Discrete Fourier Transform

Valentina Hubeika, Jan Černocký

DCGM FIT BUT Brno, {ihubeika,cernocky}@fit.vutbr.cz

Diskrete Fourier transform (DFT)

We have just one problem with DFS that needs to be solved. Infinite length of signal and finite length of computed spectrum. DFT transforms a sequence of length N to other sequence of length N – we will see that it is a transform of one period of the input signal to one period in DFS. The procedure is the following:

1. periodize a sequence $x[n]$ of length N : $\tilde{x}[n] = x[\mod_N(n)]$.
2. find DFS coefficients: $\tilde{X}[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn}$. Note, only one period of periodic signal $x[\mod_N(n)]$ is taken, therefore, we can work just with original sequence $x[n]$. Step 1. is taken to fulfill requirements for DFS computing.
3. resulting sequence is windowed again to the length N:

$$X[k] = R_N[k] \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn}$$

Usually we find this formula with no windowing function as computing only through one period is assumed, $X[k]$ for $k = [0, N-1]$:

$$\boxed{X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn}}$$

$X[k]$ is a projection/image of DFT, denoted as $x[n] \stackrel{DFT}{\longrightarrow} X[k]$. Inverse DFT for samples $n = [0, N-1]$ is obtained in the same manner (periodization of DFT spectrum, inverse DFS application, windowing of the resulting periodic signal) :

$$\boxed{x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{+j\frac{2\pi}{N}kn},}$$

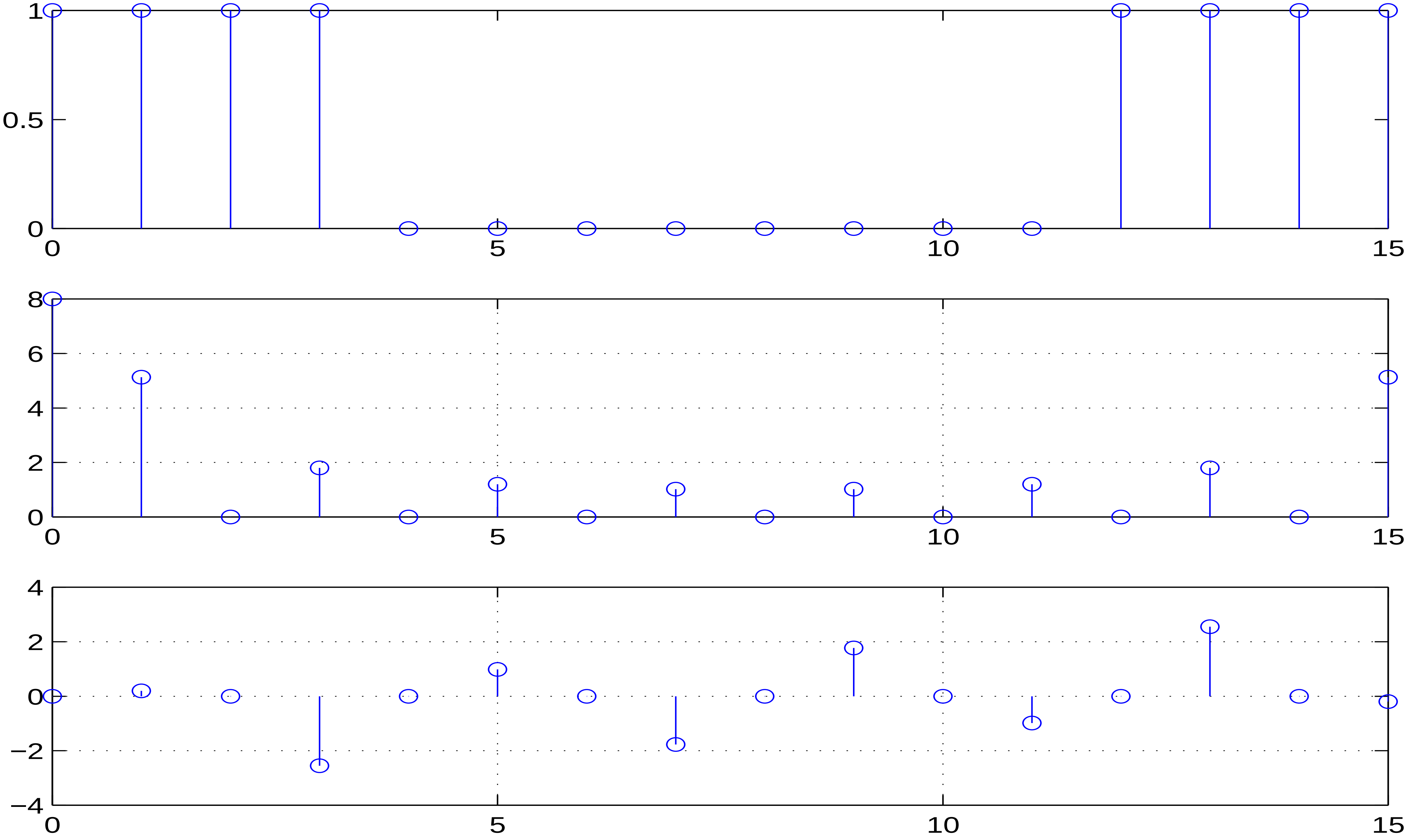
we denote $X[k] \stackrel{DFT^{-1}}{\longrightarrow} x[n]$

