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# Simulation Subsumption in Ramsey-based Büchi Automata Universality and Inclusion Testing*

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**Abstract.** There are two main classes of methods for checking universality and language inclusion of Büchi-automata: Rank-based methods and Ramsey-based methods. While rank-based methods have a better worst-case complexity, Ramsey-based methods have been shown to be quite competitive in practice [9, 8]. It was shown in [9] (for universality checking) that a simple subsumption technique, which avoids exploration of certain cases, greatly improves the performance of the Ramsey-based method. Here, we present a much more general subsumption technique for the Ramsey-based method, which is based on using simulation preorder on the states of the Büchi-automata. This technique applies to both universality and inclusion checking, yielding a substantial performance gain over the previous simple subsumption approach of [9].

## 1 Introduction

Universality and language-inclusion checking are important problems in the theory of automata with significant applications, e.g., in model-checking. More precisely, the problem of checking whether an implementation meets a specification can be formulated as a language inclusion problem. The behavior of the implementation is represented by an automaton $\mathcal{A}$, the specification is given by an automaton $\mathcal{B}$, and one checks whether $\mathcal{L}(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{B})$. As one is generally interested in non-halting computations, automata are used as acceptors of languages over *infinite words*. In this paper, we concentrate on *Büchi automata*, where accepting runs are those containing some accepting state infinitely often.

A naïve inclusion-checking algorithm involves complementation: One has that $\mathcal{L}(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{B})$ iff $\mathcal{L}(\mathcal{A}) \cap \overline{\mathcal{L}(\mathcal{B})} = \emptyset$. However, the complementary automaton $\overline{\mathcal{B}}$ is, in the worst case, exponentially bigger than the original automaton $\mathcal{B}$. Hence, direct complementation should be avoided.

Among methods that keep the complementation step implicit, *Rank-based* and *Ramsey-based* methods have recently gained interest. The former uses a

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rank-based analysis of rejecting runs [13], leading to a simplified complementation procedure. The latter is based on Büchi's original combinatorial Ramsey-based argument for showing closure of $\omega$-regular languages under complementation [3]. Notice that a high worst-case complexity is unavoidable, since both universality and language-inclusion testing are PSPACE-complete problems.

However, in practice, subsumption techniques can often greatly speed up universality/inclusion checking by avoiding the exploration of certain cases that are subsumed by other cases. Recently, [4] described a simple set-inclusion-based subsumption technique that speeds up the rank-based method for both universality and language inclusion checking. Using this technique, [4] is capable of handling automata several orders of magnitude larger than previously possible. Similarly, [9] improved the Ramsey-based method (but only for universality checking) by a simple subsumption technique that compares finite labeled graphs (using set-inclusion on the set of arcs, plus an order on the labels; see the last paragraph in Section 3).

*Our contribution.* We improve the Ramsey-based approach in a twofold way. First, we show how to employ simulation preorder to generalize the simple subsumption technique of [9] for Ramsey-based universality checking. Second, we introduce a simulation-based subsumption relation for Ramsey-based language inclusion checking, thus extending the theory of subsumption to the realm of Ramsey-based inclusion checking. Note that the proposed use of simulations is significantly different from just using simulations to reduce Büchi automata (quotienting), followed by an application of the original approach. The reason is that, when reducing automata, one has to use simulation equivalences which tend to be much smaller (and hence less helpful) than simulation preorders, which are used in our method. Therefore, our approach takes full advantage of the asymmetry of simulation preorder, making it more general than just quotienting beforehand the automaton w.r.t. the induced equivalence.

Experimental results show that our algorithm based on simulation subsumption significantly and consistently outperforms the algorithm based on the original subsumption of [9]. We perform the evaluation on Büchi automata models of several mutual exclusion algorithms (the largest having several thousands of states and tens of thousands of transitions), random Büchi automata generated from LTL formulae, and Büchi automata generated from the random model of [17]. In many cases, the difference between the two approaches is very significant. For example, our approach finishes an experiment on the Bakery algorithm in minutes, while the original approach cannot handle it in hours. In the largest examples generated from LTL formulae, our approach is on average 20 times faster than the original one when testing universality and more than 1900 times faster when testing language inclusion. All relevant information is provided online [20], enabling interested readers to reproduce our experiments.

## 2 Preliminaries

A *Büchi Automaton (BA)* $\mathcal{A}$ is a tuple $(\Sigma, Q, I, F, \delta)$ where $\Sigma$ is a finite alphabet, $Q$ is a finite set of states, $I \subseteq Q$ is a non-empty set of *initial* states, $F \subseteq Q$ is

a set of *accepting* states, and $\delta \subseteq Q \times \Sigma \times Q$ is the transition relation. For convenience, we write $p \xrightarrow{a} q$ instead of $(p, a, q) \in \delta$.

A *run* of $\mathcal{A}$ on a word $w = a_1 a_2 \ldots \in \Sigma^\omega$ *starting* in a state $q_0 \in Q$ is an infinite sequence $q_0 q_1 q_2 \ldots$ such that $q_{j-1} \xrightarrow{a_j} q_j$ for all $j > 0$. The run is *accepting* iff $q_i \in F$ for infinitely many $i$. The *language of* $\mathcal{A}$ is the set $\mathcal{L}(\mathcal{A}) = \{w \in \Sigma^\omega \mid \mathcal{A} \text{ has an accepting run on } w \text{ starting from some } q_0 \in I\}$.

A *path* in $\mathcal{A}$ on a finite word $w = a_1 \ldots a_n \in \Sigma^+$ is a finite sequence $q_0 q_1 \ldots q_n$ where $q_{j-1} \xrightarrow{a_j} q_j$ for all $0 < j \leq n$. The path is *accepting* iff $q_i \in F$ for some $0 \leq i \leq n$. We define the following predicates for $p, q \in Q$: (1) $p \overset{w}{\underset{F}{\Longrightarrow}} q$ iff there is an accepting path on $w$ from $p$ to $q$. (2) $p \overset{w}{\Longrightarrow} q$ iff there is a path (not necessarily accepting) on $w$ from $p$ to $q$. (3) $p \overset{w}{\not\Longrightarrow} q$ iff there is no path on $w$ from $p$ to $q$.

Define $E = Q \times \{0, 1, -1\} \times Q$ and let $G_\mathcal{A}$ be the largest subset of $2^E$ whose elements contain exactly one member of $\{\langle p, 0, q\rangle, \langle p, 1, q\rangle, \langle p, -1, q\rangle\}$ for every $p, q \in Q$. Each element in $G_\mathcal{A}$ is a $\{0, 1, -1\}$-arc-labeled graph on $Q$.

For each pair of states $p, q \in Q$, we define the following three sets of languages: (1) $\mathcal{L}(p, 1, q) = \{w \in \Sigma^+ \mid p \overset{w}{\underset{F}{\Longrightarrow}} q\}$, (2) $\mathcal{L}(p, 0, q) = \{w \in \Sigma^+ \mid p \overset{w}{\Longrightarrow} q \wedge \neg\left(p \overset{w}{\underset{F}{\Longrightarrow}} q\right)\}$, (3) $\mathcal{L}(p, -1, q) = \{w \in \Sigma^+ \mid p \overset{w}{\not\Longrightarrow} q\}$. As in [9], the language of a graph $g \in G_\mathcal{A}$ is defined as the intersection of the languages of arcs in $g$, i.e., $\mathcal{L}(g) = \bigcap_{\langle p,a,q\rangle \in g} \mathcal{L}(p, a, q)$. For each word $w \in \Sigma^+$ and each pair $p, q \in Q$, there exists exactly one arc $\langle p, a, q\rangle$ such that $w \in \mathcal{L}(p, a, q)$. Therefore, the languages of the graphs in $G_\mathcal{A}$ form a partition of $\Sigma^+$, since they are the intersections of the languages of the arcs. Let $Y_{gh}$ be the $\omega$-regular language $\mathcal{L}(g) \cdot \mathcal{L}(h)^\omega$.

**Lemma 1.** *(1)* $\Sigma^\omega = \bigcup_{g,h \in G_\mathcal{A}} Y_{gh}$. *(2) For* $g, h \in G_\mathcal{A}$ *s.t.* $\mathcal{L}(g), \mathcal{L}(h) \neq \emptyset$, *either* $Y_{gh} \cap \mathcal{L}(\mathcal{A}) = \emptyset$ *or* $Y_{gh} \subseteq \mathcal{L}(\mathcal{A})$. *(3)* $\overline{\mathcal{L}(\mathcal{A})} = \bigcup_{g,h \in G_\mathcal{A} \wedge Y_{gh} \cap \mathcal{L}(\mathcal{A}) = \emptyset} Y_{gh}$.

In fact, Lemma 1 is a relaxed version of the lemma proved by a Ramsey-based argument described in [15, 8, 9]. A proof can be found in Appendix A.

## 3 Ramsey-based Universality Testing

Based on Lemma 1, one can construct an algorithm for checking universality of BA [8]. This type of algorithm is said to be Ramsey-based, since the proof of Lemma 1 relies on the infinite Ramsey theorem. Lemma 1 implies that $\mathcal{L}(\mathcal{A})$ is universal iff $\forall g, h \in G_\mathcal{A} : \; Y_{gh} \subseteq \mathcal{L}(\mathcal{A})$. Since $\mathcal{L}(g) = \emptyset$ or $\mathcal{L}(h) = \emptyset$ implies $Y_{gh} \subseteq \mathcal{L}(\mathcal{A})$, it suffices to build and check graphs with nonempty languages in $G_\mathcal{A}$ when testing universality.

As proposed in [8, 9, 12], the set $G^f_\mathcal{A} = \{g \in G_\mathcal{A} \mid \mathcal{L}(g) \neq \emptyset\}$ can be generated iteratively as follows. First, given $g, h \in G_\mathcal{A}$, their composition $g; h$ is defined as

$$
\begin{aligned}
&\{\langle p, -1, q\rangle \mid \forall t \in Q : (\langle p, a, t\rangle \in g \wedge \langle t, b, q\rangle \in h) \rightarrow (a = -1 \vee b = -1)\} \cup \\
&\{\langle p, 0, q\rangle \mid \exists r \in Q : \langle p, 0, r\rangle \in g \wedge \langle r, 0, q\rangle \in h \wedge \\
&\qquad \wedge \; \forall t \in Q : (\langle p, a, t\rangle \in g \wedge \langle t, b, q\rangle \in h) \rightarrow (a \neq 1 \wedge b \neq 1)\} \cup \\
&\{\langle p, 1, q\rangle \mid \exists r \in Q : \langle p, a, r\rangle \in g \wedge \langle r, b, q\rangle \in h \wedge \neg(a \neq 1 \wedge b \neq 1)\}.
\end{aligned}
$$

For all $a \in \Sigma$, define the single-character graph $g_a = \{\langle p, -1, q\rangle \mid q \notin \delta(p,a)\} \cup \{\langle p, 0, q\rangle \mid p \in (Q \setminus F) \wedge q \in (\delta(p,a) \setminus F)\} \cup \{\langle p, 1, q\rangle \mid q \in \delta(p,a) \wedge \{p,q\} \cap F \neq \emptyset\}$. Let $G^1_{\mathcal{A}} = \{g_a \mid a \in \Sigma\}$. As shown in [7] (Lemma 3.1.1), one can obtain $G^f_{\mathcal{A}}$ by repeatedly composing graphs in $G^1_{\mathcal{A}}$ until a fixpoint is reached:

**Lemma 2.** *A graph $g$ is in $G^f_{\mathcal{A}}$ iff $\exists g_1, \ldots, g_n \in G^1_{\mathcal{A}} : \; g = g_1; \ldots; g_n$.*

It remains to sketch how to check that no pair $\langle g, h\rangle$ of graphs $g, h \in G^f_{\mathcal{A}}$ is a counterexample to universality, which, by Point 3 of Lemma 1, reduces to testing $Y_{gh} \cap \mathcal{L}(\mathcal{A}) \neq \emptyset$. The so called *lasso-finding test*, proposed in [9], can be used for this purpose. The lasso-finding test of a pair of graphs $\langle g, h\rangle$ checks the existence of a *lasso*, i.e., a path in $g$ from some state $p \in I$ to some state $q \in F$ and a path in $h$ from $q$ to itself. Equivalently, we consider a pair of graphs $\langle g, h\rangle$ to pass the *lasso-finding test* (denoted by $LFT(g,h)$) iff there is an arc $\langle p, a_0, q_0\rangle$ in $g$ and an infinite sequence of arcs $\langle q_0, a_1, q_1\rangle, \langle q_1, a_2, q_2\rangle, \ldots$ in $h$ s.t. $p \in I$, $a_i \in \{0,1\}$ for all $i \geq 0$, and $a_j = 1$ for infinitely many $j \in \mathbb{N}$. The following lemma was proved in [9] (we provide a considerably simplified proof in Appendix B.

**Lemma 3.** *$\mathcal{L}(\mathcal{A})$ is universal iff $LFT(g,h)$ for all $g, h \in G^f_{\mathcal{A}}$.*

To be more specific, the procedure for the lasso-finding test works as follows. It (1) finds all 1-SCCs (strongly connected components that contain only $\{0,1\}$-labeled arcs and at least one of the arcs is 1-labeled) in $h$, (2) records the set of states $T_h$ from which there is an $\{0,1\}$-labeled path to some state in some 1-SCC, (3) records the set of states $H_g$ such that for all $q \in H_g$, there exists an arc $\langle p, a, q\rangle \in g$ for some $p \in I$ and $a \geq 0$, and then (4) checks if $H_g \cap T_h \neq \emptyset$. We have $LFT(g,h)$ iff $H_g \cap T_h \neq \emptyset$. This procedure is *polynomial* in the number of $\{0,1\}$-labeled arcs in $g$ and $h$.

Finally, Algorithm 1 gives a naïve universality test obtained by combining the above principles for generating $G^f_{\mathcal{A}}$ and using $LFT$. A more efficient version of the algorithm is given in [9], using the following idea. For $f, g, h \in G_{\mathcal{A}}$, we say that $g \sqsubseteq h$ iff for each arc $\langle p, a, q\rangle \in g$, there is an arc $\langle p, a', q\rangle \in h$ such that $a \leq a'$. If $g \sqsubseteq h$, we have that (1) $LFT(f,g) \implies LFT(f,h)$ and (2) $LFT(g,f) \implies LFT(h,f)$ for all $f \in G_{\mathcal{A}}$. Since the algorithm searches for counterexamples to universality, the tests on $h$ are subsumed by the tests on $g$, and thus $h$ can be discarded. We refer to this method, which is based on the relation $\sqsubseteq$, as *subsumption*, in contrast to our more general *simulation subsumption* which is described in the next section.

