Practical Using of VRT's

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ABSTRACT *This paper is focused on experience with using variance reduction techniques in discrete simulation experiments. Particular server queuing systems are discussed. The paper describes experiments involving following techniques that are suitable to use in a simulation of server queuing systems: antithetic variates, common random numbers, and joint application of antithetic variates and common random numbers.*

Keywords: discrete simulation, variance reduction techniques, server queuing systems

1. Introduction

Simulation as a process can consume a lot of computer time, despite advances in computer technology. These advances, however, have allowed the simulation of more complicated systems. Moreover, even when simpler systems are simulated, it is possible to gain more confidence in results of simulation. One way, to generate results quicker and with more confidence for discrete simulation processes, is using some sophisticated statistical techniques for processing of simulation results. This paper is focused on experience with an application of these techniques in server queuing systems (SQS's).

2. Variance Reduction Techniques

In the literature the efficiency of a variance reduction technique (VRT) is usually measured by the decrease of the variance of the estimator of the mean. For the reliability of an estimate can be measured by confidence limits of a form as in (1).

$$\bar{x} \pm a\left(\sigma/\sqrt{n}\right) \tag{1}$$

where $\bar{x}$ is average of n independent observations, σ is the standard deviation of an individual observation and a is suitable constant.

It is necessary to measure the decrease of the standard deviation due to applying a VRT. If it is required that the desired reliability is predetermined, then the variance should be measured. Of course a variance reduction implies a standard deviation reduction, and vice versa. The variance reduction (VR) is defined (as a percentage) as

$$\left[\left(\sigma_0^2 - \sigma_1^2\right)/\sigma_0^2\right] \times 100\% \tag{2}$$

where σ_0^2 and σ_1^2 are the old and new variance, respectively. Besides the reduction of the standard deviation or variance it is necessary to take into account the extra computer time that a VRT may require. Further, one should not forget the time taken to construct the computer program.

2.1 Antithetic Variates

The antithetic variates technique tries to create negative correlation between observations, generating one observation from the random number r and the other observation from its "antithetic" partner $(1-r)$. So the variance of the estimator $\bar{x}$ is given by

$$\mathrm{var}\left(\bar{x}\right) = \frac{1}{4}\left[\mathrm{var}\left(x_1\right) + \mathrm{var}\left(x_2\right) + 2\,\mathrm{cov}\left(x_1, x_2\right)\right] \tag{3}$$

Hence the variance of $\bar{x}$ decrease if x_1 and x_2 are negatively correlated, or equivalently if $\mathrm{cov}\left(x_1, x_2\right)$ is negative.

2.2 Common Random Numbers

It is usual to simulate more than one system since it is desirable to choose between several systems. In that case one is not interested in the absolute values of the system responses but in the differences between system responses. The systems are compared under the same circumstances. The general expression for the variance of the difference between two systems is given by

$$\mathrm{var}\left(x - y\right) = \mathrm{var}\left(x\right) + \mathrm{var}\left(y\right) - 2\,\mathrm{cov}\left(x, y\right) \tag{4}$$

Hence the variance of the estimated difference is decreased if the covariance term in (4) can be made positive. Such a positive covariance is created by use of the same random numbers if it is assumed that both systems react to the stochastic input variables in the same direction.

2.3 Joint Application of Antithetic Variates and Common Random Numbers

It is possible to apply more than one VRT in a problem. Some authors recommend the combination of antithetic variates and common random numbers.

The variance is given by

$$\mathrm{var}\left(\bar{d}\right) = \mathrm{var}\left(\bar{x}\right) + \mathrm{var}\left(\bar{y}\right) - 2\,\mathrm{cov}\left(\bar{x}, \bar{y}\right) \tag{5}$$

Antithetic variates reduce the variance of the average of the estimated responses of particular system, i.e., $\mathrm{var}\left(\bar{x}\right)$ and $\mathrm{var}\left(\bar{y}\right)$ are decreased. The use of the same random numbers is supposed to create a positive covariance between $\bar{x}$ and $\bar{y}$.