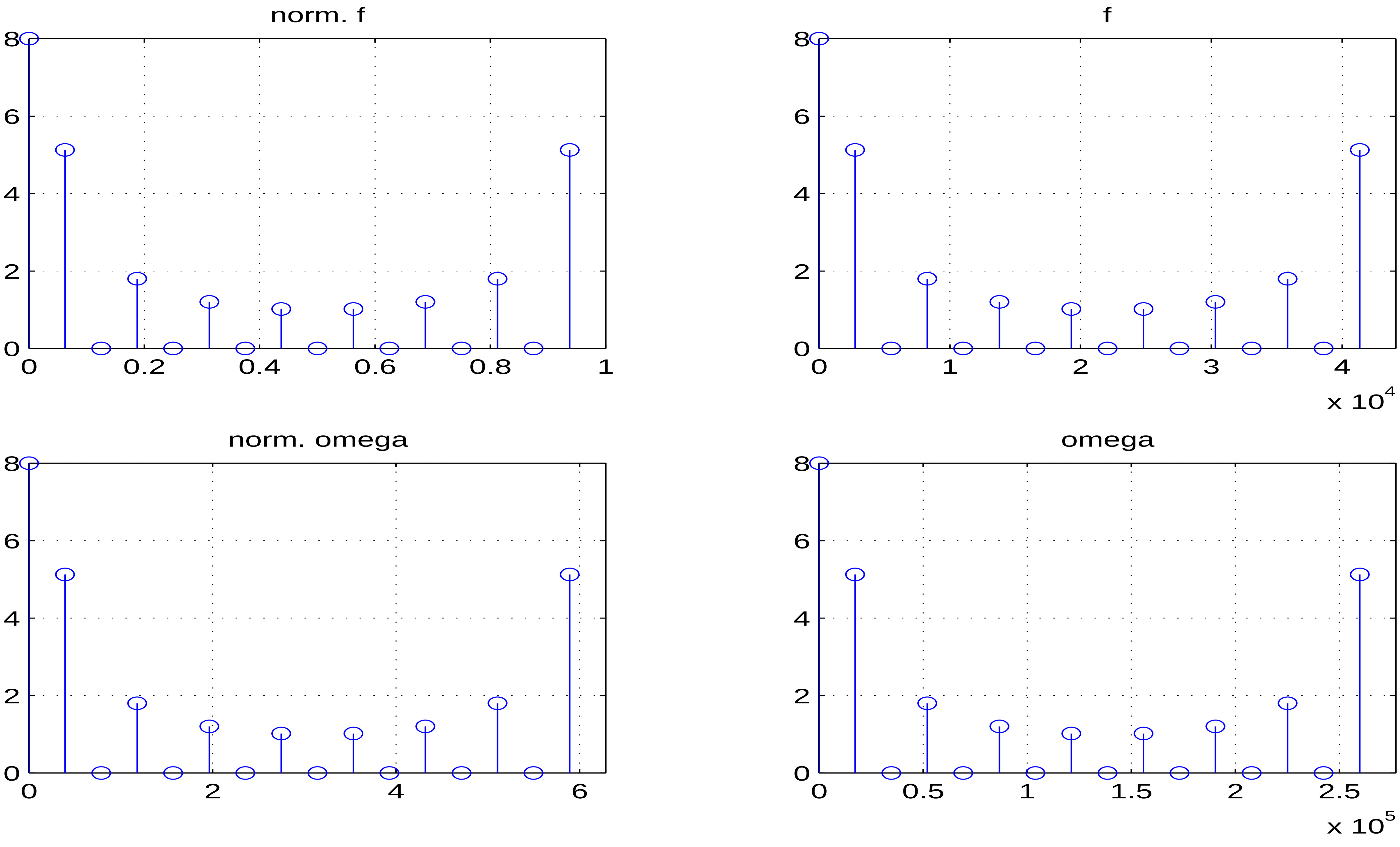
Frequency axis in DFT

N samples of DFT are placed from 0 approaching sampling frequency:

- sampling frequency is N.
- we have N samples placed from 0 to $N-1$.
- thus for samples $X[k]$ holds:
  - normalized frequency $\frac{k}{N}$ to $\frac{N-1}{N}$.
  - normalized circle frequency $2\pi\frac{k}{N}$ to $2\pi\frac{N-1}{N}$
  - regular frequency $\frac{k}{N}F_s$ to $\frac{N-1}{N}F_s$
  - circle frequency $\frac{k}{N}2\pi F_s$ to $\frac{N-1}{N}2\pi F_s$

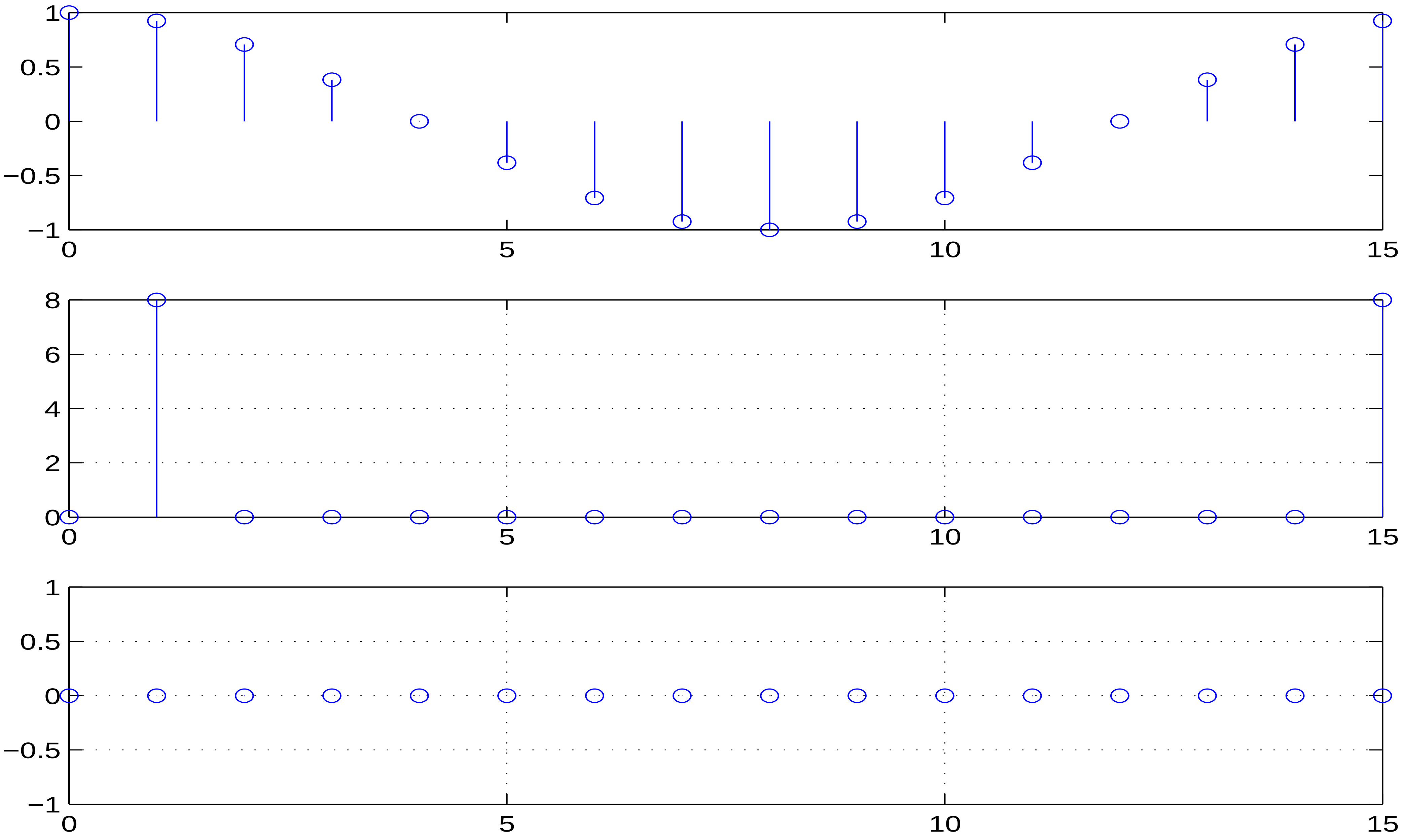
Example 1: $N = 16$, shifted square signal of length 8, $F_s =$44100 Hz.