## 4 Improving Universality Testing via Simulation

In this section, we describe our technique to use simulation-based subsumption in order to accelerate the Ramsey-based universality test [9] for Büchi automata.

A *simulation* on a BA $\mathcal{A} = (\Sigma, Q, I, F, \delta)$ is a relation $R \subseteq Q \times Q$ such that $pRr$ only if (1) $p \in F \implies r \in F$, and (2) for every transition $p \xrightarrow{a} p'$, there is a transition $r \xrightarrow{a} r'$ such that $p'Rr'$. It can be shown that there exists

**Algorithm 1**: *Ramsey-based Universality Checking*

**Input**: A BA $\mathcal{A} = (\Sigma, Q, I, F, \delta)$, the set of all single-character graphs $G^1_{\mathcal{A}}$
**Output**: TRUE if $\mathcal{A}$ is universal. Otherwise, FALSE.

```
1 Next := G^1_A; Processed := ∅;
2 while Next ≠ ∅ do
3 |   Pick and remove a graph g from Next;
4 |   foreach h ∈ Processed do
5 |   |   if ¬LFT(g,h) ∨ ¬LFT(h,g) ∨ ¬LFT(g,g) then return FALSE;
6 |   Add g to Processed;
7 |   foreach h ∈ G^1_A do if g;h ∉ Processed then Add g;h to Next;
8 return TRUE;
```

a unique maximal simulation, which is a preorder (called *simulation preorder* and denoted by $\preceq_{\mathcal{A}}$, or just $\preceq$ when $\mathcal{A}$ is clear from the context), computable in time $\mathcal{O}(|\Sigma||Q||\delta|)$ [10, 11]. The relation $\simeq = \preceq \cap \succeq$ is called *simulation equivalence*.

If $\mathcal{A}$ is interpreted as an automaton over finite words, $\preceq$ implies language containment, and quotienting w.r.t. $\simeq$ preserves the regular language. If $\mathcal{A}$ is interpreted as a BA, then the particular type of simulation defined above is called *direct simulation*. It implies $\omega$-language containment, and (unlike for fair simulation [6]) quotienting w.r.t. $\simeq$ preserves the $\omega$-regular language of $\mathcal{A}$.

Our method for accelerating the Ramsey-based universality test [9] of $\mathcal{A}$ is based on two optimizations which we describe below together with some intuition underlying their correctness. We formally prove the correctness of these optimizations in Lemmas 4–8, presented afterwards.

*Optimization 1.* The first optimization is based on the observation that the subsumption relation $\sqsubseteq$ from [9] can be weakened by exploiting simulation preorder. We call our weaker notion the *simulation-subsumption-based relation*, written $\sqsubseteq^{\forall\exists}$. The idea is as follows: While in $\sqsubseteq$ one requires that arcs of the form $\langle p, a, q\rangle$ can only be subsumed by arcs of the form $\langle p, a', q'\rangle$ with $a \leq a' \wedge q' = q$, we generalize this notion by replacing the last equality with simulation. This gives rise to the definition of $\sqsubseteq^{\forall\exists}$ below.

**Definition 1.** *For any $g, h \in G_{\mathcal{A}}$, we say that $g \sqsubseteq^{\forall\exists} h$ iff for every arc $\langle p, a, q\rangle \in g$, there exists an arc $\langle p, a', q'\rangle \in h$ such that $a \leq a'$ and $q \preceq q'$.*

*Optimization 2.* The second optimization builds on the fact that even the structure of the particular graphs in $G_{\mathcal{A}}$ can be simplified via simulation-subsumption, allowing us to replace some $\{0, 1\}$-labeled arcs by negative arcs. Since the complexity of the lasso-finding test, subsumption-checking, and graph-composition is polynomial in the number of $\{0, 1\}$-arcs, having smaller graphs reduces the cost of these operations.

For the purpose of reducing graphs, we define a (possibly non-deterministic) operation *Min* that maps each graph $f \in G_{\mathcal{A}}$ to a graph $Min(f) \in G_{\mathcal{A}}$ such that

$Min(f) \leqslant^* f$. Here, $g \leqslant^* h$ means that $g$ is either equal to $h$, or it is a reduced version of $h$ that can be derived from $h$ by weakening some of the arcs that are subsumed (simulation-smaller) by other arcs present both in $g$ and $h$. We define $\leqslant^*$ below.

**Definition 2.** *For any graphs $g, h \in G_{\mathcal{A}}$, we write $g \leqslant h$ iff there exist arcs $\langle p, a, q\rangle \in h$ and $\langle p, a', q'\rangle \in g \cap h$ s.t. $a \leq a'$, $q \preceq q'$, and $g = (h \setminus \{\langle p, a, q\rangle\}) \cup \{\langle p, a'', q\rangle\}$ where $a'' \leq a$. The relation $\leqslant^*$ is the transitive closure of $\leqslant$.*

We write $G^m_{\mathcal{A}} = \{g \in G_{\mathcal{A}} \mid \exists h \in G^f_{\mathcal{A}} :\ g \leqslant^* h\}$ to denote the set of reduced versions of graphs with nonempty languages. In practice, *Min* can be implemented such that it returns a graph which is as $\leqslant^*$-small as possible (meaning that as many arcs as possible will be restricted down to $-1$).

*Correctness of the Optimizations.* We now prove the correctness of our optimizations in a formal way. The correctness of the second optimization follows directly from 1) the observation that $\leqslant$ implies $\sqsubseteq^{\forall\exists}$-equivalence (Lemma 4), and 2) the fact that $G^m_{\mathcal{A}}$ is closed under composition (Lemma 8).

Let $\simeq^{\forall\exists} = \sqsubseteq^{\forall\exists} \cap (\sqsubseteq^{\forall\exists})^{-1}$. The lemma below follows by transitivity.

**Lemma 4.** *For any $g, h \in G_{\mathcal{A}}$, if $g \leqslant^* h$ then $g \simeq^{\forall\exists} h$.*

In particular, minimized graphs are $\sqsubseteq^{\forall\exists}$-equivalent to their original version. Notice that the relation $\leqslant^*$ does not preserve the language of graphs (and often for $g \leqslant^* h$, $\mathcal{L}(g) = \emptyset$ when $\mathcal{L}(h) \neq \emptyset$).

The correctness of the first optimization follows from the following observations. First, the lasso-finding test is $\sqsubseteq^{\forall\exists}$-monotonic, i.e., if $\sqsubseteq^{\forall\exists}$-smaller (pairs of) graphs pass the test, then so do $\sqsubseteq^{\forall\exists}$-bigger (pairs of) graphs (Lemma 7). In particular, for graphs $f, g, h \in G^f_{\mathcal{A}}$ such that $g \sqsubseteq^{\forall\exists} h$, $LFT(g, f) \implies LFT(h, f)$ and $LFT(f, g) \implies LFT(f, h)$. Therefore, we can ignore all lasso-finding tests related to the bigger $h$. Second, graph-composition is also $\sqsubseteq^{\forall\exists}$-monotonic (Lemma 6): Composing $\sqsubseteq^{\forall\exists}$-smaller graphs always yields $\sqsubseteq^{\forall\exists}$-smaller graphs. Thus, we can also ignore all lasso-finding tests related to any extension $h; f$ of $h$, for some $f \in G^f_{\mathcal{A}}$.

We begin by proving an auxiliary lemma—used to prove Lemma 6—which says that minimized graphs are in some sense complete w.r.t. simulation-bigger states.

**Lemma 5.** *Let $g$ be a graph in $G^m_{\mathcal{A}}$. We have that $\langle p, a, q\rangle \in g \ \wedge\ p \preceq p'$ implies $\exists \langle p', a', q'\rangle \in g :\ a \leq a' \ \wedge\ q \preceq q'$.*

*Proof.* If $a = -1$, the lemma trivially holds (e.g., by taking $q' = q$). Assume therefore $a \in \{0, 1\}$. From $g \in G^m_{\mathcal{A}}$, there is some $g' \in G^f_{\mathcal{A}}$ such that $g \leqslant^* g'$. Since $\mathcal{L}(g') \neq \emptyset$ and $a \in \{0, 1\}$, there is some word $w \in \mathcal{L}(g')$ such that $p \overset{w}{\Longrightarrow} q$. Since $p \preceq p'$, there is some $q''$ such that $p' \overset{w}{\Longrightarrow} q''$, $q \preceq q''$, and if $p \overset{w}{\underset{F}{\Longrightarrow}} q$, then $p' \overset{w}{\underset{F}{\Longrightarrow}} q''$. Since $w \in \mathcal{L}(g')$, $\langle p', a'', q''\rangle \in g'$ for $a \leq a''$. From Lemma 4, we get that there is an arc $\langle p', a', q'\rangle \in g$ such that $a \leq a'' \leq a'$ and $q \preceq q'' \preceq q'$. □

The lemma below states that composing minimized graphs is a $\sqsubseteq^{\forall\exists}$-monotonic operation. We actually prove a slightly stronger property, since we do not require that all graphs are minimal.

**Lemma 6.** *Let $f, g, f' \in G_{\mathcal{A}}$ and $g' \in G^m_{\mathcal{A}}$ be graphs s.t. $f \sqsubseteq^{\forall\exists} f'$ and $g \sqsubseteq^{\forall\exists} g'$. Then $f;g \sqsubseteq^{\forall\exists} f';g'$.*

*Proof.* We consider an arc $\langle p, c, r\rangle$ in $f;g$ and show that $f';g'$ must contain a larger arc w.r.t. $\sqsubseteq^{\forall\exists}$. The case $c = -1$ is trivial. For $c \in \{0,1\}$, there must be arcs $\langle p, a, q\rangle \in f$ and $\langle q, b, r\rangle \in g$ where $a, b \in \{0,1\}$ and $c = \max(\{a, b\})$. Since $f \sqsubseteq^{\forall\exists} f'$, there is an arc $\langle p, a', q'\rangle \in f'$ with $a \leq a'$ and $q \preceq q'$. Since $g \sqsubseteq^{\forall\exists} g'$, there is an arc $\langle q, b', r'\rangle \in g'$ with $b \leq b'$ and $r \preceq r'$. Since $g' \in G^m_{\mathcal{A}}$, Lemma 5 implies that there is an arc $\langle q', b'', r''\rangle \in g'$ s.t. $b \leq b' \leq b''$ and $r \preceq r' \preceq r''$. Thus $\langle p, c'', r''\rangle \in f';g'$ where $c = \max(\{a, b\}) \leq \max(\{a', b''\}) \leq c''$ and $r \preceq r''$. □

Below, we prove a lemma allowing us to replace lasso-finding tests on graphs by lasso-finding tests on (minimized versions of) smaller graphs. For the sake of generality, we prove a somewhat stronger property.

**Lemma 7.** *Let $e$, $f$, $g$, $h$ be graphs in $G_{\mathcal{A}}$ such that $\{f, h\} \cap G^m_{\mathcal{A}} \neq \emptyset$, $e \sqsubseteq^{\forall\exists} g$, and $f \sqsubseteq^{\forall\exists} h$. Then $LFT(e, f) \implies LFT(g, h)$.*

*Proof.* If $LFT(e, f)$, there exist an arc $\langle p, a_0, q_0\rangle \in e$ and an infinite sequence of arcs $\langle q_0, a_1, q_1\rangle$, $\langle q_1, a_2, q_2\rangle$, ... in $f$ s.t. $p \in I$, $a_i \in \{0,1\}$ for all $i$, and $a_j = 1$ for infinitely many $j$. By the premise $e \sqsubseteq^{\forall\exists} g$, there is $\langle p, a'_0, q'_0\rangle \in g$ s.t. $a_0 \leq a'_0$ and $q_0 \preceq q'_0$ (Property 1). We now show how to construct an infinite sequence $q'_0 a'_1 q'_1 a'_2 q'_2 \cdots$ that satisfies the following (Property 2): $\langle q'_n, a'_{n+1}, q'_{n+1}\rangle \in h$, $a_{n+1} \leq a'_{n+1}$, and $q_n \preceq q'_n$ for all $n \geq 0$. We do this by proving that every finite sequence $q'_0 a'_1 q'_1 \ldots q'_{k-1} a'_k q'_k$ satisfying Property 2 can be extended by one step to length $k+1$ while preserving Property 2. Moreover, such a sequence can be started (case $k = 0$) since for $k = 0$, Property 1 implies Property 2 as $q'_1$ is not in the sequence then. For the extension, we distinguish two (non-exclusive) cases:

1. $f \in G^m_{\mathcal{A}}$. Since $\langle q_k, a_{k+1}, q_{k+1}\rangle \in f$ and $q_k \preceq q'_k$ (by Property 2), Lemma 5 implies that there exists an arc $\langle q'_k, a, q\rangle \in f$ such that $a_{k+1} \leq a$ and $q_{k+1} \preceq q$. Since $f \sqsubseteq^{\forall\exists} h$, there must be some arc $\langle q'_k, a'_{k+1}, q'_{k+1}\rangle \in h$ such that $a_{k+1} \leq a \leq a'_{k+1}$ and $q_{k+1} \preceq q \preceq q'_{k+1}$.
2. $h \in G^m_{\mathcal{A}}$. Since $\langle q_k, a_{k+1}, q_{k+1}\rangle \in f$ and $f \sqsubseteq^{\forall\exists} h$, there is some arc $\langle q_k, a, q\rangle \in h$ s.t. $a_{k+1} \leq a$ and $q_{k+1} \preceq q$. Since $q_k \preceq q'_k$ (by Property 2) and $\langle q_k, a, q\rangle \in h$, Lemma 5 implies that there is an arc $\langle q'_k, a'_{k+1} q'_{k+1}\rangle \in h$ such that $a_{k+1} \leq a \leq a'_{k+1}$ and $q_{k+1} \preceq q \preceq q'_{k+1}$.

To conclude, there exist an arc $\langle p, a'_0, q'_0\rangle \in g$ and an infinite sequence of arcs $\langle q'_0, a'_1, q'_1\rangle$, $\langle q'_1, a'_2, q'_2\rangle$, ... in $h$ such that $p \in I$ and $a'_i \in \{0,1\}$ for all $i$ and $a'_j = 1$ for infinitely many $j$. Hence, $LFT(g, h)$ holds. □

Finally, we show that the set $G^m_{\mathcal{A}}$ is closed under composition.

