty} \ L(G)$.

- **Proof** (Sketch): Let $K \in RE$. There exists a SCG, G, such that $L = L$(G). Construct a ε-free SCG,

  $G = (V, P, S, \{\#\} \cup T)$, so that $L \ {}_{\#}\!\blacktriangleleft_{\infty} \ L(G)$.

Homomorphism *h* :

h(A) = A for every nonterminal A

$h(a) = a$ for every terminal a

$h(\varepsilon) = Y$

Coincidental Extension 4/6

***P* constructed by performing the next six steps**:

I. add $(Z) \rightarrow (YS\$)$ to P

II. for every $(A_1, \ldots, A_n) \rightarrow (x_1, \ldots, x_n) \in P$, add
$(A_1, \ldots, A_n, \$) \rightarrow (h(x_1), \ldots, h(x_n), \$)$ to P

III. add $(Y, \$) \rightarrow (YY, \$)$ to P

IV. for every $a, b, c \in T$,
add $(\langle a \rangle, \langle b \rangle, \langle c \rangle, \$) \rightarrow (\langle 0a \rangle, \langle 0b \rangle, \langle 0c \rangle, \S)$ to P

V. for every $a, b, c, d \in T$, add
$(Y, \langle 0a \rangle, Y, \langle 0b \rangle, Y, \langle 0c \rangle, \S) \rightarrow (\#, \langle 0a \rangle, X, \langle 0b \rangle, Y, \langle 0c \rangle, \S)$,
$(\langle 0a \rangle, \langle 0b \rangle, \langle 0c \rangle, \S) \rightarrow (\langle 4a \rangle, \langle 1b \rangle, \langle 2c \rangle, \S)$,
$(\langle 4a \rangle, X, \langle 1b \rangle, Y, \langle 2c \rangle, \S) \rightarrow (\langle 4a \rangle, \#, \langle 1b \rangle, X, \langle 2c \rangle, \S)$,
$(\langle 4a \rangle, \langle 1b \rangle, \langle 2c \rangle, \langle d \rangle, \S) \rightarrow (a, \langle 4b \rangle, \langle 1c \rangle, \langle 2d \rangle, \S)$,
$(\langle 4a \rangle, \langle 1b \rangle, \langle 2c \rangle, \S) \rightarrow (a, \langle 1b \rangle, \langle 3c \rangle, \S)$,
$(\langle 1a \rangle, X, \langle 3b \rangle, Y, \S) \rightarrow (\langle 1a \rangle, \#, \langle 3b \rangle, \#, \S)$
to P

Coincidental Extension 5/6

VI. for every $a, b \in T$, add

$(\langle 1a \rangle, X, \langle 3b \rangle, \S) \rightarrow (a, \#, b, \#)$ to P.

G generates every $y \in L(G)$ in this way

$$Z \Rightarrow YS\$ \Rightarrow^{+} x\$ \Rightarrow v\S \Rightarrow^{+} z\S \Rightarrow y$$

where $v \in (T\{Y\}^{+})^{+}\{\$\}$. In addition,

$$v = u_0\langle 0a_1\rangle u_1\langle 0a_2\rangle u_2\langle 0a_3\rangle \ldots u_{n-1}\langle a_n\rangle u_n\S$$

if and only if $a_1a_2a_3\ldots a_n \in L(\mathrm{G})$

Coincidental Extension 6/6

In greater detail, $v\S \Rightarrow^+ z\S \Rightarrow y$ can be expressed as

$$Y^i\langle 0a_1\rangle Y^i\langle 0a_2\rangle Y^i\langle 0a_3\rangle \ldots Y^i\langle a_n\rangle Y^{i-1}\S$$

$$\Rightarrow^i \#^i\langle 0a_1\rangle X^i\langle 0a_2\rangle Y^i\langle 0a_3\rangle Y^i\langle a_4\rangle \ldots Y^i\langle a_n\rangle Y^{i-1}\S$$