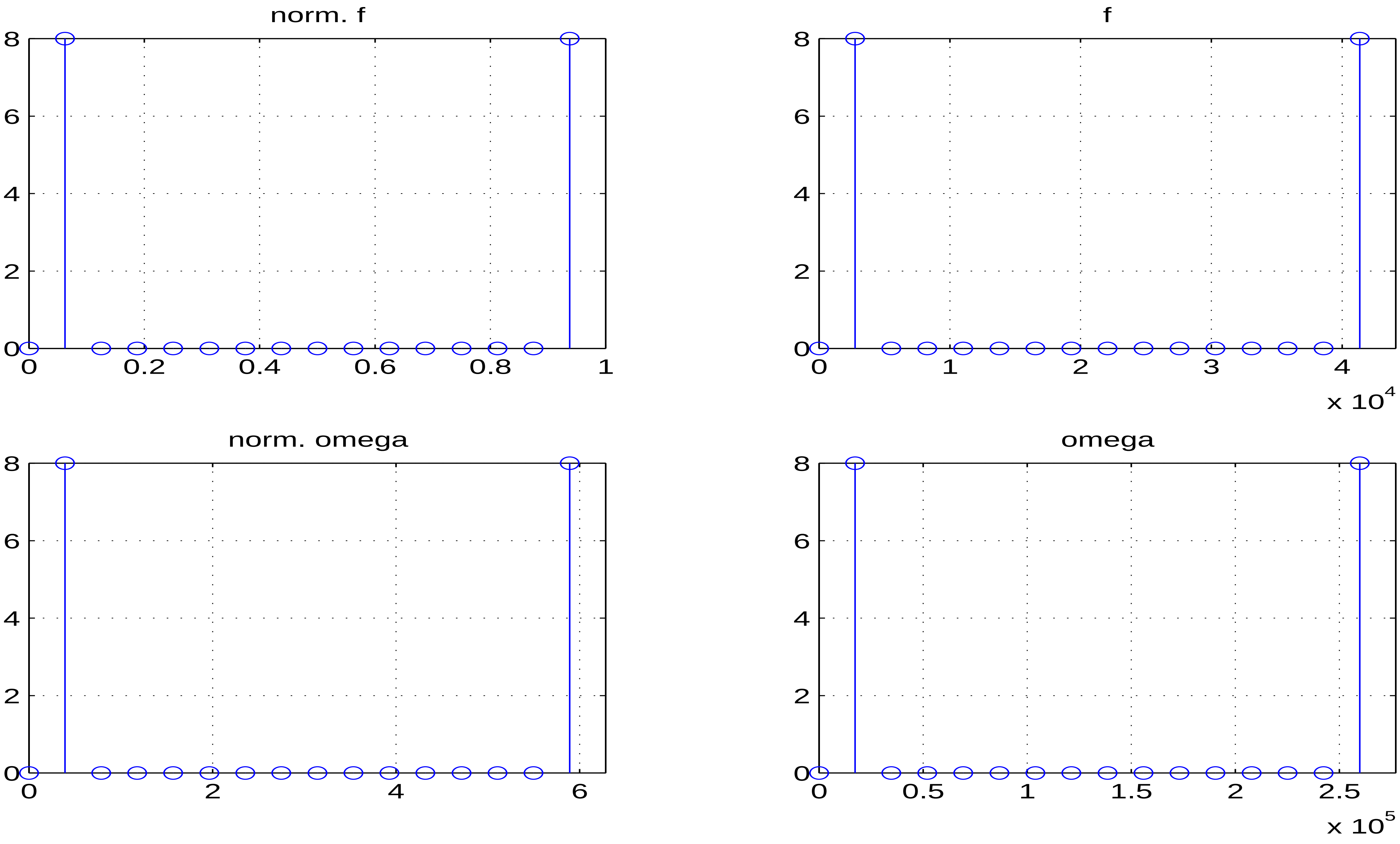




norm. f
f
x 10^4
norm. omega
omega
x 10^5

Example 2: One period of a harmonic signal, $N = 16$, F_s =44100 Hz, $\omega_1 = \frac{2\pi}{16}$ rad




norm. f
f
x 10^4
norm. omega
omega
x 10^5

Properties of DFT

Image of a real sequence

similarly to FS:

$$X[k] = X^{\star}[N-k]$$

- $X[0]$ would be complex conjugate to $X[N]$, but $X[N]$ does not exist. Recall that according to DFS definition, $X[0]$ is a sum of discrete samples, that is a direct component.
- If N is even, then:

$$X\left[\frac{N}{2}\right] = X^{\star}\left[N-\frac{N}{2}\right] = X^{\star}\left[\frac{N}{2}\right].$$

  is complex conjugate to itself, thus it must be real.

Ilustration: see previous examples.

Linearity

$$x_1[n] \stackrel{DFT}{\longrightarrow} X_1[k]$$

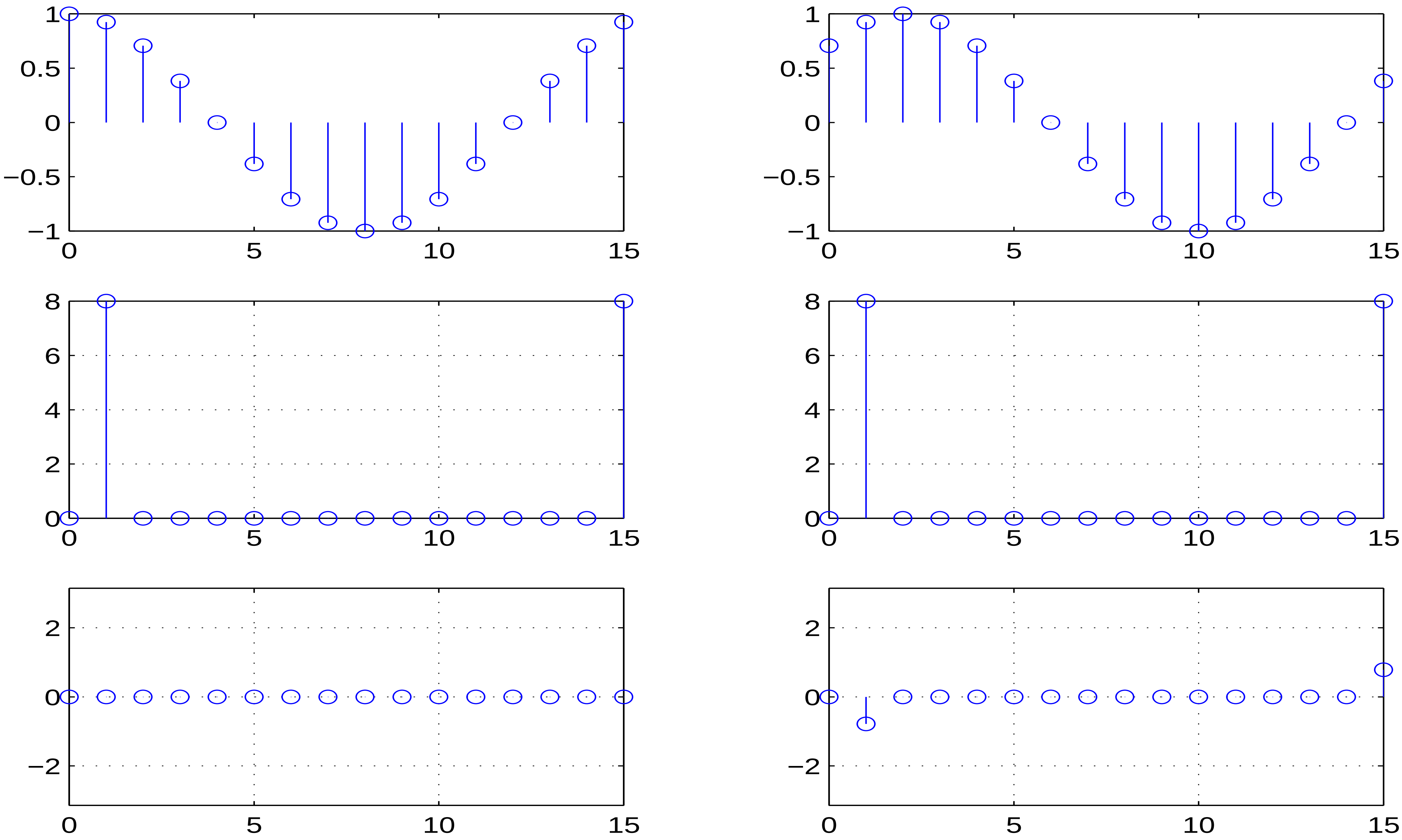
$$x_2[n] \stackrel{DFT}{\longrightarrow} X_2[k]$$