**Lemma 8.** *$G^m_{\mathcal{A}}$ is closed under composition. That is, $\forall e, f \in G^m_{\mathcal{A}} : e;f \in G^m_{\mathcal{A}}$.*

*Proof.* As $e, f \in G^m_{\mathcal{A}}$, there are $g, h \in G^f_{\mathcal{A}}$ with $e \leqslant^* g$ and $f \leqslant^* h$. We will show that $e; f \leqslant^* g; h$ (notice that this does not directly follow from Lemma 6, since $\sqsubseteq^{\forall\exists}$ does not imply $\leqslant^*$). Since by Lemma 2, $g; h \in G^f_{\mathcal{A}}$, this will give $e; f \in G^m_{\mathcal{A}}$. By the definition of $\leqslant^*$, there are $g_0, h_0, g_1, h_1, \ldots, g_n, h_n \in G^m_{\mathcal{A}}$ s.t. $g_0 = g, h_0 = h, g_n = e, h_n = f$, and for each $i : 1 \leq i \leq n$, $g_i \leqslant g_{i-1}$ and $h_i \leqslant h_{i-1}$. We will show that for any $i : 1 \leq i \leq n$, $g_i; h_i \leqslant^* g_{i-1}; h_{i-1}$ which implies that $e; f \leqslant^* g; h$.

Since $g_i \leqslant g_{i-1}$, for every arc $\langle p, a, q\rangle \in g_i$, $\langle p, a', q\rangle \in g_{i-1}$ with $a \leq a'$. Since $h_i \leqslant h_{i-1}$, for every arc $\langle q, b, r\rangle \in h_i$, $\langle q, b', r\rangle \in h_{i-1}$ with $b \leq b'$. Therefore, by the definition of composition, for each $\langle p, c, r\rangle \in g_i; h_i$, we have $\langle p, c', r\rangle \in g_{i-1}; h_{i-1}$ with $c \leq c'$. To prove that $g_i; h_i \leqslant^* g_{i-1}; h_{i-1}$, it remains to show that there is also $\langle p, \bar{c}, \bar{r}\rangle \in g_i; h_i \cap g_{i-1}; h_{i-1}$ with $c' \leq \bar{c}$ and $r \preceq \bar{r}$. The case when $c = c'$ is trivial. If $c < c'$, then $0 \leq c'$ and thus there are $\langle p, a, q\rangle \in g_{i-1}$ and $\langle q, b, r\rangle \in h_{i-1}$ s.t. $c' = \max(\{a, b\})$. Since $g_i \leqslant g_{i-1}$, there is $\langle p, \bar{a}, \bar{q}\rangle \in g_i \cap g_{i-1}$ with $a \leq \bar{a}$ and $q \preceq \bar{q}$. By Lemma 5 and as $h_i \in G^m_{\mathcal{A}}$, there is also $\langle \bar{q}, b', r'\rangle \in h_i$ with $b \leq b'$ and $r \preceq r'$. Since $h_i \leqslant h_{i-1}$, there is $\langle \bar{q}, \bar{b}, \bar{r}\rangle \in h_i \cap h_{i-1}$ where $b' \leq \bar{b}$ and $r' \preceq \bar{r}$. Together with $\langle p, \bar{a}, \bar{q}\rangle \in g_i \cap g_{i-1}$, this implies that there is $\langle p, \bar{c}, \bar{r}\rangle \in g_i; h_i \cap g_{i-1}; h_{i-1}$ with $\max(\{\bar{a}, \bar{b}\}) \leq \bar{c}$ and $r' \preceq \bar{r}$. Since $c' = \max(\{a, b\}) \leq \max(\{\bar{a}, \bar{b}\}) \leq \bar{c}$ and $r \preceq r' \preceq \bar{r}$, $\langle p, \bar{c}, \bar{r}\rangle$ is the wanted arc. □

*The Algorithm.* Algorithm 2 gives a complete description of our approach to universality testing of BA. In this algorithm, Lines 4, 5, 14, and 15 implement Optimization 1; Lines 1 and 13 implement Optimization 2. Overall, the algorithm works such that for each graph in $G^f_{\mathcal{A}}$, a minimization of some $\sqsubseteq^{\forall\exists}$-smaller graph is generated and used in the lasso-finding tests (and only minimizations of graphs $\sqsubseteq^{\forall\exists}$-smaller than those in $G^f_{\mathcal{A}}$ are generated and used). The correctness of the algorithm is stated in Theorem 1, which is proved in Appendix C using the closure of $G^m_{\mathcal{A}}$ under composition stated in Lemma 8, the monotonicity from Lemma 6, and the preservation of lasso-finding tests from Lemma 7.

**Theorem 1.** *Algorithm 2 terminates. It returns* TRUE *iff* $\mathcal{A}$ *is universal.*

## 5 Language Inclusion of BA

Let $\mathcal{A} = (\Sigma, Q_{\mathcal{A}}, I_{\mathcal{A}}, F_{\mathcal{A}}, \delta_{\mathcal{A}})$ and $\mathcal{B} = (\Sigma, Q_{\mathcal{B}}, I_{\mathcal{B}}, F_{\mathcal{B}}, \delta_{\mathcal{B}})$ be two BA. Let $\preceq_{\mathcal{A}}$ and $\preceq_{\mathcal{B}}$ be the maximal simulations on $\mathcal{A}$ and $\mathcal{B}$, respectively. We first introduce some further notations from [8] before explaining how to extend our approach from universality to language inclusion checking. Define the set $E_{\mathcal{A}} = Q_{\mathcal{A}} \times Q_{\mathcal{A}}$. Each element in $E_{\mathcal{A}}$ is an edge $\langle p, q\rangle$ asserting that there is a path from $p$ to $q$ in $\mathcal{A}$. Define the language of an edge $\langle p, q\rangle$ as $\mathcal{L}(p, q) = \{w \in \Sigma^+ \mid p \overset{w}{\Longrightarrow} q\}$.

Define $S_{\mathcal{A},\mathcal{B}} = E_{\mathcal{A}} \times G_{\mathcal{B}}$. We call $\mathbf{g} = \langle \overline{g}, g\rangle$ a *supergraph* in $S_{\mathcal{A},\mathcal{B}}$. For any supergraph $\mathbf{g} \in S_{\mathcal{A},\mathcal{B}}$, its language $\mathcal{L}(\mathbf{g})$ is defined as $\mathcal{L}(\overline{g}) \cap \mathcal{L}(g)$. Let $Z_{\mathbf{gh}}$ be the $\omega$-regular language $\mathcal{L}(\mathbf{g}) \cdot \mathcal{L}(\mathbf{h})^{\omega}$. We say $Z_{\mathbf{gh}}$ is *proper* if $\overline{g} = \langle p, q\rangle$ and $\overline{h} = \langle q, q\rangle$ for some $p \in I_{\mathcal{A}}$ and $q \in F_{\mathcal{A}}$. Notice that, by the definition of properness, every proper $Z_{\mathbf{gh}}$ is contained in $\mathcal{L}(\mathcal{A})$. The following is a relaxed version of Lemma 4 in [8] (the constraints of being a proper $Z_{\mathbf{gh}}$ are weaker than those in [8]).

**Algorithm 2**: *Simulation-optimized Ramsey-based Universality Checking*

**Input**: A BA $\mathcal{A} = (\Sigma, Q, I, F, \delta)$, the set of all single-character graphs $G^1_{\mathcal{A}}$.
**Output**: TRUE if $\mathcal{A}$ is universal. Otherwise, FALSE.

```
1  Next := {Min(g) | g ∈ G^1_A}; Init := ∅;
2  while Next ≠ ∅ do
3      Pick and remove a graph g from Next;
4      if ∃f ∈ Init : f ⊑^{∀∃} g then Discard g and continue;
5      Remove all graphs f from Init s.t. g ⊑^{∀∃} f;
6      Add g into Init;
7  Next := Init; Processed := ∅;
8  while Next ≠ ∅ do
9      Pick a graph g from Next;
10     if ¬LFT(g, g) ∨ ∃h ∈ Processed : ¬LFT(h, g) ∨ ¬LFT(g, h) then
           return FALSE;
11     Remove g from Next and add it to Processed;
12     foreach h ∈ Init do
13         f = Min(g; h);
14         if ∃k ∈ Processed ∪ Next : k ⊑^{∀∃} f then Discard f and continue;
15         Remove all graphs k from Processed ∪ Next s.t. f ⊑^{∀∃} k;
16         Add f into Next;
17 return TRUE;
```

**Lemma 9.** *(1)* $\mathcal{L}(\mathcal{A}) = \bigcup\{Z_{\mathbf{gh}} \mid Z_{\mathbf{gh}}$ *is proper*$\}$*. (2) For all non-empty proper* $Z_{\mathbf{gh}}$*, either* $Z_{\mathbf{gh}} \cap \mathcal{L}(\mathcal{B}) = \emptyset$ *or* $Z_{\mathbf{gh}} \subseteq \mathcal{L}(\mathcal{B})$*. (3)* $\mathcal{L}(\mathcal{A}) \cap \overline{\mathcal{L}(\mathcal{B})} = \bigcup\{Z_{\mathbf{gh}} \mid Z_{\mathbf{gh}}$ *is proper and* $Z_{\mathbf{gh}} \cap \mathcal{L}(\mathcal{B}) = \emptyset\}$*.*

The above lemma implies that $\mathcal{L}(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{B})$ iff for all $\mathbf{g}, \mathbf{h} \in S_{\mathcal{A},\mathcal{B}}$, either $Z_{\mathbf{gh}}$ is not proper or $Z_{\mathbf{gh}} \subseteq \mathcal{L}(\mathcal{B})$. Since $\mathcal{L}(\mathbf{g}) = \emptyset$ or $\mathcal{L}(\mathbf{h}) = \emptyset$ implies $Z_{\mathbf{gh}} \subseteq \mathcal{L}(\mathcal{B})$, it is sufficient to build and check only supergraphs with nonempty languages (whose set we denote $S^f_{\mathcal{A},\mathcal{B}}$) when checking language inclusion.

Supergraphs in $S^f_{\mathcal{A},\mathcal{B}} = \{\mathbf{g} \in S_{\mathcal{A},\mathcal{B}} \mid \mathcal{L}(\mathbf{g}) \neq \emptyset\}$ can be built as follows. First, supergraphs $\mathbf{g} = \langle\langle p_{\mathbf{g}}, q_{\mathbf{g}}\rangle, g\rangle$ and $\mathbf{h} = \langle\langle p_{\mathbf{h}}, q_{\mathbf{h}}\rangle, h\rangle$ in $S_{\mathcal{A},\mathcal{B}}$ are *composable* iff $q_{\mathbf{g}} = p_{\mathbf{h}}$. Then, their composition $\mathbf{g};\mathbf{h}$ is defined as $\langle\langle p_{\mathbf{g}}, q_{\mathbf{h}}\rangle, g; h\rangle$. For all $a \in \Sigma$, define the set of single-character supergraphs $S^a = \{\langle\langle p, q\rangle, g_a\rangle \mid q \in \delta_{\mathcal{A}}(p, a)\}$. Let $S^1_{\mathcal{A},\mathcal{B}} := \bigcup_{a\in\Sigma} S^a$. As in universality checking, one can obtain $S^f_{\mathcal{A},\mathcal{B}}$ by repeatedly composing composable supergraphs in $S^1_{\mathcal{A},\mathcal{B}}$ until a fixpoint is reached.

A method to check whether a pair of supergraphs $\langle \mathbf{g}, \mathbf{h}\rangle$ is a counterexample to $\mathcal{L}(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{B})$, i.e., a test whether $Z_{\mathbf{gh}}$ is both proper and disjoint from $\mathcal{L}(\mathcal{B})$, was proposed in [8]. A pair of supergraphs $\langle \mathbf{g} = \langle\overline{g}, g\rangle, \mathbf{h} = \langle\overline{h}, h\rangle\rangle$ passes the *double-graph test* (denoted $DGT(\mathbf{g}, \mathbf{h})$) iff $Z_{\mathbf{gh}}$ is not proper or $LFT(g, h)$. The following lemma is due to [8] .

**Lemma 10.** $\mathcal{L}(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{B})$ *iff* $DGT(\mathbf{g}, \mathbf{h})$ *for all* $\mathbf{g}, \mathbf{h} \in S^f_{\mathcal{A},\mathcal{B}}$*.*

Analogously to the universality checking algorithm in Section 3, a language inclusion checking algorithm can be obtained by combining the above principles for generating supergraphs in $S^f_{\mathcal{A},\mathcal{B}}$ and using the double-graph test (cf. Appendix D).

## 6 Improving Language Inclusion Testing via Simulation

Here we describe our approach of utilizing simulation-based subsumption techniques to improve the Ramsey-based language inclusion test.

In order to be able to use simulation-based subsumption as in Section 4, we lift the subsumption relation $\sqsubseteq^{\forall\exists}$ to supergraphs as follows: Let $\mathbf{g} = \langle\langle p_\mathbf{g}, q_\mathbf{g}\rangle, g\rangle$ and $\mathbf{h} = \langle\langle p_\mathbf{h}, q_\mathbf{h}\rangle, h\rangle$ be two supergraphs in $S_{\mathcal{A},\mathcal{B}}$. Let $\mathbf{g} \sqsubseteq^{\forall\exists}_S \mathbf{h}$ iff $p_\mathbf{g} = p_\mathbf{h}$, $q_\mathbf{g} \succeq_\mathcal{A} q_\mathbf{h}$, and $g \sqsubseteq^{\forall\exists} h$. Define $\simeq^{\forall\exists}_S$ as $\sqsubseteq^{\forall\exists}_S \cap (\sqsubseteq^{\forall\exists}_S)^{-1}$.

Since we want to work with supergraphs that are minimal w.r.t. $\sqsubseteq^{\forall\exists}_S$, we need to change the definition of properness and the respective double-graph test. We say that $Z_\mathbf{gh}$ is *weakly proper* iff $\overline{g} = \langle p, q\rangle$ and $\overline{h} = \langle q_1, q_2\rangle$ where $p \in I_\mathcal{A}$, $q_2 \in F_\mathcal{A}$, $q \succeq_\mathcal{A} q_1$, and $q_2 \succeq_\mathcal{A} q_1$.

**Definition 3.** *Supergraphs* $\mathbf{g}, \mathbf{h} \in S_{\mathcal{A},\mathcal{B}}$ pass the relaxed double-graph test*, written* $RDGT(\mathbf{g}, \mathbf{h})$*, iff either (1)* $Z_\mathbf{gh}$ *is not weakly proper, or (2)* $LFT(g, h)$*.*

The following Lemma 11 is needed to prove the central Theorem 2.