$$\Rightarrow \#^i\langle 4a_1\rangle X^i\langle 1a_2\rangle Y^i\langle 2a_3\rangle Y^i\langle a_4\rangle \ldots Y^i\langle a_n\rangle Y^{i-1}\S$$

$$\Rightarrow^i \#^i\langle 4a_1\rangle \#^i\langle 1a_2\rangle X^i\langle 2a_3\rangle Y^i\langle a_4\rangle \ldots Y^i\langle a_n\rangle Y^{i-1}\S$$

$$\Rightarrow \#^i a_1 \#^i\langle 4a_2\rangle X^i\langle 1a_3\rangle Y^i\langle 2a_4\rangle \ldots Y^i\langle a_n\rangle Y^{i-1}\S$$

$$\Rightarrow^i \#^i a_1 \#^i\langle 4a_2\rangle \#^i\langle 1a_3\rangle X^i\langle 2a_4\rangle \ldots Y^i\langle a_n\rangle Y^{i-1}\S$$

$$\Rightarrow \#^i a_1 \#^i a_2 \#^i\langle 4a_3\rangle X^i\langle 1a_4\rangle Y^i\langle 2a_5\rangle \ldots Y^i\langle a_n\rangle Y^{i-1}\S$$

$$\vdots$$

$$\#^i a_1 \#^i a_2 \#^i a_3 \ldots \langle 4a_{n-2}\rangle \#^i\langle 1a_{n-1}\rangle X^i\langle 2a_n\rangle Y^{i-1}\S$$

$$\Rightarrow \#^i a_1 \#^i a_2 \#^i a_3 \ldots a_{n-2} \#^i\langle 1a_{n-1}\rangle X^i\langle 3a_n\rangle Y^{i-1}\S$$

$$\Rightarrow^{i-1} \#^i a_1 \#^i a_2 \#^i a_3 \ldots \#^i a_{n-2} \#^i\langle 1a_{n-1}\rangle \#^j X \#^k\langle 3a_n\rangle \#^{i-1}\S$$

$$\Rightarrow \#^i a_1 \#^i a_2 \#^i a_3 \ldots \#^i a_{n-2} \#^i a_{n-1} \#^i a_n \#^i$$

- **Corollary**: Let $K \in RE$. Then, there exists a ε-free SCG, G, such that $K\ _\# \blacktriangleleft\ L(G)$.

Use in Theoretical Computer Science

Use in Theoretical Computer Science

- **Corollary**: For every language $K \in RE$, there exists a homomorphism h and a language $H \in \varepsilon$*-free* ***SC*** such that $K = h(H)$.
- In a complex way, this result was proved on page 245 in [Greibach, S. A. and Hopcroft, J. E.: Scattered Context Grammars. *J. Comput. Syst. Sci.* 3, 232-247 (1969)]

Future Investigation

Future Investigation: ***k-limited coincidental extension***

- Let k be a non-negative integer.
- For a symbol, #, and a string, $x = a_1a_2\ldots a_{n-1}a_n$, any string of the form $\#^ia_1\#^ia_2\#^i\ldots\#^ia_{n-1}\#^ia_n\#^i$, where $k \geq i \geq 0$, is a *k-limited coincidental #-extension* of x.
- A language, K, is a coincidental a *k-limited #-extension* of L if every string of K represents a k-limited coincidental extension of a string in L and the deletion of all #s in K results in L, symbolically written as $L\ {}_{k\geq\#}\blacktriangleleft\ K$

Example

- For $X = \{\#^ia\#^ib\#^i\text{: } 2 \geq i \geq 0\} \cup \{\#^ic\#^id\#^i\text{: } 4 \geq i \geq 0\}$ and $Y = \{ab, cd\}$,

$$Y\ {}_{4\geq\#}\blacktriangleleft\ X$$

Very Important Open Problem

Important Open Problem: ε-*free* ***SC*** = ***CS*** ?

- Does there exist a non-negative integer *k*, such that for every $L \in CS$, $L\ {}_{k\geq\#}\blacktriangleleft\ L(H)$ for some ε-free SCG, *H*?
- If so, I know how to prove ε**-free** ***SC*** = ***CS*** ☺.

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