$$ax_1[n] + bx_2[n] \stackrel{DFT}{\longrightarrow} aX_1[k] + bX_2[k]$$

Image of a circulary shifted sequence

$$x[n] \stackrel{DFT}{\longrightarrow} X[k]$$

$$R_N x[\mod_N(n-m)] \stackrel{DFT}{\longrightarrow} X[k]e^{-j\frac{2\pi}{N}mk}$$



1
0.5
0
−0.5
−1
0
5
10
15
8
6
4
2
−2

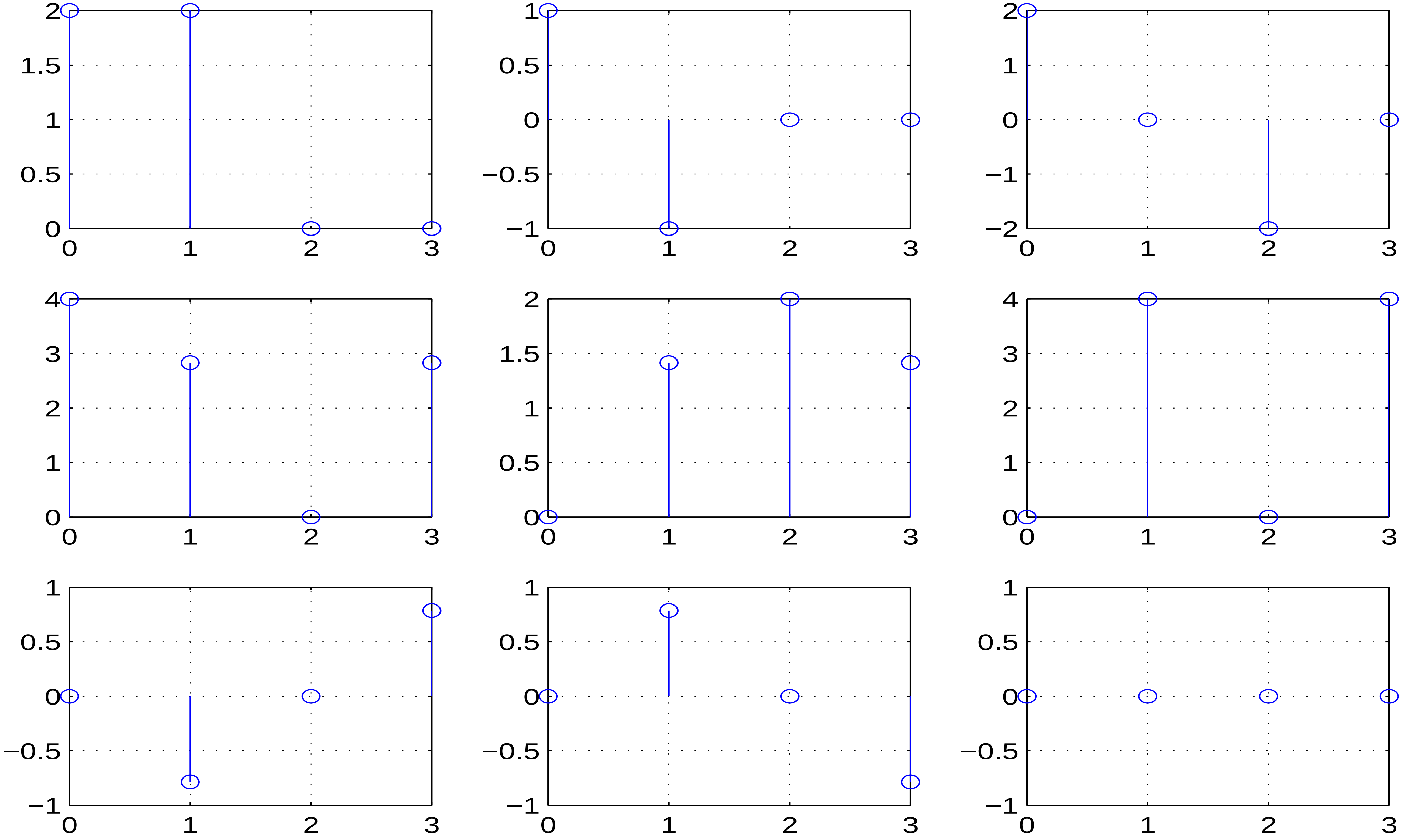
Image of circular convolution

$$x_1[n] \xrightarrow{DFT} X_1[k]$$

$$x_2[n] \xrightarrow{DFT} X_2[k]$$

$$x_1[n] \circledN x_2[n] \xrightarrow{DFT} X_1[k]X_2[k]$$

Similarly as for regular FT convolution of two signals corresponded to multiplication of their spectra in frequency, the DFT image of a circular convolution is a product of DFT coefficients of the convoluted signals.


2
1.5
1
0.5
0
0 1 2 3
1
0.5
0
−0.5
−1
0 1 2 3
2
1
0
−1
−2
0 1 2 3
4
3
2
1
0
0 1 2 3
2
1.5
1
0.5
0
0 1 2 3
4
3
2
1
0
0 1 2 3
1
0.5
0
−0.5
−1
0 1 2 3
1
0.5
0
−0.5
−1
0 1 2 3
1
0.5
0
−0.5
−1
0 1 2 3

Fast Fourier transform FFT

Computing of DFT according to:

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn}$$

requires $2N^2$ operations (multiplication or addition) with complex numbers. Cooley and Tukey invented an efficient algorithm for DFT and its inverse with $N = 2^k$, where k is an integer: **Fast Fourier transform – FFT**. The number of operations becomes only $N \log_2 N$. FFT recursively breaks the transform into two $N/2$ transforms processing a pair of samples producing a pair of coefficients in each step.

Example: pro N $= 1024$, $2N^2 = 2$ MOPS, $N \log_2 N = 10$ kOPS

FFT produces the same output as DFT !