**Lemma 11.** *Every weakly proper* $Z_\mathbf{gh}$ *is contained in* $\mathcal{L}(\mathcal{A})$*.*

**Theorem 2.** $\mathcal{L}(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{B})$ *iff* $RDGT(\mathbf{g}, \mathbf{h})$ *for all* $\mathbf{g}, \mathbf{h} \in S^f_{\mathcal{A},\mathcal{B}}$*.*

Furthermore, we lift the notions of $\leqslant^*$ and *Min* from Section 4 from graphs to supergraphs. For any two supergraphs $\mathbf{g} = \langle\overline{g}, g\rangle, \mathbf{h} = \langle\overline{h}, h\rangle$ from $S_{\mathcal{A},\mathcal{B}}$, we write $\mathbf{g} \leqslant^*_S \mathbf{h}$ iff $\overline{g} = \overline{h}$ and $g \leqslant^* h$. Then $S^m_{\mathcal{A},\mathcal{B}} = \{\mathbf{g} \in S_{\mathcal{A},\mathcal{B}} \mid \exists \mathbf{h} \in S^f_{\mathcal{A},\mathcal{B}} : \mathbf{g} \leqslant^*_S \mathbf{h}\}$. $Min_S(\mathbf{g})$ again computes a graph that is $\leqslant^*_S$-smaller than $\mathbf{g}$. It is a possibly non-deterministic operation such that $Min_S(\overline{g}, g) = \langle\overline{g}, Min(g)\rangle$.

Like in Section 4, it is now possible to prove a closure of $S^m_{\mathcal{A},\mathcal{B}}$ under composition as well as preservation of the double-graph test on $\sqsubseteq^{\forall\exists}_S$-larger supergraphs (cf. Appendix E). What slightly differs is the monotonicity of the composition, which is caused by the additional composability requirement. To cope with it, we define a new relation $\trianglelefteq^{\forall\exists}$, weakening $\sqsubseteq^{\forall\exists}_S$: For $\mathbf{g} = \langle\langle p, q\rangle, g\rangle, \mathbf{h} = \langle\langle p', q'\rangle, h\rangle \in S_{\mathcal{A},\mathcal{B}}$, $\mathbf{g} \trianglelefteq^{\forall\exists} \mathbf{h}$ iff $p' \preceq p$, $q' \preceq q$, and $g \sqsubseteq^{\forall\exists} h$. Notice that $\sqsubseteq^{\forall\exists}_S$ indeed implies $\trianglelefteq^{\forall\exists}$. From the definitions of $\sqsubseteq^{\forall\exists}_S$, $\trianglelefteq^{\forall\exists}$, and Lemma 6, we then easily get the following relaxed monotonicity lemma.

**Lemma 12.** *For any* $\mathbf{e}, \mathbf{f}, \mathbf{g} \in S_{\mathcal{A},\mathcal{B}}$ *and* $\mathbf{h} \in S^m_{\mathcal{A},\mathcal{B}}$ *with* $\mathbf{e} \sqsubseteq^{\forall\exists}_S \mathbf{g}$*, and* $\mathbf{f} \trianglelefteq^{\forall\exists} \mathbf{h}$ *where* $\mathbf{e}, \mathbf{f}$ *and* $\mathbf{g}, \mathbf{h}$ *are composable,* $\mathbf{e};\mathbf{f} \sqsubseteq^{\forall\exists}_S \mathbf{g};\mathbf{h}$*.*

Now we show that it is safe to work with $\sqsubseteq^{\forall\exists}_S$-smaller supergraphs in the incremental supergraph construction. Given supergraphs $\mathbf{e}$, $\mathbf{g}$, $\mathbf{h}$ s.t. $\mathbf{e} \sqsubseteq^{\forall\exists}_S \mathbf{g}$

**Algorithm 3**: *Optimized Ramsey-based Language Inclusion Checking*

**Input**: BA $\mathcal{A} = (\Sigma, Q_\mathcal{A}, I_\mathcal{A}, F_\mathcal{A}, \delta_\mathcal{A})$, $\mathcal{B} = (\Sigma, Q_\mathcal{B}, I_\mathcal{B}, F_\mathcal{B}, \delta_\mathcal{B})$, and the set $S^1_{\mathcal{A},\mathcal{B}}$.
**Output**: TRUE if $\mathcal{L}(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{B})$. Otherwise, FALSE.

```
1  Next := {Min_S(g) | g ∈ S^1_{A,B}}; Init := ∅;
2  while Next ≠ ∅ do
3      Pick and remove a supergraph g from Next;
4      if ∃f ∈ Init : f ⊑^{∀∃}_S g then Discard g and continue;
5      Remove all supergraphs f from Init s.t. g ⊑^{∀∃}_S f;
6      Add g into Init;
7  Next := Init; Processed := ∅;
8  while Next ≠ ∅ do
9      Pick a supergraph g from Next;
10     if ¬RDGT(g, g) ∨ ∃h ∈ Processed : ¬RDGT(h, g) ∨ ¬RDGT(g, h) then
           return FALSE;
11     Remove g from Next and add it to Processed;
12     foreach h ∈ Init where ⟨g, h⟩ are composable do
13         f := Min_S(g; h);
14         if ∃k ∈ Processed ∪ Next : k ⊑^{∀∃}_S f then Discard f and continue;
15         Remove all supergraphs k from Processed ∪ Next s.t. f ⊑^{∀∃}_S k;
16         Add f into Next;
17 return TRUE;
```

and $\mathbf{g}$, $\mathbf{h}$ are composable, one can always find a supergraph $\mathbf{f}$ satisfying the preconditions of Lemma 12—excluding the situation of only the bigger supergraphs $\mathbf{g}$, $\mathbf{h}$ being composable. Fortunately, it is possible to show that the needed supergraph $\mathbf{f}$ always exists in $S^m_{\mathcal{A},\mathcal{B}}$. Moreover, since only 1-letter supergraphs are used on the right of the composition, all supergraphs needed as right operands in the compositional construction can always be found in the minimization of $S^1_{\mathcal{A},\mathcal{B}}$, which can easily be generated.

Algorithm 3 is our simulation-optimized algorithm for BA inclusion-checking. Its correctness is stated below and proved in Appendix E.

**Theorem 3.** *Algorithm 3 terminates. It returns* TRUE *iff* $\mathcal{L}(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{B})$.

## 7 Experimental Results

We have implemented both our simulation subsumption algorithms and the original ones of Fogarty and Vardi [8, 9] in Java. For universality testing, we compared our algorithm to the one in [9].[1] For language inclusion testing, we compared

[1] The algorithm in [9] uses the simple subsumption relation $\sqsubseteq$, and also a different search strategy than our algorithm. To take into account the latter, we have also included a combination of our search strategy with the simple subsumption into our experiments. We evaluated the new search strategy with random automata of size

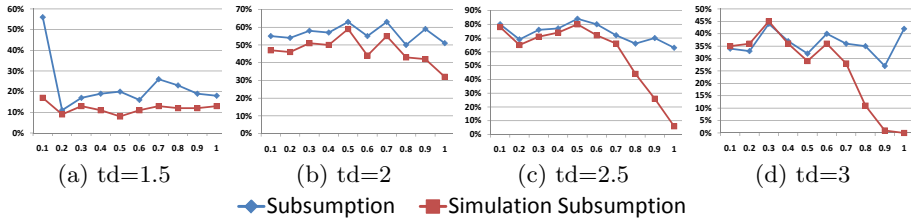


**Fig. 1.** Timeout cases of universality checking on Tabakov and Vardi's random model. The vertical axis is the percentage of tasks that cannot be finished within the timeout period and the horizontal axis is the acceptance density ($ad$).

our simulation subsumption algorithm to a restricted version that uses the simple subsumption relation of [9]. (The language inclusion checking algorithm described in [8] does not use any subsumption at all and performed much worse.) We refer interested readers to [20] for all relevant materials needed to reproduce the results. A description of the machines that we used is given in Appendix F.

*Universality Testing.* We have two sets of experiments. In Experiment 1, we use Tabakov and Vardi's random model. This model fixes the alphabet size to 2 and uses two parameters: *t*ransition *d*ensity (i.e. the average number of outgoing transitions per alphabet symbol) and *a*cceptance *d*ensity (percentage of accepting states). We use this approach with $td = 0.5, 1, \ldots, 4$ and $ad = 0.1, 0.2, \ldots, 1.0$, and generate 100 random BA for each combination of $td$ and $ad$. In Figure 1, we compare the number of timeouts between the two approaches when the number of states is 100 and the timeout period is 3500 seconds[2]. We only list cases with $td = 1.5, 2.0, 2.5, 3.0$, since in the other cases both methods can complete most of the tasks within the timeout period. For the two most difficult configurations ($td = 2.5, 3.0$), the difference between the two approaches gets larger as $ad$ increases. The trend shown in Figure 1 stayed the same when the number of states increases. We refer the interested readers to Appendix F for results of automata with the number of states ranging from 10 to 200.

Experiment 2 uses BA from random LTL formulae. We generate only valid formulae (in the form $f \vee \neg f$), thus ensuring that the corresponding BA are universal. We only focus on valid formulae since most BA generated from random LTL formulae can be recognized as non-universal in almost no time. Thus, the results would not have been interesting. We generate LTL formulae with lengths 12, 18, 24, and 30 and 1–4 propositions (which corresponds to automata with

20. While our search strategy with the simple subsumption $\sqsubseteq$ is already on average about 20% faster than [9], the main improvement we witness in our experiments is achieved by using the simulation subsumption $\sqsubseteq^{\forall\exists}$.

[2] Our approach can be slightly slower than [9] in some rare cases, due to the overhead of computing simulation. For those cases, the simulation relation is sparse and does not yield any speedup. However, since there are efficient algorithms for computing simulation, the relative overhead is quite small on difficult instances.

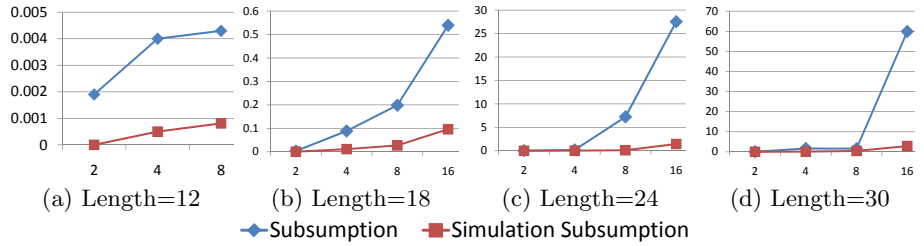


**Fig. 2.** Universality checking on automata generated from random LTL. Each point in the figure is the average result of 100 different instances. The horizontal axis is the size of the alphabet. The vertical axis is the average execution time in seconds.

| | Original | | Error | | Subsumption | | Simulation Sub. | |
|---|---|---|---|---|---|---|---|---|
| | Tr. | St. | Tr. | St. | $\mathcal{L}(E)\subseteq\mathcal{L}(O)$ | $\mathcal{L}(O)\subseteq\mathcal{L}(E)$ | $\mathcal{L}(E)\subseteq\mathcal{L}(O)$ | $\mathcal{L}(O)\subseteq\mathcal{L}(E)$ |
| Bakery | 2697 | 1506 | 2085 | 1146 | >1day | >1day | 32m15s(F) | 49m57s |
| Peterson | 34 | 20 | 33 | 20 | 2.7s(T) | 1.4s(F) | 0.9s(T) | 0.2s(F) |
| Dining phil. Ver.1 | 212 | 80 | 464 | 161 | >1day | >1day | 11m52s(F) | >1day |
| Dining phil. Ver.2 | 212 | 80 | 482 | 161 | >1day | >1day | 14m40s(F) | >1day |
| Fisher Ver.1 | 1395 | 634 | 3850 | 1532 | >1day | oom | 1m23s(F) | >1day |
| Fisher Ver.2 | 1395 | 634 | 1420 | 643 | >1day | >1day | 8s(T) | 8s(T) |
| MCS queue lock | 21503 | 7963 | 3222 | 1408 | >1day | >1day | 1h36m(T) | 6m22s(F) |

**Table 1.** Language inclusion checking on mutual exclusion algorithms. The columns "Original" and "Error" refer to the original, resp., erroneous, model. We test inclusion for both directions. ">1day" = the task cannot be finished within one day. "oom" = required memory > 8GB. If a task completes successfully, we record the run time and the fact whether language inclusion holds or not ("T" or "F", respectively).

alphabet sizes 2, 4, 8, and 16). For each configuration, we generate 100 BA[3]. The results are shown in Figure 2. The difference between the two approaches is quite large in this set of experiments. With formulae of length 30 and 4 propositions, our approach is 20 times faster than the original subsumption based approach.

*Language Inclusion Testing.* We again have two sets of experiments. In Experiment 1, we inject (artificial) errors into models of several mutual exclusion algorithms from [14][4], translate both the original and the modified version into BA, and test whether the control flow (w.r.t. the occupation of the critical section) of the two versions is the same. In these models, a state $p$ is accepting iff the critical section is occupied by some process in $p$. The results are shown in Table 1. The results of a slightly different version where all states are accepting is given in in Appendix F. In both versions, our approach has significantly and consistently better performance than the basic subsumption approach.

[3] We do not have formulae having length 12 and 4 propositions because our generator [18] requires that $(\textit{length of formula}/3) > \textit{number of propositions}$.

[4] The models in [14] are based on guarded commands. We modify the models by randomly weakening or strengthening the guard of some commands.

In Experiment 2, we translate two randomly generated LTL formulae to BA $\mathcal{A}$, $\mathcal{B}$ and then check whether $\mathcal{L}(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{B})$. We use formulae having length 10, 15, and 20 and 1–3 propositions (which corresponds to BA of alphabet sizes 2, 4, and 8). For each length of formula and each number of propositions, we generate 10 pairs of BA. The relative results are shown in Figure 3. The difference in performance of using the basic subsumption and the simulation subsumption is again quite significant here. For the largest cases (with formulae having length 20 and 3 propositions), our approach is 1914 times faster than the basic subsumption approach.

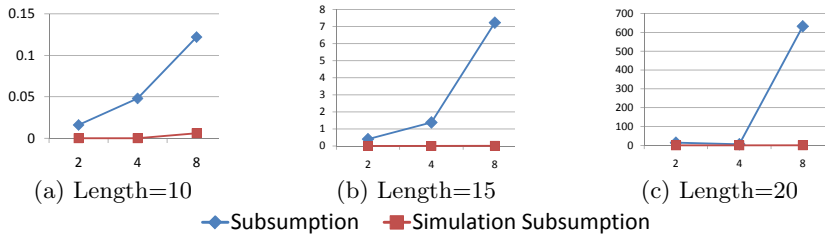


**Fig. 3.** Language inclusion checking on automata generated from random LTL. Each point in the figure is the average result of 10 different instances. The horizontal axis is the size of the alphabet. The vertical axis is the average execution time in seconds.