3. Practical using of VRT's

This paper deals with three VRT's: antithetic variates (AV), common random numbers (CRN), and joint application of antithetic variates and common random numbers (JAC). The paper examines their efficiency when they are applied on simple server queuing systems. Examples are designed in C++ using an independent VRT module and SIMLIB. SIMLIB is used for a model building and simulation running, the VRT module is used for an input parameters synchronisation and output statistics (more information can be found in [6]).

3.1 Experimental Results

All examples are server queuing systems with mean of interarrival time given by exponential distribution and the mean of service time given by exponential distribution. Experiments were designed in the following way. The estimated mean of the service time varied usually from 1.0 to 10.0. A dependency of the variance reduction on a various system load was observed.

One particular experiment was designed in the following way. The variance reduction was computed for each value of the service time by (2). More details in [6].

Two selected examples of a practical usage follows:

3.1.1 Experiment 1

In this experiment there was applied method of antithetic variates on single server queuing system with these parameters:

- interarrival time - 10.0
- service time - from 1.0 to 10.0 step 1.0.

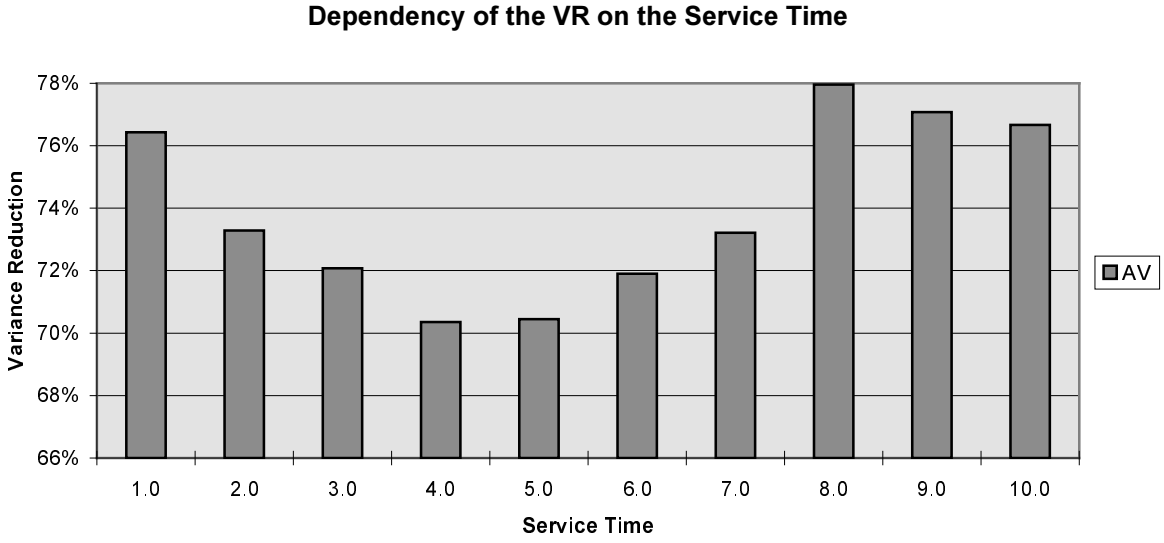

Figure 1: Variance reduction of AV

As shown in figure 1, yielded variance reduction is stabile without a significant dependency on the system load. Its values varied from 70 to 78 %.

3.1.2 Experiment 2

Common random numbers and JAC applied on single server queuing system and double servers queuing system with these parameters:

a) single SQS: interarrival time - 10.0
service time - 5.0
double SQS: interarrival time - 10.0
service time - from 1.0 to 10.0 step 1.0.