Computating FS and FT (with continuous time) using DFT

We are interested how to compute a spectral representation (coefficients of FS or FT) just using DFT.

first let us summarize what we compute using DFT:

- the signal is **sampled**, thus spectrum is periodic (eventhough we compute only one period of spectrum with N samples (1, 2π, F_s, $2\pi F_s$, according to the type of frequency).
- signal is periodic (by N samples) (eventhough we consider only one period for computating of DFT), spectrum is thus **sampled (discrete)**. The step in spectrum is $\frac{1}{N}$, $\frac{2\pi}{N}$, $\frac{F_s}{N}$, $\frac{2\pi F_s}{N}$ according to the type of frequency.
- signal is **windowed** – the spectrum of the window occurs also in DFT image, $x(t)w(t) \longrightarrow X(j\omega) \star W(j\omega)$.

Computation of coefficients FS using DFT

To remind, for a continuous-time signal with period T_1, FS coefficients are:

$$c_k = \frac{1}{T_1}\int_{T_1} x(t)e^{-jk\omega_1 t}dt,$$

If such signal is sampled with sampling period T, and T_1 then contains N samples, we can aproximate the integral using:

$$c_k \approx \frac{1}{NT}\sum_{n=0}^{N-1} x(nT)e^{-jk\frac{2\pi}{NT}nT}T = \frac{T}{NT}\sum_{n=0}^{N-1} x(nT)e^{-jk\frac{2\pi}{N}n} = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-jkn\frac{2\pi}{N}}.$$

This definition resembles the DFT formula with the only difference that we have to divide the c_k by the number of samples N:

$$c_k = \frac{X[k]}{N}.$$

The equation can be used only when the following restrictions are satisfied:

1. we can compute only coefficients c_k for $k < \frac{N}{2}$ (second half is mirrored to the first one).
2. sampling theorem must be satisfied: last non-zero coefficient of "analog signal" is for

$$k_{max} < \frac{N}{2},$$

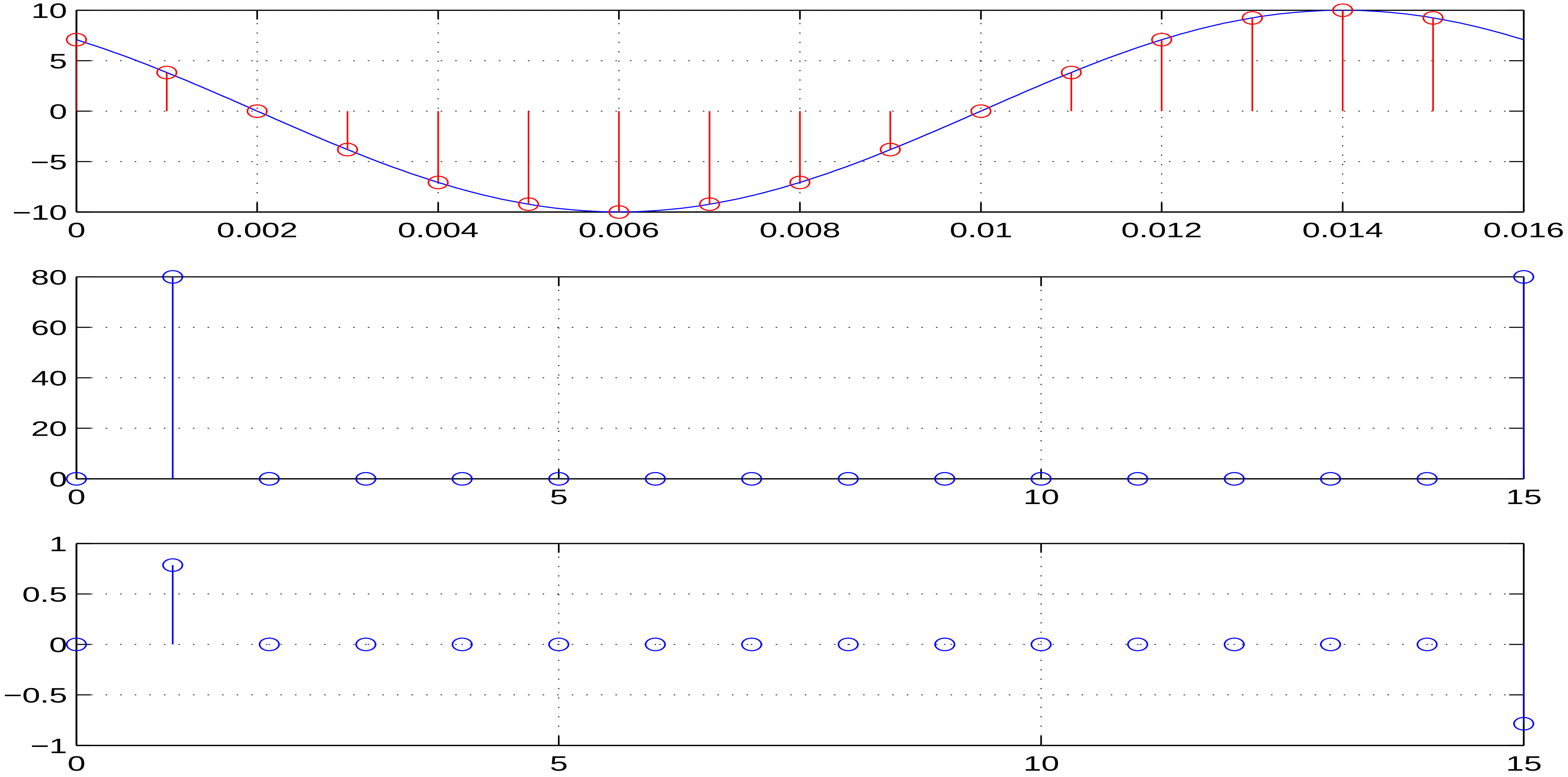
otherwise aliasing ocures. We must to realize that N now corresponds to the sampling frequency, so the above equation is equivalent to:

$$\omega_{max} < \frac{\Omega_s}{2}.$$

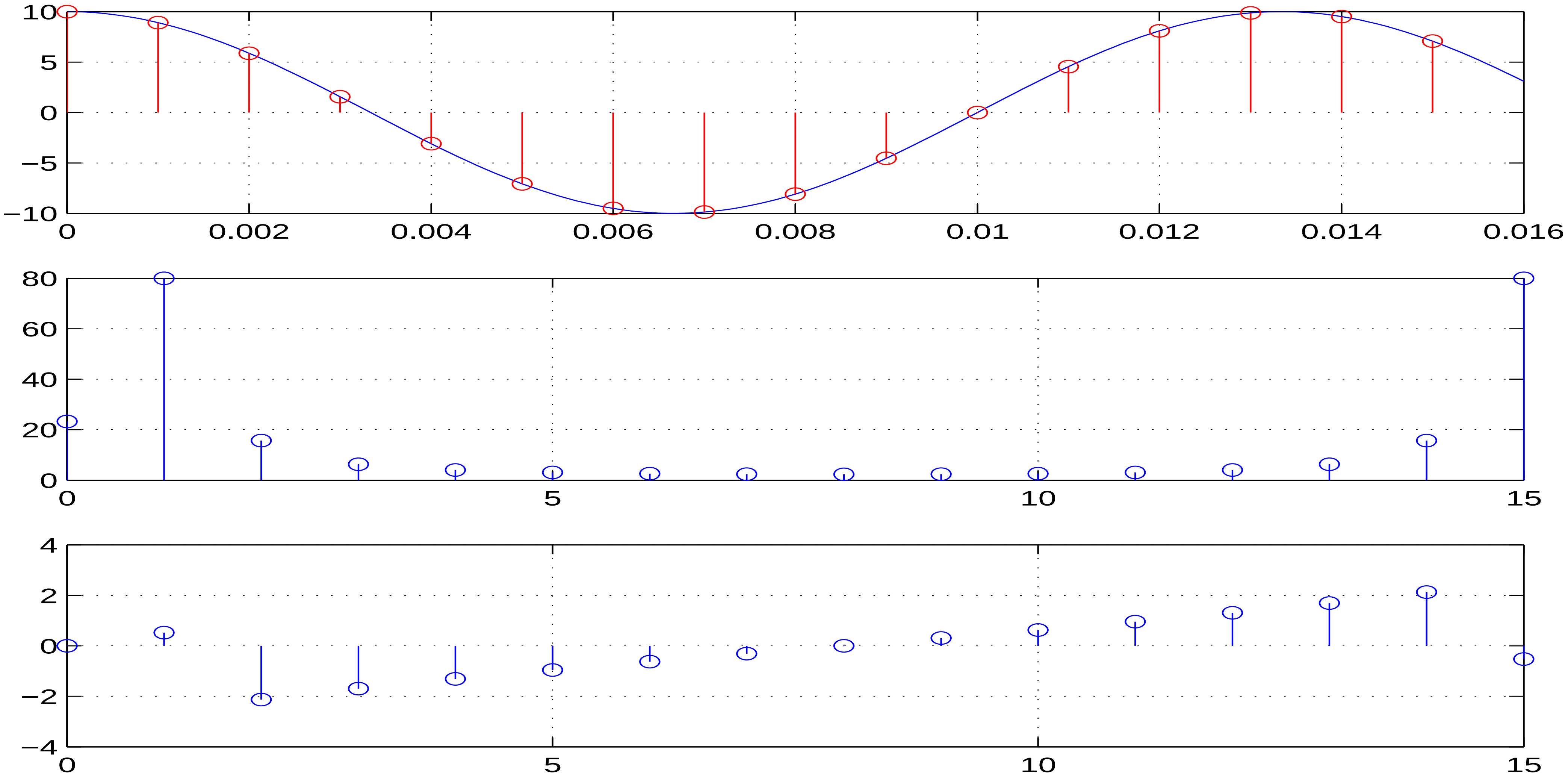
3. N samples must fit into exactly one period of the signal. When more periods – m, we need to make a small modification:

$$c_k = \frac{S[mk]}{N}$$

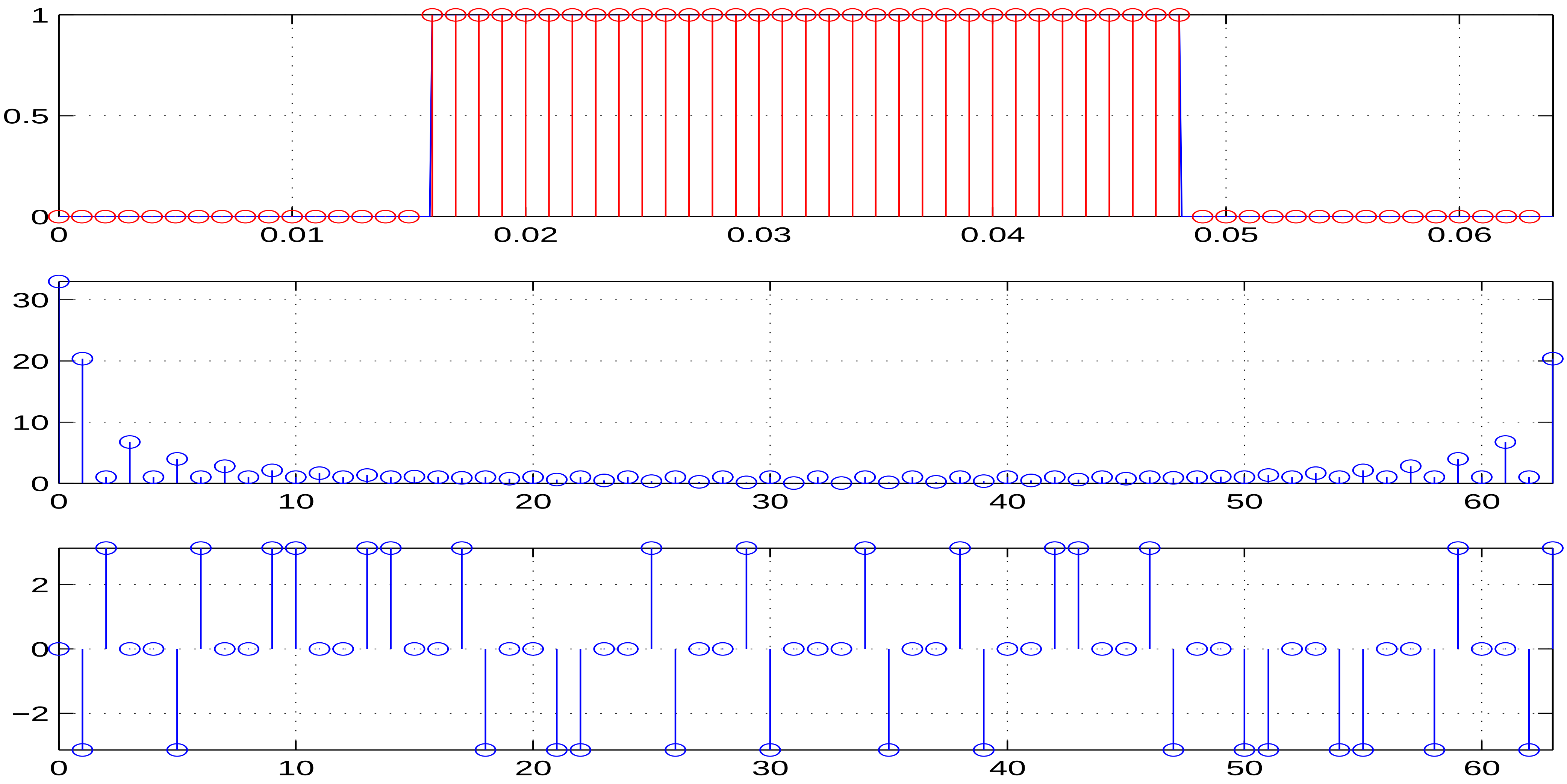
Example 1: signal with continuous time $x(t) = 10\cos(125\pi t + \pi/4)$ sampled at 1 kHz. Compute coefficients of FS using DFT. Period $T_1 = \frac{2\pi}{125\pi} = 0.016$. Number of samples for computation is
$\frac{T_1}{T} = 0.016/0.001 = 16$. Theoretic values of the coefficients are $c_1 = 5e^{j\pi/4}$, $c_{-1} = 5e^{-j\pi/4}$



Example 2: signal with continuous time $x(t) = 10\cos(150\pi t)$ sampled at 1 kHz. Compute coefficients of FS using DFT. We don't know the pe-
riod of the signal, we can choose $N = 16$. Theoretic values of coefficients are $c_1 = 5$, $c_{-1} = 5$



Example 3: signal with continuous time: periodic sequence of square impulses with $D = 1$, $\vartheta = 32$ ms, $T_1 = 64$ ms, sampled at 1 kHz. Compute coefficients of FS using DFT. Theoretic values of coefficients are $c_k = \frac{D\vartheta}{T_1}\text{sinc}(\frac{\vartheta}{2}k\omega_1)$.