## 8 Conclusions and Extensions

We have introduced simulation-based subsumption techniques for Ramsey-based universality/language-inclusion checking for nondeterministic BA. We evaluated our approach by a wide set of experiments, showing that the simulation-subsumption approach consistently outperforms the basic subsumption of [9].

Our techniques can be extended in several ways. Weaker simulations for BA have been defined in the literature, e.g., delayed/fair simulation [6], or their multipebble extensions [5]. One possibility is to quotient the BA w.r.t. (multipebble) delayed simulation, which (unlike quotienting w.r.t. fair simulation) preserves the language. Furthermore, in our language inclusion checking algorithm, the subsumption w.r.t. direct simulation $\preceq_{\mathcal{A}}$ on $\mathcal{A}$ can be replaced by the weaker delayed simulation (but not by fair simulation). Moreover, in the definition of being *weakly proper* in Section 6, in the condition $q \succeq_{\mathcal{A}} q_1$, the $\succeq_{\mathcal{A}}$ can be replaced by any other relation that implies $\omega$-language containment, e.g., fair simulation. On the other hand, delayed/fair simulation cannot be used for subsumption in the automaton $\mathcal{B}$ in inclusion checking (nor in universality checking), since Lemma 6 does not carry over.

Next, our language-inclusion algorithm does not currently exploit any simulation preorders *between* $\mathcal{A}$ and $\mathcal{B}$. Of course, direct/delayed/fair simulation between the respective initial states of $\mathcal{A}$ and $\mathcal{B}$ is sufficient (but not necessary) for language inclusion. However, it is more challenging to exploit simulation preorders between internal states of $\mathcal{A}$ and internal states of $\mathcal{B}$.

Finally, it is easy to see that the proposed simulation subsumption technique can be built over *backward simulation preorder* too. It would, however, be interesting to evaluate such an approach experimentally. Further, one could then also try to extend the framework to use some form of *mediated preorders* combining forward and backward simulation preorders like in the approach of [2].

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## A Proof of Lemma 1 from Section 2

Define the function $\mathcal{H}(A) = \{\{x, y\} \mid x, y \in A \wedge x \neq y\}$, where $A$ is an countably infinite set. The following is a suitable version of the infinite Ramsey theorem.

**Lemma 13.** *Let $A$ be an countably infinite set, and let $B = \mathcal{H}(A)$. For any partitioning of $B$ into finitely many classes $B_1, \ldots, B_m$, there exists an infinite subset $A'$ of $A$ s.t. $\mathcal{H}(A') \subseteq B_k$, for some $k$.*

*Proof (Lemma 1).* We first show Point 1, i.e., $\Sigma^\omega = \bigcup\{Y_{gh} \mid g, h \in G_\mathcal{A}\}$. Let $w = a_0 a_1 \ldots$ be any $\omega$-word. We have to show that there exist graphs $g, h \in G_\mathcal{A}$ s.t. $w \in Y_{gh}$.

Consider prefixes $w_i = a_0 \ldots a_i$ of $w$. Each $w_i$ belongs to the language of some graph in $G_\mathcal{A}$, and, being the number of such graphs finite, there exists a graph $g \in G_\mathcal{A}$ s.t. $w_i \in \mathcal{L}(g)$ for infinitely many $i$'s. Let $A = \mathcal{L}(g)$.

For any $w_i, w_j \in \mathcal{L}(g)$, $i \leq j$, let $w_j \ominus w_i = a_{i+1} \ldots a_j$ (and define $w_j \ominus w_i$ as $w_i \ominus w_j$ if $i > j$). Let $B = \mathcal{H}(A)$ be the set of unordered pairs of strings from $\mathcal{L}(g)$. Consider the partitioning of $B = \bigcup_{h \in G_\mathcal{A}} B_h$ into finitely many classes, where each class $B_h$ is defined as: $\{w_i, w_j\} \in B_h$ iff $w_j \ominus w_i \in \mathcal{L}(h)$.

By the infinite-Ramsey theorem, there exists a graph $h$ in $G_\mathcal{A}$ and an infinite subset $A'$ of $\mathcal{L}(g)$ s.t. $\mathcal{H}(A') \subseteq B_h$, i.e., for every $w_i, w_j$ in $A'$, $w_j \ominus w_i$ belongs to $\mathcal{L}(h)$. Thus, it is possible to split the word $w$ as follows:

$$w = a_0 \ldots a_{i_0 - 1} \quad a_{i_0} \ldots a_{i_1 - 1} \quad a_{i_1} \ldots a_{i_2 - 1} \quad a_{i_2} \ldots$$

where $a_0 \ldots a_{i_0 - 1} \in \mathcal{L}(g)$, and $a_{i_k} \ldots a_{i_{k+1} - 1} \in \mathcal{L}(h)$ for $k \geq 0$. (Here, $i_0$ is the least index $i$ s.t. $w_i \in A'$, and, inductively, $i_{k+1}$ is the least index $i > i_k$ s.t. $w_i \in A'$.) Hence, $w \in Y_{gh}$.

We now show Point 2, i.e., $\forall g, h \in G_\mathcal{A}$ with $\mathcal{L}(g), \mathcal{L}(h) \neq \emptyset$, either $Y_{gh} \cap \mathcal{L}(\mathcal{A}) = \emptyset$ or $Y_{gh} \subseteq \mathcal{L}(\mathcal{A})$. We prove that if a word $w \in Y_{gh}$ is in $\mathcal{L}(\mathcal{A})$, then *every* word $w' \in Y_{gh}$ is in $\mathcal{L}(\mathcal{A})$ as well.

The crucial observation is that, from the definition of $\mathcal{L}(g)$, all words in $\mathcal{L}(g)$ share the same reachability properties in $\mathcal{A}$. More precisely, $w \in \mathcal{L}(g)$ implies that, for any pair of states $p, q \in Q$, $w \in \mathcal{L}(p, a, q)$ with $\langle p, a, q\rangle \in g$. This implies that, for any two $w, w' \in \mathcal{L}(g)$, and for any two $p, q \in Q$, 1) $w$ induces a path between $p$ and $q$ iff $w'$ does so, i.e., $p \stackrel{w}{\Longrightarrow} q$ iff $p \stackrel{w'}{\Longrightarrow} q$, and 2) $w$ induces an accepting path between $p$ and $q$ iff $w'$ does so, i.e., $p \stackrel{w}{\underset{F}{\Longrightarrow}} q$ iff $p \stackrel{w'}{\underset{F}{\Longrightarrow}} q$.

Thus, assume that $w \in Y_{gh}$ is in $\mathcal{L}(\mathcal{A})$ and let $w'$ be any other word in $Y_{gh}$. We have to show $w' \in \mathcal{L}(\mathcal{A})$. Let $\pi = q_0 q_1 \ldots$ be an accepting run of $\mathcal{A}$ over $w$, thus witnessing $w$ being accepted. We can decompose $w$ in $w = w_0 w_1 w_2 \ldots$, with $w_0 \in \mathcal{L}(g)$ and $w_i \in \mathcal{L}(h)$, for all $i > 0$. Moreover, the above decomposition of $w$ induces a decomposition of $\pi$ in $q_{i_0} \ldots q_{i_1} \ldots q_{i_2} \ldots$ (where $i_0 = 0$) s.t. $q_{i_k} \stackrel{w_k}{\Longrightarrow} q_{i_{k+1}}$ for any $k \geq 0$, and $q_{i_k} \stackrel{w_k}{\underset{F}{\Longrightarrow}} q_{i_{k+1}}$ for infinitely many $k > 0$.

We now decompose $w' = w'_0 w'_1 w'_2 \ldots$ as we did with $w$ above, i.e., $w'_0 \in \mathcal{L}(g)$ and $w'_i \in \mathcal{L}(h)$, for $i > 0$, and we show how to build an accepting run $\pi' = q'_0 q'_1 \ldots$

over $w'$. We take $q'_0 := q_0$. By $w'_0 \in \mathcal{L}(g)$ and the crucial observation 1) above, $q_0 \overset{w'_0}{\Longrightarrow} q_{i_1}$, hence we make the two runs $\pi$ and $\pi'$ synchronize at $i_1$, i.e., $q'_{i_1} := q_{i_1}$. Similarly, for any $k > 0$, by $w'_k \in \mathcal{L}(h)$ and the crucial observation 1), then $q_{i_k} \overset{w'_k}{\Longrightarrow} q_{i_{k+1}}$, thus $q'_{i_{k+1}} := q_{i_{k+1}}$, for $k > 0$. Hence, the two paths $\pi$ and $\pi'$ synchronize at indices $i_k$, for $k \geq 0$. Moreover, if $q_{i_k} \overset{w_k}{\underset{F}{\Longrightarrow}} q_{i_{k+1}}$ for infinitely many positive $k$'s, then, by the crucial observation 2), $q'_{i_k} \overset{w'_k}{\underset{F}{\Longrightarrow}} q'_{i_{k+1}}$ for infinitely many positive $k$'s as well. Hence, if $\pi$ is an accepting run, then $\pi'$ is an accepting run. Thus, $w' \in \mathcal{L}(\mathcal{A})$.

Finally, Point 3 follows directly from Points 1 and 2.

□

## B Proofs of Lemmas from Section 3

The following auxiliary lemma has been proved in [7].

**Lemma 14.** *(Lemma 3.1.1 in [7])* $\forall g, h \in G_{\mathcal{A}} : \mathcal{L}(g) \cdot \mathcal{L}(h) \subseteq \mathcal{L}(g; h)$.

We now prove Lemma 2, which states the relationship between graphs in $G^f_{\mathcal{A}}$ and $G^1_{\mathcal{A}}$.

*Proof (Lemma 2).* Since the languages of the graphs in $G_{\mathcal{A}}$ form a partition of $\Sigma^+$, we have that $w \in \mathcal{L}(g)$ and $w \in \mathcal{L}(h)$ iff $\mathcal{L}(g) = \mathcal{L}(h)$ iff $g = h$. Let $h$ be a graph with a non-empty language and let $w = a_1 a_2 a_3 \dots a_n$ be a word in $\mathcal{L}(h)$. The word $w$ is also in the language of the graph $g_{a_1}; g_{a_2}; \dots; g_{a_n}$. Therefore, $h = g_{a_1}; g_{a_2}; \dots; g_{a_n}$. Conversely, assume $h = g_{a_1}; g_{a_2}; \dots; g_{a_n}$ with $a_i \in \Sigma$ for $1 \leq i \leq n$. By Lemma 14, the word $a_1 a_2 a_3 \dots a_n$ is in $\mathcal{L}(h)$. Therefore, $h \in G^f_{\mathcal{A}}$. □

Before proving Lemma 3, we first prove the lemma below.

**Lemma 15.** *For* $g, h \in G^f_{\mathcal{A}}$, $LFT(g, h)$ *iff* $Y_{gh} \cap \mathcal{L}(\mathcal{A}) \neq \emptyset$.

*Proof.* For the "if" direction, assume that $Y_{gh} \cap \mathcal{L}(\mathcal{A}) \neq \emptyset$. Let $uv^\omega$ be a word in $Y_{gh} \cap \mathcal{L}(\mathcal{A})$ s.t. $u \in \mathcal{L}(g)$ and $v \in \mathcal{L}(h)$. Since $uv^\omega \in \mathcal{L}(\mathcal{A})$, we have that there exists an infinite sequence of states $p, q_0, q_1, \dots$ in $Q$ s.t. (1) $p \overset{u}{\Longrightarrow} q_0$, $p \in I$ and (2) $q_i \overset{v}{\Longrightarrow} q_{i+1}$ for all $i \in \mathbb{N}$ and $q_j \overset{v}{\underset{F}{\Longrightarrow}} q_{j+1}$ for infinitely many $j \in \mathbb{N}$. From (1) and $u \in \mathcal{L}(g)$, there exists an arc $\langle p, a_0, q_0 \rangle$ in $g$ for $a_0 \in \{0, 1\}$, $p \in I$. From (2) and $v \in \mathcal{L}(h)$, there exists an infinite sequence of arcs $\langle q_0, a_1, q_1 \rangle, \langle q_1, a_2, q_2 \rangle \dots \langle q_{k-1}, a_k, q_k \rangle \dots$ in $h$ where $a_i \in \{0, 1\}$ for all $i \in \mathbb{N}$, and $a_j = 1$ for infinitely many $j \in \mathbb{N}$. Therefore, we have $LFT(g, h)$.

For the "only if" direction, assume that $LFT(g, h)$. There exists an infinite sequence of states $p, q_0, q_1, \dots$ in $Q$ s.t. (1) the arc $\langle p, a_0, q_0 \rangle$ in $g$ for $a_0 \in \{0, 1\}$, $p \in I$ and (2) the arcs $\langle q_0, a_1, q_1 \rangle, \langle q_1, a_2, q_2 \rangle \dots \langle q_{k-1}, a_k, q_k \rangle \dots$ in $h$ where $a_i \in \{0, 1\}$ for all $i \in \mathbb{N}$, and $a_j = 1$ for infinitely many $j \in \mathbb{N}$. Let $uv^\omega$ be any word in

$Y_{gh}$ s.t. $u \in \mathcal{L}(g)$ and $v \in \mathcal{L}(h)$—such a word must exist as $g, h \in G^f_{\mathcal{A}}$. From (1) and (2), we have $p \overset{u}{\Longrightarrow} q_0$ for $p \in I$, $q_i \overset{v}{\Longrightarrow} q_{i+1}$ for all $i \in \mathbb{N}$ and $q_j \overset{v}{\underset{F}{\Longrightarrow}} q_{j+1}$ for infinitely many $j \in \mathbb{N}$. Hence, $uv^\omega \in \mathcal{L}(\mathcal{A})$, and thus we have $Y_{gh} \cap \mathcal{L}(\mathcal{A}) \neq \emptyset$. □

Now, we can prove Lemma 3.

