Scattered Context Grammars with erasing productions

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Content

- Definitions and Motivation
- Erasing nonterminals in a *k*-limited way
  - Theorem
  - Example
  - Proof (Basic Idea)

Scattered Context Grammar (SCG)

$$G = (V, T, P, S)$$

- V is an finite alphabet
- T is a set of terminals, $T \subset V$
- S is the start symbol, $S \in V - T$
- P is a finite set of productions of the form

$$(A_1, \dots, A_n) \rightarrow (x_1, \dots, x_n),$$

where $A_1, \dots, A_n \in V - T, x_1, \dots, x_n \in V^*$

$$\mathcal{L}(SC) = \mathcal{L}(RE)$$

Propagating SCG

$$G = (V, T, P, S)$$

- V, T, S like in SCG
- P is finite set of productions of the form

$$(A_1, \dots, A_n) \to (x_1, \dots, x_n),$$

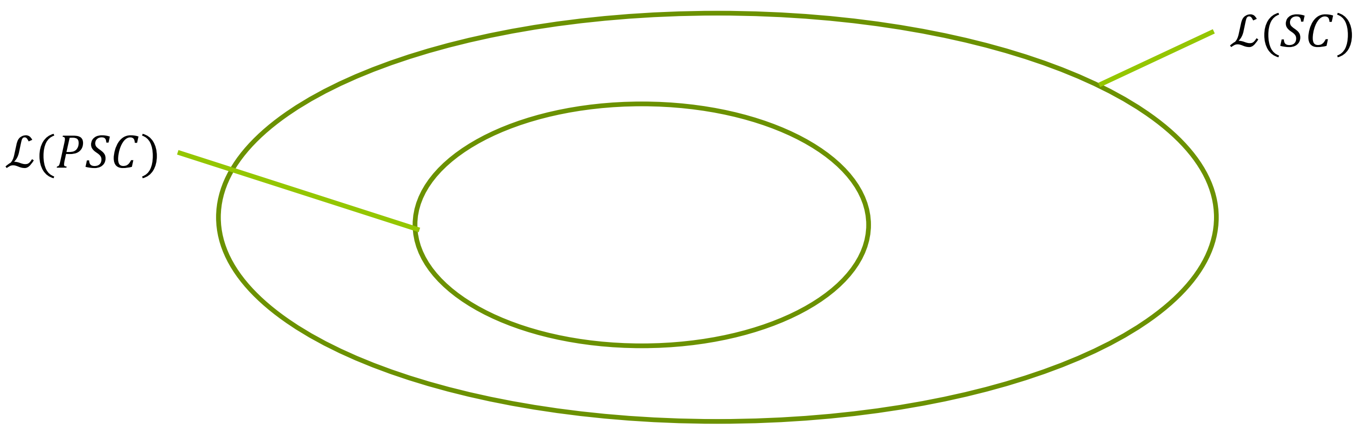
where $A_1, \dots, A_n \in V - T, x_1, \dots, x_n \in V^+$

- no erasing productions (EP)

$$\mathcal{L}(CF) \subset \mathcal{L}(PSC) \subseteq \mathcal{L}(CS) \subset \mathcal{L}(RE)$$

Motivation

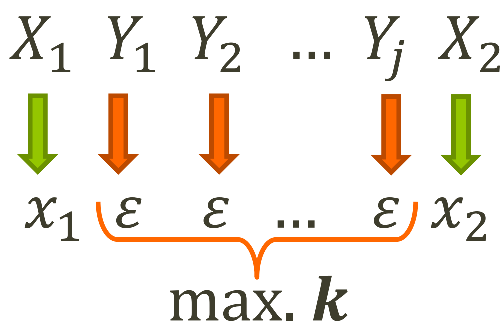
⇒ We cannot convert **any** SCG with EP
to an **equivalent** SCG without EP



but …

Erasing nonterminals in a *k*-limited way

- between any two symbols from which G derives non-empty string, there occur no more than ***k*** nonterminals from which G derives empty string



$X_i, Y_i \in V - T, x_i \in T^+$

Symbols erased during derivation

- $\check{A}$ symbol is erased during derivation
- $\hat{A}$ symbol is not erased during derivation

Erasing nonterminals in a k-limited way:

G erases its nonterminals in a k-limited way if for every $y \in L(G)$ there exists derivation satisfies these properties:

1. Every $x = uAvBw, \hat{A}, \hat{B}, \check{v}$, satisfies $|v| \leq k$
2. Every $x = uAw, \hat{A}$, satisfies: if $\check{u}$ or $\check{w}$, then $|u| \leq k$ or $|w| \leq k$

Theorem:

For every SC grammar G, which erases its nonterminals in a k-limited way, there exists a PSC grammar $\bar{G}$, such that $L(G) = L(\bar{G})$

Proof (Basic Idea):

$\bar{G}$ simulates G using nonterminals of the special form $\langle \dots \rangle$. In each nonterminal of this form, in every derivation step, $\bar{G}$ records a substring of the corresponding sentential form of G

Example

$$G = (\{S, A, A', B, B', C, C', a, b, c\}, \{a, b, c\}, P, S)$$

$$P = \{\ p_1: (S) \rightarrow (AA'BB'CC'),$$
$$p_2: (A, B, C) \rightarrow (aA, bB, cC),$$
$$p_3: (A, B, C) \rightarrow (a, b, c),$$
$$p_4: (A', B', C') \rightarrow (\varepsilon, \varepsilon, \varepsilon)\}$$

$$L(G) = \{a^n b^n c^n | n \geq 1\}$$

$$S \Rightarrow ABC \Rightarrow aAA'bBB'cCC' \Rightarrow aaA'bbB'ccC' \Rightarrow aabbcc$$

Proof (Basic Idea)

$G = (V, T, P, S)$ is a SCG which erases nonterminals in *k*-limited way.

$p: (A_1, \dots, A_n) \to (x_1, \dots, x_n) \in P$

$\text{len}(p) = n$
$\text{lhs}(p) = A_1, \dots, A_n$
$\text{rhs}(p) = x_1, \dots, x_n$

$\text{pos}(a_1 \dots a_i \dots a_n, i) = a_i$

Proof (Basic Idea)

For every $p = (A_1, \ldots, A_n) \rightarrow (x_1, \ldots, x_n) \in P$ let $\lfloor p, i \rfloor$ denote $A_i \rightarrow x_i \; \forall \; 1 \leq i \leq n$

$\Psi = \{\lfloor p, i \rfloor : p \in P, 1 \leq i \leq \mathrm{len}(p)\}$

$\Psi' = \{\lfloor p, i \rfloor' : \lfloor p, i \rfloor \in \Psi\}$

$\overline{N_1} = \{\langle x \rangle : x \in (V - T)^* \cup (V - T)^* T (V - T)^*, |x| \leq 2k + 1\}$

Example

$$\Psi = \{ \lfloor p_1, 1 \rfloor, \lfloor p_2, 1 \rfloor, \lfloor p_2, 2 \rfloor, \lfloor p_2, 3 \rfloor, \lfloor p_3, 1 \rfloor, \lfloor p_3, 2 \rfloor, \lfloor p_3, 3 \rfloor, \lfloor p_4, 1 \rfloor, \lfloor p_4, 2 \rfloor, \lfloor p_4, 3 \rfloor \}$$

$$\Psi' = \{\lfloor p_1, 1 \rfloor', \dots, \lfloor p_4, 3 \rfloor'\}$$

$$\overline{N_1} = \{ \langle\ \rangle, \langle S \rangle, \langle A \rangle, \langle A' \rangle, \langle B \rangle, \langle B' \rangle, \langle C \rangle, \langle C' \rangle, \langle SS \rangle, \langle SA \rangle, \dots, \langle C'C' \rangle, \langle SSS \rangle, \dots, \langle C'C'C' \rangle, \langle a \rangle, \langle b \rangle, \langle c \rangle, \langle aS \rangle, \langle aA \rangle, \dots, \langle cC' \rangle, \langle Sa \rangle, \langle Aa \rangle, \dots, \langle C'a \rangle, \langle aSS \rangle, \dots, \langle aC'C' \rangle, \langle SaS \rangle, \dots, \langle C'aC' \rangle, \langle aC'C' \rangle, \langle SSa \rangle, \dots, \langle C'C'a \rangle\}$$

Proof (Basic Idea)

$\forall \langle x \rangle \in \overline{N_1}$ and $\lfloor p, i \rfloor \in \Psi$:

$\text{lhs}-\text{replace}(\langle x \rangle, \lfloor p, i \rfloor) = \{\langle x_1 \lfloor p, i \rfloor x_2 \rangle : x_1, x_2 \in V^*,$
$x_1 \text{lhs}(\lfloor p, i \rfloor) x_2 = x\}$

$\overline{N_2} = \{\langle x \rangle : \langle x \rangle \in \text{lhs}-\text{replace}(\langle y \rangle, \lfloor p, i \rfloor), \langle y \rangle \in \overline{N_1}, \lfloor p, i \rfloor \in \Psi\}$

$\forall \langle x \rangle \in \overline{N_1}$ *and* $\lfloor p, i \rfloor' \in \Psi'$:

$\text{insert}(\langle x \rangle, \lfloor p, i \rfloor') = \{\langle x_1 \lfloor p, i \rfloor' x_2 \rangle : x_1, x_2 \in V^*,\ x_1 x_2 = x\}$

$\overline{N_2}' = \{\langle x \rangle : \langle x \rangle \in \text{insert}(\langle y \rangle, \lfloor p, i \rfloor'), \langle y \rangle \in \overline{N_1}, \lfloor p, i \rfloor' \in \Psi'\}$

Example

$$\text{lhs}-\text{replace}(\langle aA\rangle, \lfloor p_3, 1\rfloor) = \{\langle a\lfloor p_2, 1\rfloor\rangle\}$$

$$\begin{aligned}\overline{N_2} = \{ & \langle\lfloor p_1, 1\rfloor\rangle, \langle\lfloor p_2, 1\rfloor\rangle, \langle\lfloor p_3, 1\rfloor\rangle, \dots, \langle\lfloor p_4, 3\rfloor\rangle, \\ & \langle\lfloor p_1, 1\rfloor S\rangle, \langle S\lfloor p_1, 1\rfloor\rangle, \dots, \\ & \langle a\lfloor p_1, 1\rfloor\rangle, \langle a\lfloor p_2, 1\rfloor\rangle, \langle a\lfloor p_3, 1\rfloor\rangle, \dots, \langle c\lfloor p_4, 3\rfloor\rangle, \\ & \langle\lfloor p_1, 1\rfloor a\rangle, \langle\lfloor p_2, 1\rfloor a\rangle, \langle\lfloor p_3, 1\rfloor a\rangle, \dots, \langle\lfloor p_4, 3\rfloor c\rangle, \\ & \langle a\lfloor p_1, 1\rfloor S\rangle, \langle a\lfloor p_2, 1\rfloor S\rangle, \dots, \langle C'\lfloor p_4, 3\rfloor c\rangle \ \}\end{aligned}$$

Example

$$\text{insert}(\langle aA\rangle, \lfloor p_3, 1\rfloor') = \{\langle aA\lfloor p_3, 1\rfloor'\rangle\}$$

$$\begin{aligned}\overline{N_2{}'} = \{ & \langle a\lfloor p_1, 1\rfloor'\rangle, \langle a\lfloor p_2, 1\rfloor'\rangle, \dots, \langle c\lfloor p_4, 3\rfloor'\rangle, \\ & \langle\lfloor p_1, 1\rfloor' a\rangle, \langle\lfloor p_2, 1\rfloor' a\rangle, \dots, \langle\lfloor p_4, 3\rfloor' c\rangle, \\ & \langle a\lfloor p_1, 1\rfloor' S\rangle, \langle a\lfloor p_2, 1\rfloor' S\rangle, \dots, \langle c\lfloor p_4, 3\rfloor' C'\rangle \\ & \langle S\lfloor p_1, 1\rfloor' a\rangle, \langle S\lfloor p_2, 1\rfloor' a\rangle, \dots, \langle C'\lfloor p_4, 3\rfloor' c\rangle, \dots \ \}\end{aligned}$$

Proof (Basic Idea)

$\forall\, x = \langle x_1\rangle\langle x_2\rangle \dots \langle x_n\rangle \in (\overline{N_1} \cup \overline{N_2} \cup \overline{N_2}')^*$ and $n \geq 1$:

$\mathrm{join}(x) = x_1 x_2 \dots x_n$

$\forall\, x = \overline{N_1} \cup \overline{N_2} \cup \overline{N_2}'$:

$\mathrm{split}(x) = \{y : x = \mathrm{join}(y)\}$

Set $\bar{V} = T \cup \overline{N_1} \cup \overline{N_2} \cup \overline{N_2}' \cup \{\bar{S}\}$

Proof (Basic Idea)

Define PSG $\bar{G} = (\bar{V}, T, \bar{P}, \bar{S})$.

Construction of set $\bar{P}$:

1. $\forall p = (S) \rightarrow (x) \in P$ add $(\bar{S}) \rightarrow (\langle \lfloor p, 1 \rfloor \rangle)$ to $\bar{P}$

Proof (Basic Idea)

2. $\forall\langle x\rangle \in \overline{N_1}$

$\wedge\ \forall X \in \text{insert}(\langle x\rangle, \lfloor p, n\rfloor')$, where $p \in P, \text{len}(p) = n$

$\wedge\ \forall\langle y\rangle \in \overline{N_1}$

$\wedge\ \forall Y \in \text{lhs}-\text{replace}(\langle y\rangle, \lfloor q, 1\rfloor)$, where $q \in P$

add

A] $(X, \langle y\rangle) \rightarrow (\langle x\rangle, Y)$ and

B] $(\langle y\rangle, X) \rightarrow (Y, \langle x\rangle)$ to $\bar{P}$

C] $if\ \langle x\rangle = \langle y\rangle$ add $(X) \rightarrow (Y)$ to $\bar{P}$

D] $(X) \rightarrow (\langle x\rangle)$ to $\bar{P}$

Proof (Basic Idea)

3. $\forall \langle x \rangle \in \overline{N_1}$

$\wedge \forall X \in \text{insert}(\langle x \rangle, \lfloor p, i \rfloor')$, where $p \in P, i < \text{len}(p)$

$\wedge \forall \langle y \rangle \in \overline{N_1}$

$\wedge \forall Y \in \text{lhs}-\text{replace}(\langle y \rangle, \lfloor q, i+1 \rfloor)$, where $q \in P$

add

A] $(X, \langle y \rangle) \rightarrow (\langle x \rangle, Y)$ to $\bar{P}$

B] $if \langle x \rangle = \langle y \rangle \wedge \text{pos}(X, l) = \lfloor p, i \rfloor', \text{pos}(Y, m) = \lfloor p, i+1 \rfloor', l < m$, add $(X) \rightarrow (Y)$ to $\bar{P}$

Proof (Basic Idea)

4. $\forall \langle x_1 \lfloor p, i \rfloor x_2 \rangle \in \text{lhs}-\text{replace}(\langle x \rangle, \lfloor p, i \rfloor), \langle x \rangle \in \overline{N_1}, \lfloor p, i \rfloor \in \Psi, x_1, x_2 \in V^*$

$\wedge \forall Y \in \text{split}(x_1 \text{rhs}(\lfloor p, i \rfloor) \lfloor p, i \rfloor' x_2)$

add

$(\langle x_1 \lfloor p, i \rfloor x_2 \rangle) \to (Y)$ to $\bar{P}$

5. $\forall a \in T$

add

$(\langle a \rangle) \to (a)$ to $\bar{P}$

Example

$\langle S \rangle$

$\Rightarrow^* \langle AA' \rangle \langle BB' \rangle \langle CC' \rangle$ $(S) \rightarrow (AA'BB'CC')$

$\Rightarrow^* \langle aAA' \rangle \langle bBB' \rangle \langle cCC' \rangle$ $(A, B, C) \rightarrow (aA, bB, cC)$

$\Rightarrow^* \langle a \rangle \langle aAA' \rangle \langle b \rangle \langle bBB' \rangle \langle c \rangle \langle cCC' \rangle$ $(A, B, C) \rightarrow (aA, bB, cC)$

$\Rightarrow^* \langle a \rangle \langle aA \rangle \langle b \rangle \langle bB \rangle \langle c \rangle \langle cC \rangle$ $(A', B', C') \rightarrow (\varepsilon, \varepsilon, \varepsilon)$

$\Rightarrow^* \langle a \rangle \langle a \rangle \langle a \rangle \langle b \rangle \langle b \rangle \langle b \rangle \langle c \rangle \langle c \rangle \langle c \rangle$ $(A, B, C) \rightarrow (a, b, c)$

$\Rightarrow^* aaabbbccc$

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