Computation of spectral function using DFT

again let's remind

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt$$

We will able to compute only FT of signal which is restricted from 0 to T_1:

- if its is not, we cannot do anything.
- if it is, but elsewhere – for example from t_{start} to $t_{start} + T_1$ – we will move it to $[0, T_1]$, but we will remember it – finally, just small fix of phase will be needed.

If such signal is sampled with sampling period T, we get N samples. Integral can be aproximated, but only for some frequencies - that are multiples of Nth portion of the sampling frequency $\Omega_s = \frac{2\pi}{T}$: $k\frac{\Omega_s}{N}$. Then:

$$X(jk\frac{\Omega_s}{N}) \approx \sum_{n=0}^{N-1} x(nT)e^{-jk\frac{\Omega_s}{N}nT}T = T\sum_{n=0}^{N-1} x(nT)e^{-jk\frac{2\pi/T}{N}nT} = T\sum_{n=0}^{N-1} x[n]e^{-jkn\frac{2\pi}{N}}.$$

We again see the definition of DFT in the derived equation so for circular frequencies $k\frac{\Omega_s}{N}$ we can write:

$$X(jk\frac{\Omega_s}{N}) = TX[k]$$

Again some restrictions:

- valid only for $k < \frac{N}{2}$.
- sampling theorem must be satisfied: the maximum frequency ω_{max} in the signal spectrum must be

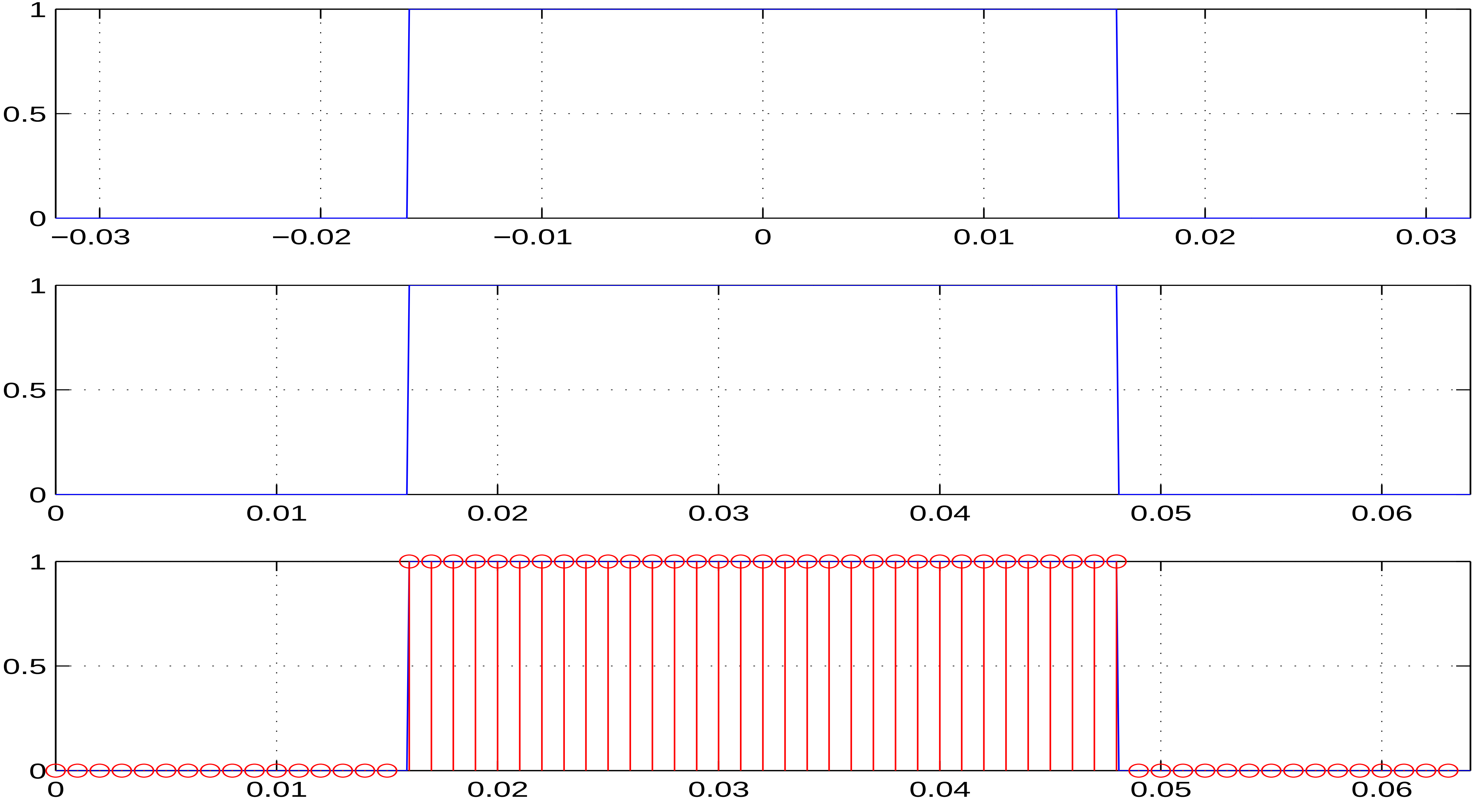
$$\omega_{max} < \frac{\Omega_s}{2}$$

otherwise aliasing occurs. When we have a signal with $\omega_{max} = \infty$ (square, ...) we should use Ω_s the highest possible so aliasing does not hurt.