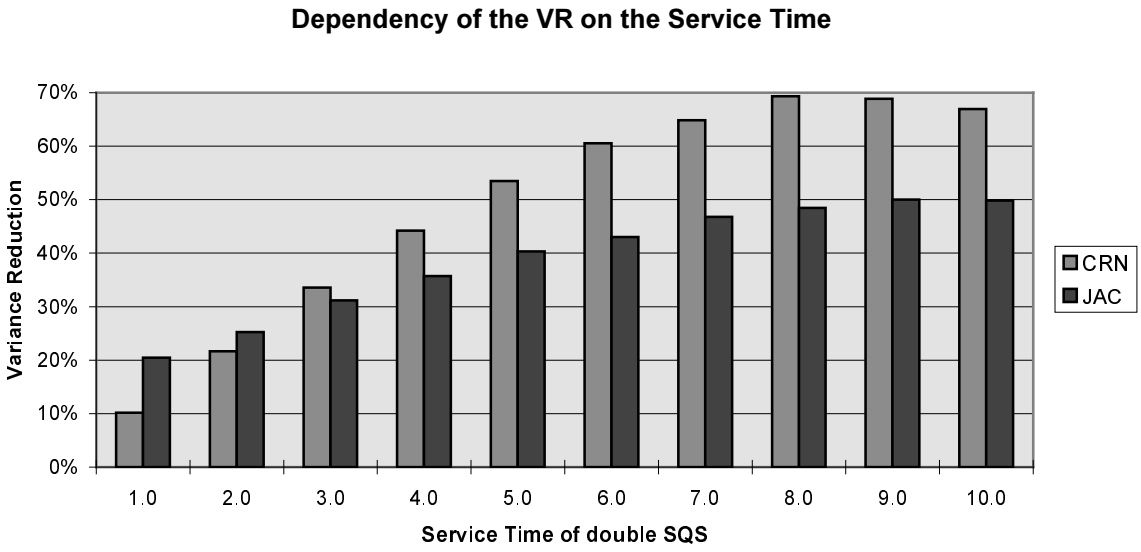

Figure 2: Variance reduction of CRN and JAC - example a

As shown in figure 2, both VRT's yield variance reduction that is directly proportional on a similarity of SQS's.

b) single SQS: interarrival time - 10.0
service time - 2.5
double SQS: interarrival time - 10.0
service time - from 1.0 to 10.0 step 1.0.

Figure 3 shows similar results as figure 2. Moreover, it is clear from this figure that the both VRT's are inversely proportional on the system load.

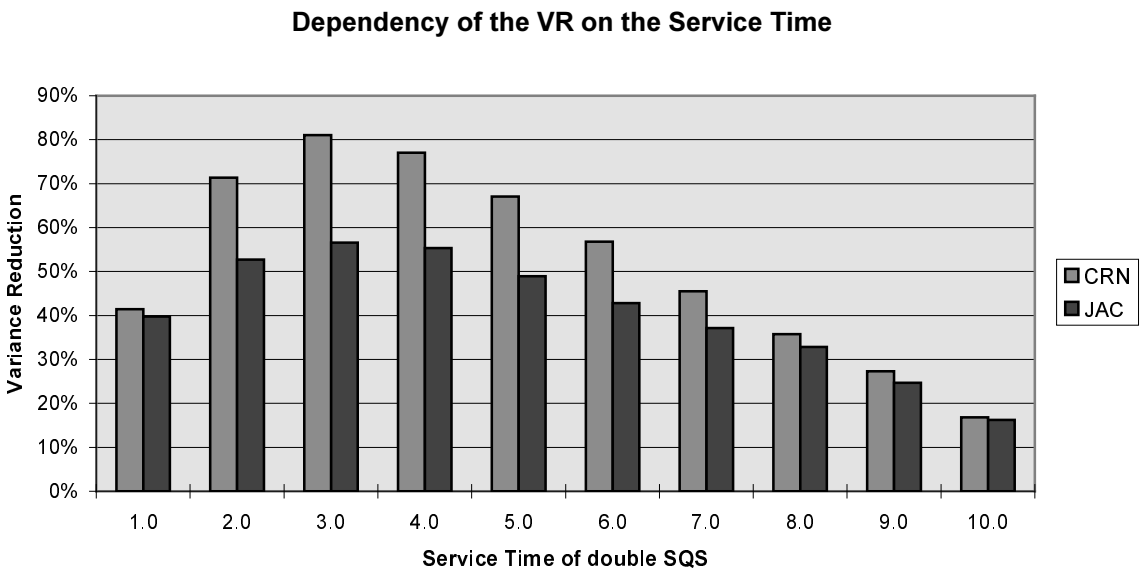

Figure 3: Variance reduction of CRN and JAC - example b

c) single SQS: interarrival time - 10.0
service time - 2.5
double SQS: interarrival time - 10.0
service time - from 0.5 to 5.0 step 0.5.

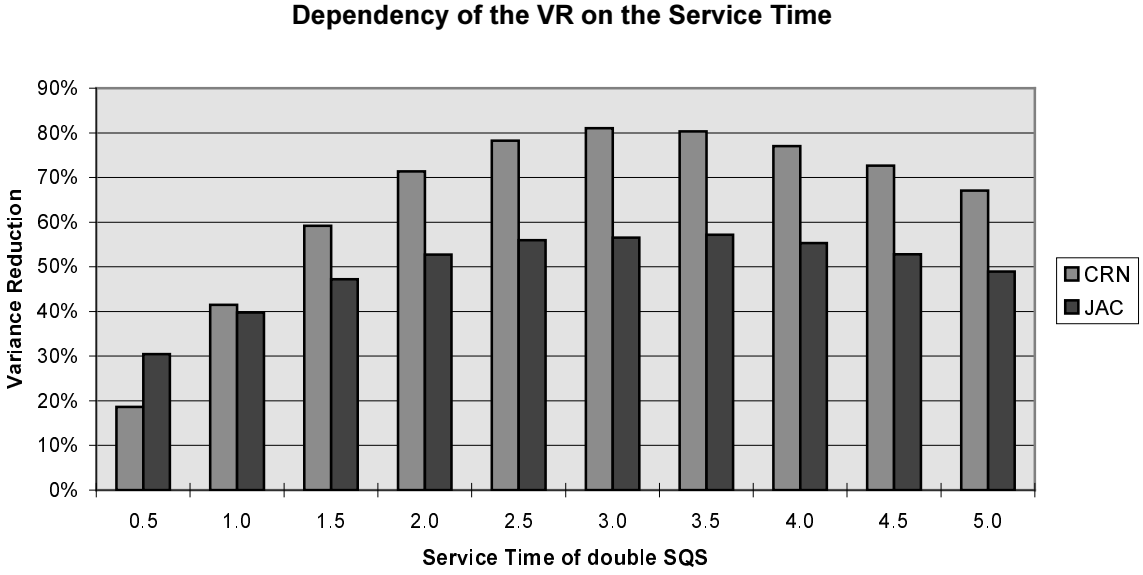

Figure 4: Variance reduction of CRN and JAC - example c

Figure 4 is the finer case of figure 3. Furthermore, it is possible to see from this figure, as from previous ones, that CRN yields higher VR than JAC for very similar systems.

4. Conclusion

This paper has evaluated three VRT's : antithetic variates, common random numbers, and joint application of antithetic variates and common random numbers.

These methods have a simple mathematical background, nevertheless they yield worthwhile variance reductions if certain assumptions are satisfied. Moreover, they do not distort the sampling process and therefore besides the average response, the dynamic behaviour of the system can be studied.

The antithetic variates technique, as shown in figure 1, is a stabile method which does not depend on the system load. This technique yields variance reduction more than 70%.

The following techniques, CRN and JAC, yield variance reduction that is directly proportional on the similarity of behaviour of SQS's. The more similar systems, the higher variance reduction. Moreover, it can be seen from figure 3 that the both VRT's are inversely proportional on the system load. The higher system load, the lower variance reduction. Furthermore, CRN gives better results than JAC if systems are very close. It is caused by the fact that the efficiency of JAC is decreased by negative cross correlations. On the other hand, JAC gives better results for non-similar systems.

All above mentioned VRT's show some potential. They are aimed to different targets. Whereas AV is designed for simulation of one system, CRN and JAC serve for a comparison of more systems. It is better to use JAC than CRN for systems which do not behave very similarly and vice versa.

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