*Proof (Lemma 3).* We have to show that $\mathcal{A}$ is universal iff for every pair of graphs $g, h$ in $G^f_{\mathcal{A}}$, $LFT(g, h)$ holds. For the "if" direction, assume that $\mathcal{A}$ is not universal. Hence, by Point 3 of Lemma 1, there must be some $g, h \in G^f_{\mathcal{A}}$ s.t. $Y_{gh} \cap \mathcal{L}(\mathcal{A}) = \emptyset$. By Lemma 15, $\neg LFT(g, h)$. For the "only if" direction, assume $\mathcal{A}$ is universal. By Point 3 in Lemma 1, $\forall g, h \in G^f_{\mathcal{A}} : \ Y_{gh} \cap \mathcal{L}(\mathcal{A}) \neq \emptyset$. By Lemma 15, we have $LFT(g, h)$ for all possible $g, h \in G^f_{\mathcal{A}}$. □

## C Correctness of Algorithm 2

**Lemma 16.** *For any $g \in G^1_{\mathcal{A}}$, there is some $f \in Init$ on Line 7 of Algorithm 2 such that $f \sqsubseteq^{\forall\exists} g$.*

*Proof.* First note that due to Lemma 4, $Min(g) \sqsubseteq^{\forall\exists} g$ for each $g \in G^1_{\mathcal{A}}$. The graph $Min(g)$ is put into *Next* on Line 1. Then, $Min(g)$ either

- stays in *Next*,
- is moved to *Init* due to Lines 3 and 6,
- is discarded due to Lines 3 and 4 which can, however, happen only if there is already some $\sqsubseteq^{\forall\exists}$-smaller graph in *Init* (which can then only be replaced by even $\sqsubseteq^{\forall\exists}$-smaller graphs on Lines 5 and 6),
- or is removed from *Init* on Line 5 when some $\sqsubseteq^{\forall\exists}$-smaller graph is put into *Init* on Line 6 (which can then in turn be replaced by even $\sqsubseteq^{\forall\exists}$-smaller graphs on Lines 5 and 6).

Since the loop terminates only when $Next = \emptyset$, the lemma clearly holds. □

**Lemma 17.** *Once any graph $f$ appears in the set Next in between of Lines 8–16 of Algorithm 2, then from this point in the run of the algorithm onwards there is always some $f' \in Next \cup Processed$ such that $f' \sqsubseteq^{\forall\exists} f$.*

*Proof.* The lemma holds since $f$ can only stay in *Next*, be moved to *Processed* on Line 11, or be removed from *Next* or *Processed* on Line 15 in which case, however, a $\sqsubseteq^{\forall\exists}$-smaller graph is put into *Next* on Line 16 (which can in turn be replaced by further $\sqsubseteq^{\forall\exists}$-smaller graphs on Lines 15 and 16). □

**Lemma 18.** *All graphs tested on the lasso-finding test within a run of Algorithm 2 are in $G^m_{\mathcal{A}}$.*

*Proof.* All graphs that participate in the lasso-finding test on Line 10 are either (1) equal to $Min(g)$ for some $g \in G^1_{\mathcal{A}}$, or (2) they are generated on Line 13. Minimizations of graphs from $G^1_{\mathcal{A}}$ are clearly in $G^m_{\mathcal{A}}$. In Case (2), the newly

generated graphs are minimizations of compositions of graphs from *Init* and graphs from *Next*. Provided that $Next \subseteq G^m_{\mathcal{A}}$, any graph generated on Line 13 is again in $G^m_{\mathcal{A}}$ by Lemma 8 and because $G^m_{\mathcal{A}}$ is clearly closed under *Min*. Since the initial value of *Next* is a subset of $G^m_{\mathcal{A}}$ (minimizations of graphs from $G^1_{\mathcal{A}}$) and since after initialization, only graphs generated on Line 13 can be added into *Next*, *Next* will always stay a subset of $G^m_{\mathcal{A}}$. Therefore, all graphs $f$ generated on Line 13 and used in the lasso-finding test on Line 10 are always in $G^m_{\mathcal{A}}$. □

Let $Processed^{end}$ denote the contents of the set *Processed* when Algorithm 2 terminates.

**Lemma 19.** *If Algorithm 2 returns* TRUE*, then for every* $g \in G^f_{\mathcal{A}}$*, there exists some* $h \in Processed^{end}$ *such that* $h \sqsubseteq^{\forall\exists} g$*.*

*Proof.* By Lemma 2, the graph $g$ is a finite composition $g_0; g_1; \ldots; g_n$ of $n+1$ graphs from $G^1_{\mathcal{A}}$. We now prove by induction that for each $0 \leq i \leq n$, there exists a graph $h_i \in Processed^{end}$ such that $h_i \sqsubseteq^{\forall\exists} g_0; g_1; \ldots; g_i$.

We start with the base case. Due to Lemma 16 and the code on Line 7, there is some $f_0 \in Next$ such that $f_0 \sqsubseteq^{\forall\exists} g_0$ when the second loop is entered for the first time. Using Lemma 17 and the fact that *Next* is empty when Algorithm 2 returns TRUE, the base case is proved.

For the inductive step, we assume that there exists a graph $h_k \in Processed^{end}$ such that $h_k \sqsubseteq^{\forall\exists} g_0; g_1; \ldots; g_k$. Due to Lemma 16, we know that there is $f_{k+1} \in Init$ such that $f_{k+1} \sqsubseteq^{\forall\exists} g_{k+1}$. By Lemmas 6 and 4, $Min(h_k; f_{k+1}) \sqsubseteq^{\forall\exists} g_0; g_1; \ldots; g_k; g_{k+1}$. The graph $Min(h_k; f_{k+1})$ is added into *Next* on Line 16 right after $h_k$ is moved to *Processed* on Line 11 (unless there is already some $\sqsubseteq^{\forall\exists}$-smaller graph in $Processed \cup Next$). From Lemma 17 and from the fact that *Next* is empty when Algorithm 2 returns TRUE, it is then clear that there is some graph $h_{k+1} \in Processed^{end}$ such that $h_{k+1} \sqsubseteq^{\forall\exists} g_0; g_1; \ldots; g_{k+1}$. □

**Lemma 20.** *Algorithm 2 eventually terminates.*

*Proof.* The set $G^1_{\mathcal{A}}$ is finite, and hence the set *Next* constructed on Line 1 is finite too. Each iteration of the loop in Lines 2–6 removes one graph from *Next*, and hence the loop must terminate. Moreover, each of the finitely many iterations of the first loop adds at most one graph to *Init*, and hence the set $Next = Init$ constructed on Line 7 is finite (if Line 7 is reached at all). Now, for the loop on Lines 8–16 not to terminate, it would have to be the case that some graph is repeatedly removed from *Next* on Line 11. Hence, this graph would have to be repeatedly added into *Next* too. However, due to Lemma 17, once some graph $g$ appears in *Next* on Line 11, there will always be some graph $h \in Processed \cup Next$ such that $h \sqsubseteq^{\forall\exists} g$. Hence, due to the test on Line 14, $g$ cannot be added to *Next* for the second time. □

Now we are finally able to prove Theorem 1.

*Proof (Theorem 1).* By Lemma 20, Algorithm 2 eventually terminates and returns a value.

First, we prove by contradiction that if TRUE is returned, $\mathcal{A}$ is universal. Assume that $\mathcal{A}$ is not universal. Then, by Lemma 3, (1) there is either some pair of graphs $g \neq h \in G^f_{\mathcal{A}}$ such that $\neg LFT(g,h)$, or (2) there is a graph $f \in G^f_{\mathcal{A}}$ such that $\neg LFT(f,f)$.

Let us start with Case (1). By Lemma 19, we have that there exist some $g', h' \in \mathit{Processed}^{end}$ such that $g' \sqsubseteq^{\forall\exists} g$ and $h' \sqsubseteq^{\forall\exists} h$. Assume that $g'$ is added to *Processed* after $h'$ (the other case being symmetrical due to the "or" used in the condition on Line 10). When $g'$ is about to be added to *Processed*, the algorithm performs a lasso-finding test on $\langle g', h' \rangle$ on Line 10. Since $\langle g, h \rangle$ fails the lasso-finding test, $g' \sqsubseteq^{\forall\exists} g$, and $h' \sqsubseteq^{\forall\exists} h$, we know by Lemma 7 that also $\neg LFT(g', h')$, and thus the algorithm returns FALSE, which leads to a contradiction.

In Case (2), by Lemma 19, we have that there exists some $f' \in \mathit{Processed}^{end}$ such that $f' \sqsubseteq^{\forall\exists} f$. A lasso-finding test on $\langle f', f' \rangle$ is performed on Line 10, right before moving $\mathbf{f}'$ to *Processed*, which happens on Line 11. Since $\neg LFT(f,f)$, $f' \sqsubseteq^{\forall\exists} f$, we know by Lemma 7 that also $\neg LFT(f', f')$, and thus the algorithm returns FALSE, which again leads to a contradiction.

On the other hand, if $\mathcal{A}$ is universal, due to Lemma 3, each pair of graphs $\langle g, h \rangle$ for $g, h \in G^f_{\mathcal{A}}$ passes the lasso-finding test. At the same time, Lemma 18 guarantees that for every pair of graphs $g', h' \in G_{\mathcal{A}}$ that are subject to the lasso-finding test in the run of Algorithm 2, $g', h' \in G^m_{\mathcal{A}}$. Therefore, there are some $g, h \in G^f_{\mathcal{A}}$ such that $g' \leqslant^* g$ and $h' \leqslant^* h$ which by Lemma 4 gives $g' \simeq^{\forall\exists} g$ and $h' \simeq^{\forall\exists} h$, and hence, $g \sqsubseteq^{\forall\exists} g'$ and $h \sqsubseteq^{\forall\exists} h'$. Since $h \in G^f_{\mathcal{A}} \subseteq G^m_{\mathcal{A}}$, Lemma 7 implies that $LFT(g', h')$. Therefore, the algorithm can only return TRUE in this case. □

## D Basic Ramsey-based Language Inclusion Checking

Below, we provide proofs of Lemmas 9 and 10.

*Proof (Lemma 9).* For Point 1, we have to prove $\mathcal{L}(\mathcal{A}) = \bigcup\{Z_{\mathbf{gh}} \mid Z_{\mathbf{gh}} \text{ is proper}\}$. For the left-to-right inclusion, we proceed as follows. Let $w$ be a word in $\mathcal{L}(\mathcal{A})$. We show that there exist supergraphs $\mathbf{g}, \mathbf{h} \in S_{\mathcal{A},\mathcal{B}}$ s.t. $w \in Z_{\mathbf{gh}}$ and $Z_{\mathbf{gh}}$ is proper. From $w \in \mathcal{L}(\mathcal{A})$, there must be a run $p \dots q \dots q \dots$ in $\mathcal{A}$ over $w$ s.t. $p \in I_{\mathcal{A}}$, $q \in F_{\mathcal{A}}$, and $q$ occurs infinitely often. Hence, we have that $w \in \mathcal{L}(p,q)\mathcal{L}(q,q)^\omega$. By Point 1 of Lemma 1, there must be a pair of graphs $g, h \in G_{\mathcal{B}}$ s.t. $w \in \mathcal{L}(g)\mathcal{L}(h)^\omega$. Hence, the argument can be proved by letting $\mathbf{g} = \langle \langle p, q \rangle, g \rangle$ and $\mathbf{h} = \langle \langle q, q \rangle, h \rangle$. We now prove the right-to-left inclusion. For any two supergraphs $\mathbf{g} = \langle \langle p, q \rangle, g \rangle, \mathbf{h} = \langle \langle q, q \rangle, h \rangle \in S_{\mathcal{A},\mathcal{B}}$ s.t. $Z_{\mathbf{gh}}$ is proper, let $w$ be a word in $Z_{\mathbf{gh}}$. We show that $w \in \mathcal{L}(\mathcal{A})$. From the definition of the language of supergraphs and $w \in Z_{\mathbf{gh}}$, we have that $w \in \mathcal{L}(p,q)\mathcal{L}(q,q)^\omega$. Moreover, by the definition of properness, we have $p \in I_{\mathcal{A}}$, $q \in F_{\mathcal{A}}$. Hence, there must be an accepting run $p \dots q \dots q \dots$ in $\mathcal{A}$ over $w$, and hence, $w \in \mathcal{L}(\mathcal{A})$.

For Point 2, we have to show for all non-empty proper $Z_{\mathbf{gh}}$ that either $Z_{\mathbf{gh}} \cap \mathcal{L}(\mathcal{B}) = \emptyset$ or $Z_{\mathbf{gh}} \subseteq \mathcal{L}(\mathcal{B})$ holds. Let $\mathbf{g} = \langle \langle p, q \rangle, g \rangle, \mathbf{h} = \langle \langle q, q \rangle, h \rangle \in S_{\mathcal{A},\mathcal{B}}$ be

two supergraphs s.t. $Z_{\mathbf{gh}}$ is non-empty and proper. From the definition of the languages of supergraphs, we have $Z_{\mathbf{gh}} \subseteq Y_{gh}$. From Point 2 of Lemma 1, either $Y_{gh} \cap \mathcal{L}(\mathcal{B}) = \emptyset$ or $Y_{gh} \subseteq \mathcal{L}(\mathcal{B})$. Therefore, either $Z_{\mathbf{gh}} \cap \mathcal{L}(\mathcal{B}) = \emptyset$ or $Z_{\mathbf{gh}} \subseteq \mathcal{L}(\mathcal{B})$.

Point 3 follows directly from Point 1 and Point 2. □

*Proof (Lemma 10).* We have to show that $\mathcal{L}(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{B})$ iff $DGT(\mathbf{g}, \mathbf{h})$ for all $\mathbf{g}, \mathbf{h} \in S^f_{\mathcal{A},\mathcal{B}}$. For the "if" direction, assume $\mathcal{L}(\mathcal{A}) \not\subseteq \mathcal{L}(\mathcal{B})$. Hence, by Point 3 of Lemma 9, there must be some $\mathbf{g}$ and $\mathbf{h}$ s.t. $Z_{\mathbf{gh}}$ is proper and $Z_{\mathbf{gh}} \cap \mathcal{L}(\mathcal{B}) = \emptyset$. By the definition of the double-graph test and Lemma 15, we have $\neg DGT(\mathbf{g}, \mathbf{h})$. For the "only if" direction, assume $\mathcal{L}(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{B})$. By Point 3 of Lemma 9, there are no $\mathbf{g}$ and $\mathbf{h}$ s.t. $Z_{\mathbf{gh}}$ is proper, $Z_{\mathbf{gh}} \cap \mathcal{L}(\mathcal{B}) = \emptyset$, and $Z_{\mathbf{gh}} \neq \emptyset$. Therefore, by the definition of the double-graph test and Lemma 15, $DGT(\mathbf{g}, \mathbf{h})$ holds for all possible $\mathbf{g}, \mathbf{h} \in G^f_{\mathcal{A}}$. □

The next lemma is an analogy of Lemma 2, allowing the incremental construction of the set $S^f_{\mathcal{A},\mathcal{B}}$.

**Lemma 21.** *A supergraph* $\mathbf{g}$ *is in* $S^f_{\mathcal{A},\mathcal{B}}$ *iff* $\exists \mathbf{g}_1, \ldots, \mathbf{g}_n \in S^1_{\mathcal{A},\mathcal{B}} : \ \mathbf{g} = \mathbf{g}_1; \ldots; \mathbf{g}_n$.

*Proof.* The lemma is a direct consequence of Lemmas 3.4.1 and 3.4.3 stated in [7] (using a slightly different notation). □

A basic language inclusion test algorithm (see Algorithm 4) can be obtained by combining Lemmas 9, 10, and 21.

**Algorithm 4**: *Ramsey-based Language Inclusion Checking (A Modified Version of The Approach in [8])*

**Input**: $\mathcal{A} = (\Sigma, Q_{\mathcal{A}}, I_{\mathcal{A}}, F_{\mathcal{A}}, \delta_{\mathcal{A}})$ and $\mathcal{B} = (\Sigma, Q_{\mathcal{B}}, I_{\mathcal{B}}, F_{\mathcal{B}}, \delta_{\mathcal{B}})$, the set $S^1_{\mathcal{A},\mathcal{B}}$
**Output**: TRUE if $\mathcal{L}(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{B})$. Otherwise, FALSE.

```
1 Next := G^1_{A,B}; Processed := ∅;
2 while Next ≠ ∅ do
3     Pick and remove a supergraph g from Next;
4     if ∃h ∈ Processed : ¬DGT(h, g) ∨ ¬DGT(g, h) ∨ ¬DGT(g, g) then
          return FALSE;
5     Add g to Processed;
6     foreach h ∈ S^1_{A,B} such that ⟨g, h⟩ is composable and g;h ∉ Processed do
          Add g;h to Next;
7 return TRUE;
```

## E Correctness of Algorithm 3

We first prove Lemma 11 and Theorem 2.

*Proof (Lemma 11).* Let $\overline{g} = \langle p, q\rangle$ and $\overline{h} = \langle q_1, q_2\rangle$, then $p \in I_{\mathcal{A}}$, $q_2 \in F_{\mathcal{A}}$, $q \succeq_{\mathcal{A}} q_1$ and $q_2 \succeq_{\mathcal{A}} q_1$ by the definition of weak properness. Given $w \in Z_{\mathbf{gh}}$, there exist $w_1 \in \mathcal{L}(p, q)$ and $w_2 \in \mathcal{L}(q_1, q_2)$ such that $w = w_1 w_2^{\omega}$. Since $q_2 \succeq_{\mathcal{A}} q_1$ and $q_2 \in F_{\mathcal{A}}$, there is a sequence $q_2, q_3, \cdots$ in $F_{\mathcal{A}}$ such that $q_n \overset{w_2}{\Longrightarrow} q_{n+1}$ for $n \geq 1$. Moreover, since $q \succeq_{\mathcal{A}} q_1$, there is a sequence $q_1', q_2', \cdots$ such that $q = q_1'$ and for $n \geq 1$ we have $q_n' \overset{w_2}{\Longrightarrow} q_{n+1}'$ and $q_{n+1}' \succeq_{\mathcal{A}} q_{n+1}$, which means $q_{n+1}' \in F_{\mathcal{A}}$ since $q_{n+1} \in F_{\mathcal{A}}$. We conclude that $w \in \mathcal{L}(\mathcal{A})$ and thus $Z_{\mathbf{gh}} \subseteq \mathcal{L}(\mathcal{A})$. □

*Proof (Theorem 2).* We show instead that there is a pair of supergraphs in $S^f_{\mathcal{A},\mathcal{B}}$ that fails the relaxed double-graph test iff $\mathcal{L}(\mathcal{A}) \not\subseteq \mathcal{L}(\mathcal{B})$.

First, assume that $\langle \mathbf{f}, \mathbf{g}\rangle$ fails the relaxed double-graph test for some $\mathbf{f}, \mathbf{g} \in S^f_{\mathcal{A},\mathcal{B}}$. Let $Z_{\mathbf{fg}}$ be the $\omega$-regular language $\mathcal{L}(\mathbf{f}) \cdot \mathcal{L}(\mathbf{g})^{\omega}$. Since $Z_{\mathbf{fg}}$ is weakly proper, $Z_{\mathbf{fg}} \subseteq \mathcal{L}(\mathcal{A})$ by Lemma 11. Moreover, since $\langle f, g\rangle$ fails the lasso-finding test, $Z_{\mathbf{fg}} \cap \mathcal{L}(\mathcal{B}) = \emptyset$ by Lemma 15. Note that $Z_{\mathbf{fg}}$ is non-empty because $\mathbf{f}, \mathbf{g} \in S^f_{\mathcal{A},\mathcal{B}}$. Thus we can conclude $\mathcal{L}(\mathcal{A}) \not\subseteq \mathcal{L}(\mathcal{B})$.

If $\mathcal{L}(\mathcal{A}) \not\subseteq \mathcal{L}(\mathcal{B})$, then by Lemma 9 and Lemma 10, there is a pair of supergraphs $\langle \mathbf{f}, \mathbf{g}\rangle \in S^f_{\mathcal{A},\mathcal{B}}$ that fail the (stronger) double graph test, i.e., the double graph test in the sense of Section 5. Therefore, $\langle \mathbf{f}, \mathbf{g}\rangle$ also fails the relaxed double graph test in the sense of the new weaker version of Definition 3. □

The following lemma states the closure of $S^m_{\mathcal{A},\mathcal{B}}$ under composition.

**Lemma 22.** *For any supergraphs $\mathbf{e}, \mathbf{f} \in S^m_{\mathcal{A},\mathcal{B}}$ that are composable, $\mathbf{e}; \mathbf{f} \in S^m_{\mathcal{A},\mathcal{B}}$.*

*Proof.* Let $\mathbf{e} = \langle\langle p, q\rangle, e\rangle$ and $\mathbf{f} = \langle\langle q, r\rangle, f\rangle$. By the definition of $S^m_{\mathcal{A},\mathcal{B}}$, there are supergraphs $\mathbf{g}, \mathbf{h} \in S^f_{\mathcal{A},\mathcal{B}}$ such that $\mathbf{g} = \langle\langle p, q\rangle, g\rangle, \mathbf{h} = \langle\langle q, r\rangle, h\rangle$ where $g, h \in G^f_{\mathcal{B}}$ and $e \leqslant^* g$ and $f \leqslant^* h$. As is shown in the proof of Lemma 8, $e; f \leqslant^* g; h$. By the definition of $S^f_{\mathcal{A},\mathcal{B}}$ and as $\mathbf{g}, \mathbf{h} \in S^f_{\mathcal{A},\mathcal{B}}$, there are words $w_1 \in \mathcal{L}(p, q) \cap \mathcal{L}(g)$ and $w_2 \in \mathcal{L}(q, r) \cap \mathcal{L}(h)$. The word $w_1.w_2$ must also be in $\mathcal{L}(p, r)$ and by Lemma 14 $w_1.w_2 \in \mathcal{L}(g; h)$. This means that $\mathcal{L}(g; h) \cap \mathcal{L}(p, r) \neq \emptyset$ and also that $g; h \in G^f_{\mathcal{B}}$. We can conclude that $\mathbf{g}; \mathbf{h} \in S^f_{\mathcal{A},\mathcal{B}} \subseteq S^m_{\mathcal{A},\mathcal{B}}$. □

Further, we state and prove a lemma about the preservation of the relaxed double-graph test on $\sqsubseteq_S^{\forall\exists}$-bigger supergraphs.

**Lemma 23.** *Let $\mathbf{e}$, $\mathbf{f}$, $\mathbf{g}$, $\mathbf{h}$ be graphs in $S_{\mathcal{A},\mathcal{B}}$ such that $\{\mathbf{f}, \mathbf{h}\} \cap S^m_{\mathcal{A},\mathcal{B}} \neq \emptyset$, $\mathbf{e} \sqsubseteq_S^{\forall\exists} \mathbf{g}$, and $\mathbf{f} \sqsubseteq_S^{\forall\exists} \mathbf{h}$. If $\langle \mathbf{e}, \mathbf{f}\rangle$ passes the relaxed double-graph test, then $\langle \mathbf{g}, \mathbf{h}\rangle$ passes the relaxed double-graph test too.*

*Proof.* We may equivalently show that if $\langle \mathbf{g}, \mathbf{h}\rangle$ fails the relaxed double-graph test, then $\langle \mathbf{e}, \mathbf{f}\rangle$ fails the relaxed double-graph test too. Assume that $\langle \mathbf{g}, \mathbf{h}\rangle$ fails the relaxed double-graph test. Then, $\mathbf{g} = \langle\langle p, q_1\rangle, g\rangle$ and $\mathbf{h} = \langle\langle q_2, q_3\rangle, h\rangle$ such that $q_1 \succeq_{\mathcal{A}} q_2$, $q_3 \succeq_{\mathcal{A}} q_2$, $p \in I_{\mathcal{A}}$, $q_3 \in F_{\mathcal{A}}$, and $\langle g, h\rangle$ fails the lasso-finding test. Let $\mathbf{e} = \langle\langle p', q_1'\rangle, e\rangle$ and $\mathbf{f} = \langle\langle q_2', q_3'\rangle, f\rangle$. From $\mathbf{e} \sqsubseteq_S^{\forall\exists} \mathbf{g}$ and $\mathbf{f} \sqsubseteq_S^{\forall\exists} \mathbf{h}$, we have that $p' = p$, $q_1' \succeq_{\mathcal{A}} q_1$, $q_2' = q_2$, $q_3' \succeq_{\mathcal{A}} q_3$. Thus $p' \in I_{\mathcal{A}}$ and $q_3' \in F_{\mathcal{A}}$. Furthermore, $q_1' \succeq_{\mathcal{A}} q_1 \succeq_{\mathcal{A}} q_2 = q_2'$ and $q_3' \succeq_{\mathcal{A}} q_3 \succeq_{\mathcal{A}} q_2 = q_2'$. Moreover, due to Lemma 7, $\langle e, f\rangle$ fails the lasso-finding test. Hence, $\langle \mathbf{e}, \mathbf{f}\rangle$ fails the relaxed double-graph test. □

Below, we provide several auxiliary lemmas used in the proof of the correctness of Algorithm 3.

**Lemma 24.** *For any* $\mathbf{g}, \mathbf{h} \in S_{\mathcal{A},\mathcal{B}}$*, if* $\mathbf{g} \leqslant_S^* \mathbf{h}$*, then* $\mathbf{h} \simeq_S^{\forall\exists} \mathbf{g}$*.*

*Proof.* Immediately by Lemma 4 and the definitions of $\sqsubseteq_S^{\forall\exists}$ and $\leqslant_S^*$. □

**Lemma 25.** *For any* $\mathbf{g} \in S^1_{\mathcal{A},\mathcal{B}}$*, there is some* $\mathbf{f} \in$ *Init on Line 7 of Algorithm 3 such that* $\mathbf{f} \sqsubseteq_S^{\forall\exists} \mathbf{g}$*.*

*Proof.* The proof is analogous to the proof of Lemma 16, it only uses Lemma 24 instead of Lemma 4. □

**Lemma 26.** *Once any graph* $\mathbf{f}$ *appears in the set Next in between of Lines 8–16 of Algorithm 3, then from this point in the run of the algorithm onwards there is always some* $\mathbf{f}' \in Next \cup Processed$ *such that* $\mathbf{f}' \sqsubseteq_S^{\forall\exists} \mathbf{f}$*.*

*Proof.* Analogous to the proof of Lemma 17. □

**Lemma 27.** *Given a supergraph* $\mathbf{g} = \langle\langle p, q\rangle, g\rangle \in S^1_{\mathcal{A},\mathcal{B}}$ *and a state* $p' \in Q_{\mathcal{A}}$ *such that* $p' \succeq_{\mathcal{A}} p$*, then there exists* $\langle\langle p', q'\rangle, g\rangle \in S^1_{\mathcal{A},\mathcal{B}}$ *such that* $q' \succeq_{\mathcal{A}} q$*.*

*Proof.* Since $\mathbf{g} \in S^1_{\mathcal{A},\mathcal{B}}$, there is some $a \in \Sigma_{\mathcal{A}} \cap \mathcal{L}(g)$ such that $p \xrightarrow{a} q$. Since $p' \succeq_{\mathcal{A}} p$, there must be some $q' \in Q_{\mathcal{A}}$ such that $p' \xrightarrow{a} q'$ and $q' \succeq_{\mathcal{A}} q$. We have $\langle p', q'\rangle \in E^1_{\mathcal{A}}$ and $\langle\langle p', q'\rangle, g\rangle \in S^1_{\mathcal{A},\mathcal{B}}$. □

Let $Processed^{end}$ denote the contents of the set *Processed* when Algorithm 3 terminates.

**Lemma 28.** *If Algorithm 3 returns* TRUE*, then for every* $\mathbf{g} \in S^f_{\mathcal{A},\mathcal{B}}$*, there exists some* $\mathbf{h} \in Processed^{end}$ *such that* $\mathbf{h} \sqsubseteq_S^{\forall\exists} \mathbf{g}$*.*

*Proof.* By Lemma 21, the supergraph $\mathbf{g}$ is a finite composition $\mathbf{g}_0; \mathbf{g}_1; \ldots; \mathbf{g}_n$ of $n+1$ supergraphs from $S^1_{\mathcal{A},\mathcal{B}}$. We now prove by induction that for each $0 \leq i \leq n$, there exists a supergraph $\mathbf{h}_i \in Processed^{end}$ such that $\mathbf{h}_i \sqsubseteq_S^{\forall\exists} \mathbf{g}_0; \mathbf{g}_1; \ldots; \mathbf{g}_i$.

We start with the base case. Due to Lemma 25 and the code on Line 7, there is some $\mathbf{f}_0 \in Next$ such that $\mathbf{f}_0 \sqsubseteq_S^{\forall\exists} \mathbf{g}_0$ when the second loop is entered for the first time. Using Lemma 26 and the fact that *Next* is empty when Algorithm 3 returns TRUE, the base case is proved.

For the inductive step, we assume that there exists a graph $\mathbf{h}_k \in Processed^{end}$ such that $\mathbf{h}_k = \langle \overline{h}_k, h_k\rangle = \langle\langle p, q_{\overline{h}_k}\rangle, h_k\rangle \sqsubseteq_S^{\forall\exists} \mathbf{g}_0; \mathbf{g}_1; \ldots; \mathbf{g}_k = \mathbf{g} = \langle \overline{g}, g\rangle = \langle\langle p, q_{\overline{g}}\rangle, g\rangle$ where $q_{\overline{h}_k} \succeq_{\mathcal{A}} q_{\overline{g}}$ and $h_k \sqsubseteq^{\forall\exists} g$. We further have $\mathbf{g}_{k+1} = \langle \overline{g}_{k+1}, g_{k+1}\rangle = \langle\langle q_{\overline{g}}, q\rangle, g_{k+1}\rangle \in S^1_{\mathcal{A},\mathcal{B}}$. Since $\mathbf{g}_{k+1} \in S^1_{\mathcal{A},\mathcal{B}}$ and $q_{\overline{h}_k} \succeq_{\mathcal{A}} q_{\overline{g}}$, due to Lemma 27, there must be some $q' \in Q_{\mathcal{A}}$ such that $q' \succeq_{\mathcal{A}} q$ and $\mathbf{e}_{k+1} = \langle\langle q_{\overline{h}_k}, q'\rangle, g_{k+1}\rangle \in S^1_{\mathcal{A},\mathcal{B}}$. We have that $\mathbf{e}_{k+1} \trianglelefteq^{\forall\exists} \mathbf{g}_{k+1}$. Due to Lemma 25, we know that there is $\mathbf{f}_{k+1} \in Init$ from Line 7 onwards such that $\mathbf{f}_{k+1} \sqsubseteq_S^{\forall\exists} \mathbf{e}_{k+1}$. $\mathbf{f}_{k+1} \sqsubseteq_S^{\forall\exists} \mathbf{e}_{k+1} \trianglelefteq^{\forall\exists} \mathbf{g}_{k+1}$ implies that $\mathbf{f}_{k+1} \trianglelefteq^{\forall\exists} \mathbf{g}_{k+1}$ (since $\sqsubseteq_S^{\forall\exists} \subseteq \trianglelefteq^{\forall\exists}$ and $\trianglelefteq^{\forall\exists}$ is obviously transitive).

Therefore, by Lemma 12, we get $\mathbf{h}_k; \mathbf{f}_{k+1} \sqsubseteq_S^{\forall\exists} \mathbf{g}; \mathbf{g}_{k+1}$. By Lemma 24, we have $\mathit{Min}_S(\mathbf{h}_k; \mathbf{f}_{k+1}) \sqsubseteq_S^{\forall\exists} \mathbf{g}; \mathbf{g}_{k+1}$. The supergraph $\mathit{Min}_S(\mathbf{h}_k; \mathbf{f}_{k+1})$ is added into *Next* on Line 16 right after $\mathbf{h}_k$ is moved to *Processed* on Line 11 (unless there is already some $\sqsubseteq_S^{\forall\exists}$-smaller graph in $\mathit{Processed} \cup \mathit{Next}$). From Lemma 26 and from the fact that *Next* is empty when Algorithm 3 returns TRUE, it is then clear that there is some graph $\mathbf{h}_{k+1} \in \mathit{Processed}^{end}$ such that $\mathbf{h}_{k+1} \sqsubseteq_S^{\forall\exists} \mathbf{g}_0; \mathbf{g}_1; \ldots; \mathbf{g}_{k+1}$. □

**Lemma 29.** *All supergraphs tested on the relaxed double-graph test within a run of Algorithm 3 are in* $S^m_{\mathcal{A},\mathcal{B}}$.

*Proof.* The proof is an analogous to the proof of Lemma 18. The class $S^m_{\mathcal{A},\mathcal{B}}$ is closed under composition by Lemma 22 and it is obviously closed under $\mathit{Min}_S$. The algorithm starts with graphs from $S^m_{\mathcal{A},\mathcal{B}}$ and creates new supergraphs using these two operations only. □

**Lemma 30.** *Algorithm 3 eventually terminates.*

*Proof.* Analogous to the proof of Lemma 20. □

Now we are finally able to prove Theorem 3.

*Proof (Theorem 3).* By Lemma 30, Algorithm 3 eventually terminates and returns a value.

First, we prove by contradiction that if TRUE is returned, $\mathcal{L}(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{B})$ holds. Assume that TRUE is returned and $\mathcal{L}(\mathcal{A}) \not\subseteq \mathcal{L}(\mathcal{B})$. Since $\mathcal{L}(\mathcal{A}) \not\subseteq \mathcal{L}(\mathcal{B})$, Theorem 2 implies that either (1) there is some pair of supergraphs $\mathbf{g} \neq \mathbf{h} \in S^f_{\mathcal{A},\mathcal{B}}$ such that $\langle \mathbf{g}, \mathbf{h} \rangle$ fails the relaxed double-graph test, or (2) there is a supergraph $\mathbf{f} \in S^f_{\mathcal{A},\mathcal{B}}$ such that $\langle \mathbf{f}, \mathbf{f} \rangle$ fails the relaxed double-graph test.

Let us start with Case (1). By Lemma 28, we have that there exist some $\mathbf{g}', \mathbf{h}' \in \mathit{Processed}^{end}$ such that $\mathbf{g}' \sqsubseteq_S^{\forall\exists} \mathbf{g}$ and $\mathbf{h}' \sqsubseteq_S^{\forall\exists} \mathbf{h}$. Assume that $\mathbf{g}'$ is added to *Processed* after $\mathbf{h}'$ (the other case being symmetrical due to the "or" used in the condition on Line 10). When $\mathbf{g}'$ is about to be added to *Processed*, the algorithm performs a relaxed double-graph test on $\langle \mathbf{g}', \mathbf{h}' \rangle$ on Line 10. Since $\langle \mathbf{g}, \mathbf{h} \rangle$ fails the relaxed double-graph test, $\mathbf{g}' \sqsubseteq_S^{\forall\exists} \mathbf{g}$, and $\mathbf{h}' \sqsubseteq_S^{\forall\exists} \mathbf{h}$, we know by Lemma 23 that $\langle \mathbf{g}', \mathbf{h}' \rangle$ also fails the relaxed double-graph test, and thus the algorithm returns FALSE, which leads to a contradiction.

In Case (2), by Lemma 28, we have that there exists some $\mathbf{f}' \in \mathit{Processed}^{end}$ such that $\mathbf{f}' \sqsubseteq_S^{\forall\exists} \mathbf{f}$. A relaxed double-graph test on $\langle \mathbf{f}', \mathbf{f}' \rangle$ is performed on Line 10, right before moving $\mathbf{f}'$ to *Processed*, which happens on Line 11. Since $\langle \mathbf{f}, \mathbf{f} \rangle$ fails the relaxed double-graph test, $\mathbf{f}' \sqsubseteq_S^{\forall\exists} \mathbf{f}$, we know by Lemma 23 that $\langle \mathbf{f}', \mathbf{f}' \rangle$ also fails the relaxed double-graph test, and thus the algorithm returns FALSE, which again leads to a contradiction.

On the other hand, if $\mathcal{L}(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{B})$, then due to Theorem 2, each pair of supergraphs $\langle \mathbf{g}, \mathbf{h} \rangle$ for any $\mathbf{g}, \mathbf{h} \in S^f_{\mathcal{A},\mathcal{B}}$ passes the relaxed double-graph test. At the same time, Lemma 29 guarantees that any two supergraphs $\mathbf{g}', \mathbf{h}'$ that are are subject to the relaxed double-graph test in the run of Algorithm 3 are from $S^m_{\mathcal{A},\mathcal{B}}$. By the definition of $S^m_{\mathcal{A},\mathcal{B}}$, there are $\mathbf{g}, \mathbf{h} \in S^f_{\mathcal{A},\mathcal{B}}$ with $\mathbf{g}' \leqslant_S^* \mathbf{g}$ and

$\mathbf{h}' \leqslant_S^* \mathbf{h}$, which by Lemma 24 gives $\mathbf{g} \sqsubseteq_S^{\forall\exists} \mathbf{g}'$ and $\mathbf{h} \sqsubseteq_S^{\forall\exists} \mathbf{h}'$. Lemma 23 implies that $\langle \mathbf{g}', \mathbf{h}' \rangle$ passes the relaxed double-graph test. Therefore, the algorithm can only return TRUE in this case. □

## F Additional Results on the Experiments

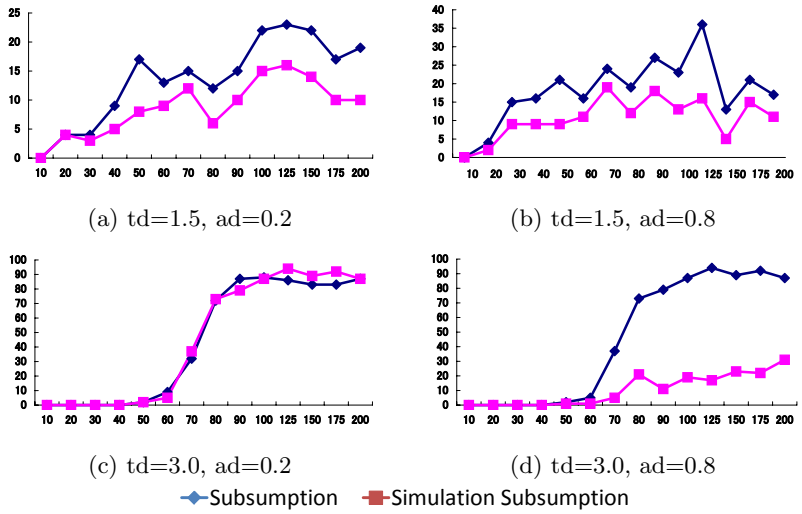



**Fig. 4.** Average Timeout Cases of Universality Checking on Tabakov and Vardi's random model. The vertical axis is the percentage of tasks that cannot be finished within the timeout period (600 seconds), and the horizontal axis is the number of the automata states, ranging from 10 to 200.

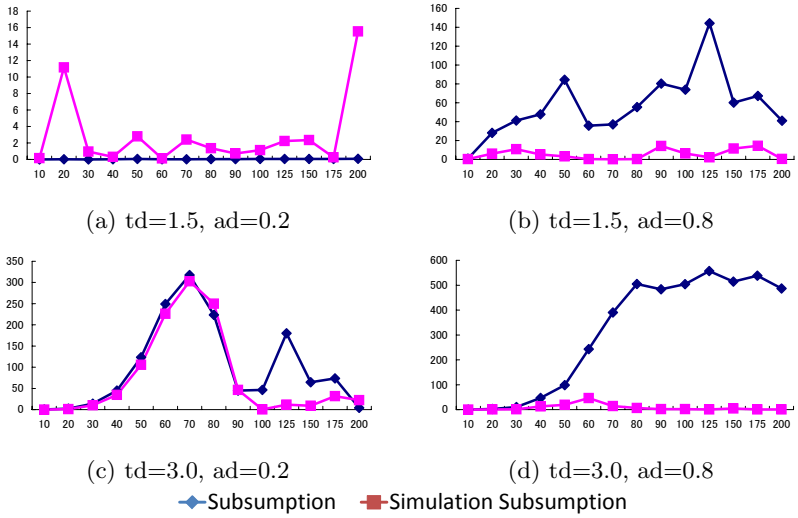
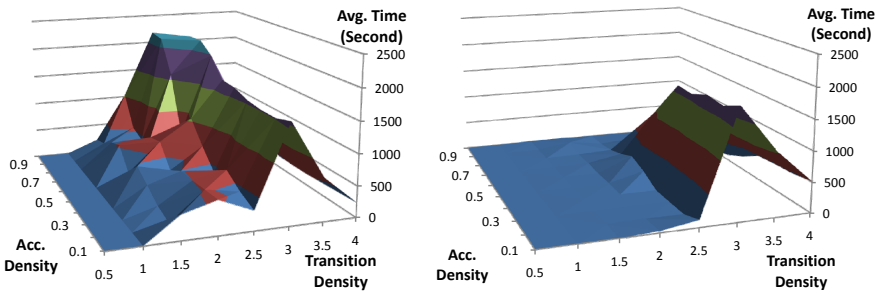



**Fig. 5.** Average Time of Universality Checking on Tabakov and Vardi's random model. The vertical axis is the average execution time in seconds, and the horizontal axis is the number of the automata states, ranging from 10 to 200.

| Experiment | Description of Machines Used |
|---|---|
| Tabakov and Vardi's (Universality) | A machine with 8 Intel(R) Xeon(R) 2.0GHz processors. Each instance of the experiment has a dedicated CPU core with 2GB memory. |
| Random LTL (Universality) | A machine with 2 Intel(R) Xeon(R) 2.66GHz processors. Each instance of the experiment has a dedicated CPU core with 2GB memory. |
| Mutual Exclusion Algorithms (Inclusion) | A machine with 8 Intel(R) Xeon(R) 2.0GHz processors (MCS queue lock, Dining phil. Ver.1) and a machine with 8 AMD Opteron(tm) 2.8GHz processors (other experiments). Each instance of the experiment has a dedicated CPU core with 8GB memory. |
| Random LTL (Inclusion) | A machine with 2 Intel(R) Xeon(R) 2.66GHz processors. Each instance of the experiment has a dedicated CPU core with 2GB memory. |

**Table 2.** Machines we used in the experiments.




(a) Subsumption (b) Simulation Subsumption

**Fig. 6.** Average time for universality checking on Tabakov and Vardi's random model. Each point in the figure is the average result of 100 Büchi automata. If both methods cannot finish the task within the timeout period (3500 seconds), we simply ignore this test while computing the average result. If only one method finishes the task within the timeout period while the other does not, we use 3500 seconds as the execution time of the failed task. From the figure, one can see that our simulation subsumption technique outperforms the basic subsumption in most of the cases.

| | Original | | Error | | Subsumption | | Simulation Sub. | |
|---|---|---|---|---|---|---|---|---|
| | Tr. | St. | Tr. | St. | $\mathcal{L}(E)\subseteq\mathcal{L}(O)$ | $\mathcal{L}(O)\subseteq\mathcal{L}(E)$ | $\mathcal{L}(E)\subseteq\mathcal{L}(O)$ | $\mathcal{L}(O)\subseteq\mathcal{L}(E)$ |
| Bakery | 2697 | 1506 | 2085 | 1146 | >1day | >1day | 2m47s(F) | 20m(F) |
| Peterson | 34 | 20 | 33 | 20 | 2.1s(T) | 0.17s(F) | 0.43s(T) | 0.08s(F) |
| Dining phil. (Ver.1) | 212 | 80 | 464 | 161 | 1m18s(F) | >1day | 3.7s(F) | >1day |
| Dining phil. (Ver.2) | 212 | 80 | 482 | 161 | 3m1s(F) | >1day | 5s(F) | >1day |
| MCS queue lock | 21503 | 7963 | 3222 | 1408 | >1day | >1day | 43s(T) | 42s(T) |

**Table 3.** Language inclusion checking on models of mutual exclusion algorithms (*where all states are accepting*). The rows "Original" and "Error" are the statistical data of the BA of the original model and the erroneous model, respectively. We test both $\mathcal{L}(E) \subseteq \mathcal{L}(O)$ and $\mathcal{L}(O) \subseteq \mathcal{L}(E)$. The result ">1day" indicates that the task cannot be finished within one day, and "oom" that the task required memory > 8GB. If a task completes successfully, we record the run time and whether language inclusion holds ("T") or not ("F").