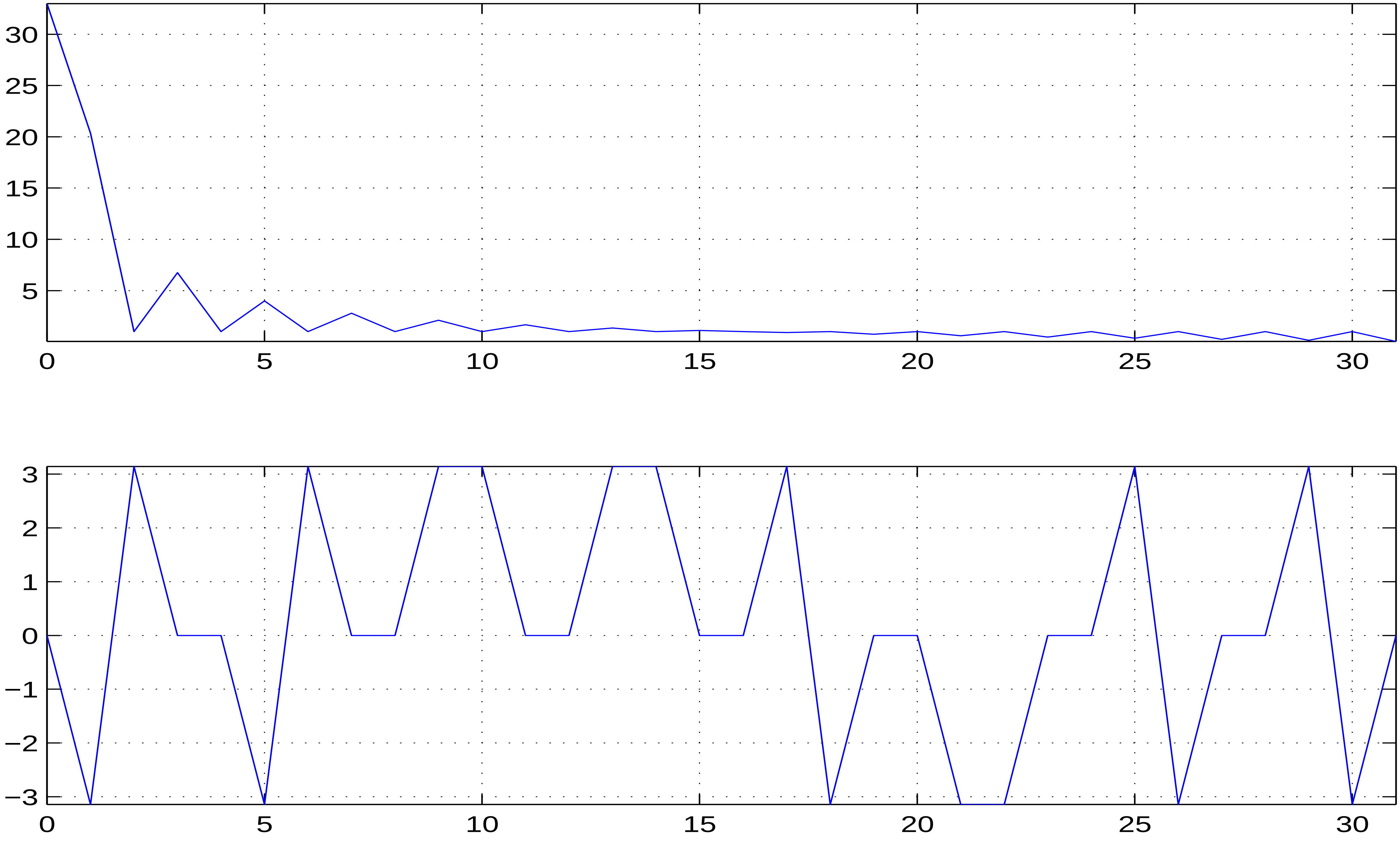
- We compute values for some certain frequencies, but we are interested in all values of the spectral function. We must interpolate, or use **zero-padding** – getting more samples in the spectrum.
- the phase need to be fixed if the signal's period was pushed to fit the interval $[0, T_1]$:

$$X(jk\frac{\Omega_s}{N}) \longrightarrow X(jk\frac{\Omega_s}{N})e^{-jk\frac{\Omega_s}{N}t_{start}}$$

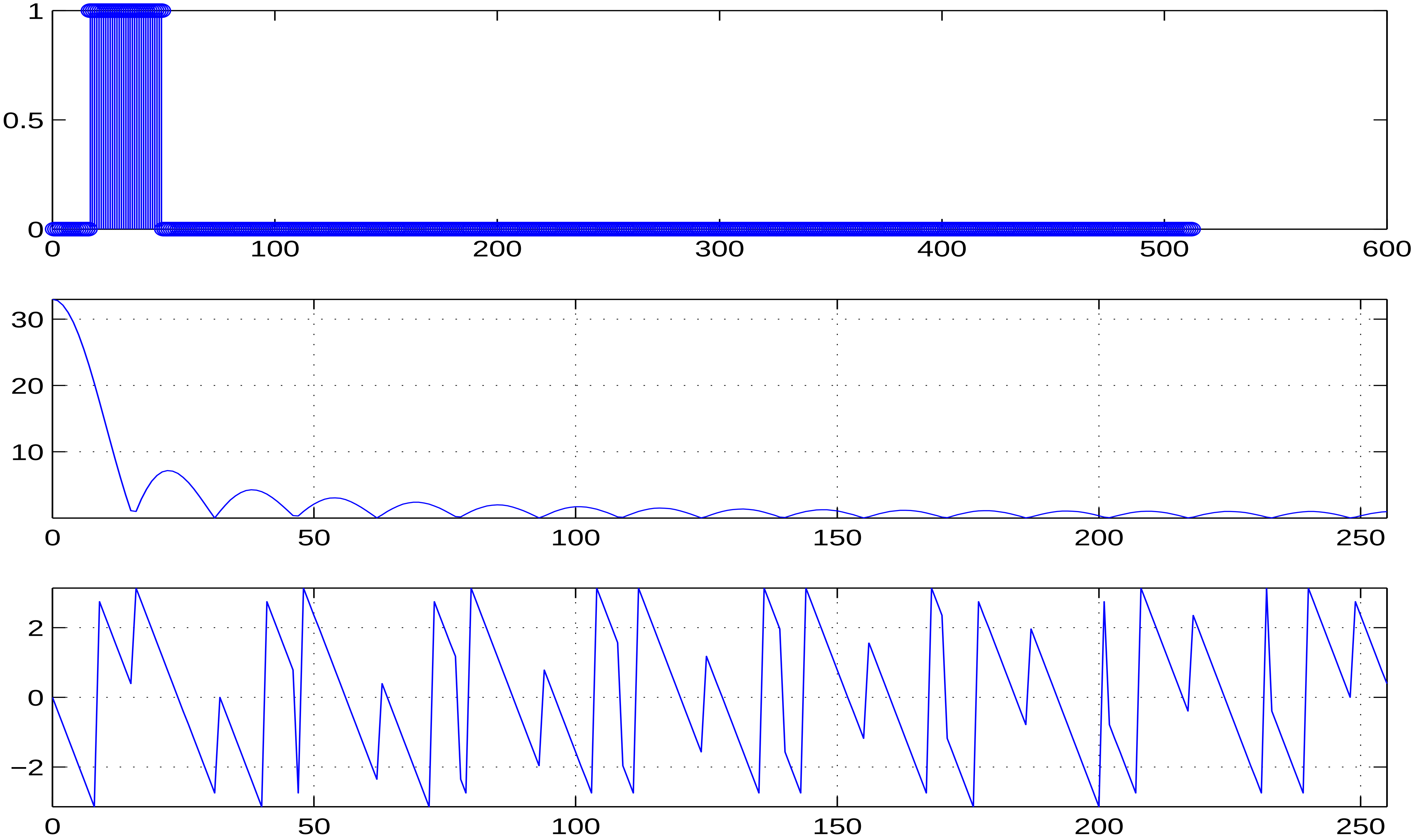
Example: square impuls s $D = 1$, $\vartheta = 32$ ms, sampled at 1 kHz. Theoretic spectral function is $X(j\omega) = D\vartheta\text{sinc}(\frac{\vartheta}{2}\omega)$.



spectral function computed for $N = 64$



zero padded and spectral function computed for $N = 512$



good frequency axis (ω), scaling (multiplied by T) and corrected phase: