Discrete Time Systems

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LTI systems

In this course, we work only with linear and time-invariant systems. We talked about them in the lexture "Systems" where we said that the system's output to an arbitrary input $x[n]$ is computed as a convolution of the input signal and the system's impulse response $h[n]$:

$$y[n] = h[n] \star x[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k].$$

For causal systems (that do not consider future samples) it reduces to:

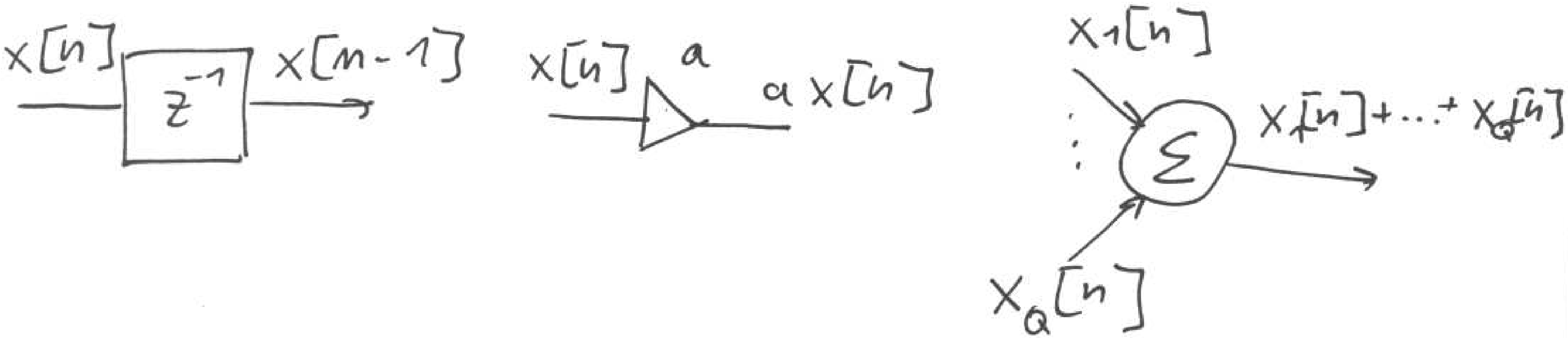
$$h[n] \star x[n] = \sum_{k=-\infty}^{n} x[k]h[n-k] = \sum_{k=0}^{\infty} h[k]x[n-k]$$

This lecture deals with discrete systems, their behaviour in frequency and implementation.

Fundamental blocks of systems

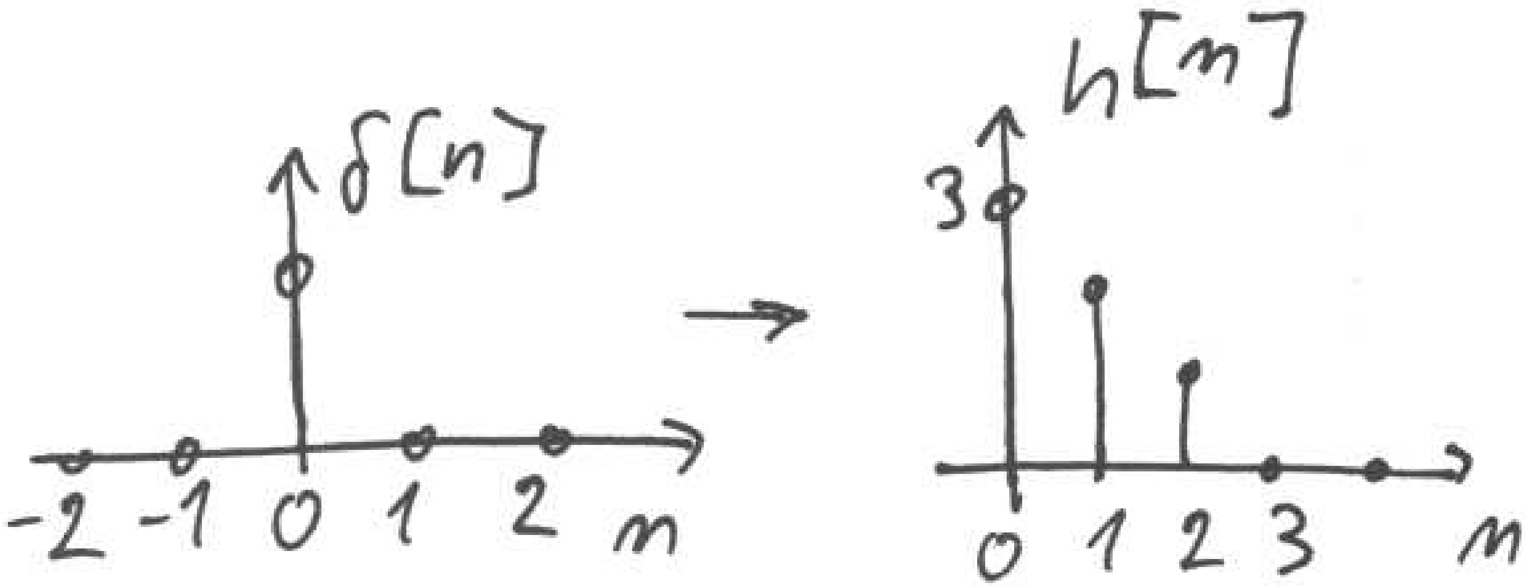
System is represented by a box with inputs $x[n]$ and outputs $y[n]$, where n is a pointer to the sample (discrete time).

Fundamental blocks are:

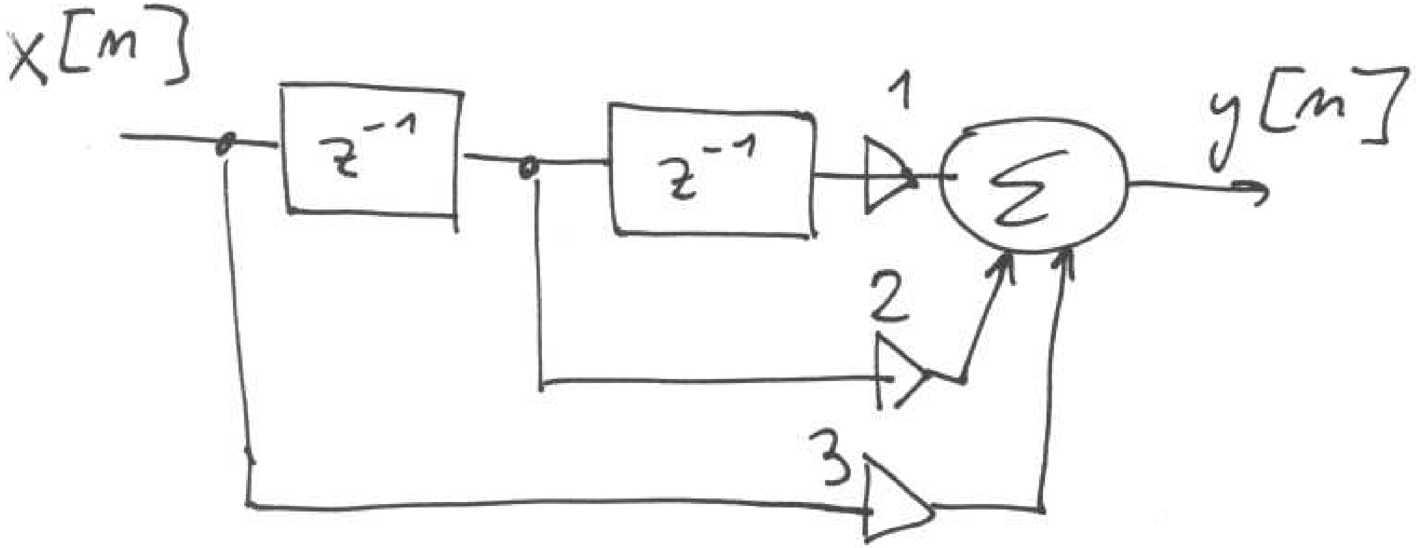


- Delay - holds a sample by one sampling period. When programming, it is a cell where the sample is placed at step n and returned at step $n+1$. Denotation z^{-1} will be explained later.
- multiplication - multiplies a sample by a coefficient.
- addition - ...

A system from a previous lecture:



can be built up as:



and we can simply verify that it corresponds to the above impulse response. We can compute its response to:

$$x[n] = \begin{cases} 2 & \text{for} \;\; n = -1 \\ -1 & \text{for} \;\; n = 0 \\ 1 & \text{for} \;\; n = -1 \\ 0 & \text{otherwise} \end{cases}$$

Frequency Characteristic of a System

The task of an LTI systems we are interested in is to modify a spectrum of an input signal. Similar as we did in previous lectures, to study its behaviour we submit a complex exponential to the system input:

$$x[n] = e^{j\omega_1 n},$$

with normalized angular frequency ω_1:

$$y[n] = h[n] \star x[n] = \sum_{k=0}^{\infty} h[k]x[n-k] = \sum_{k=0}^{\infty} h[k]e^{j\omega_1(n-k)} = e^{j\omega_1 n}\sum_{k=0}^{\infty} h[k]e^{-j\omega_1 k},$$

We see that the output signal is the input signal multiplied by a function of its angular frequency and the impulse response:

$$H(e^{j\omega_1}) = \sum_{k=0}^{\infty} h[k]e^{-j\omega_1 k}$$

We can write:

$$y[n] = x[n]H(e^{j\omega_1})$$

again, only the "width" and the initial phase of the input complex exponential is changed.

We call $H(e^{j\omega_1})$ a **transfer** and define it for an arbitrary (normalized !) frequency. We then get a **(complex) frequency characteristic** function:

$$H(e^{j\omega}) = \sum_{k=0}^{\infty} h[k] e^{-j\omega k}$$

NOTE, that the frequency characteristic is a **DTFT-projection** of impulse response:

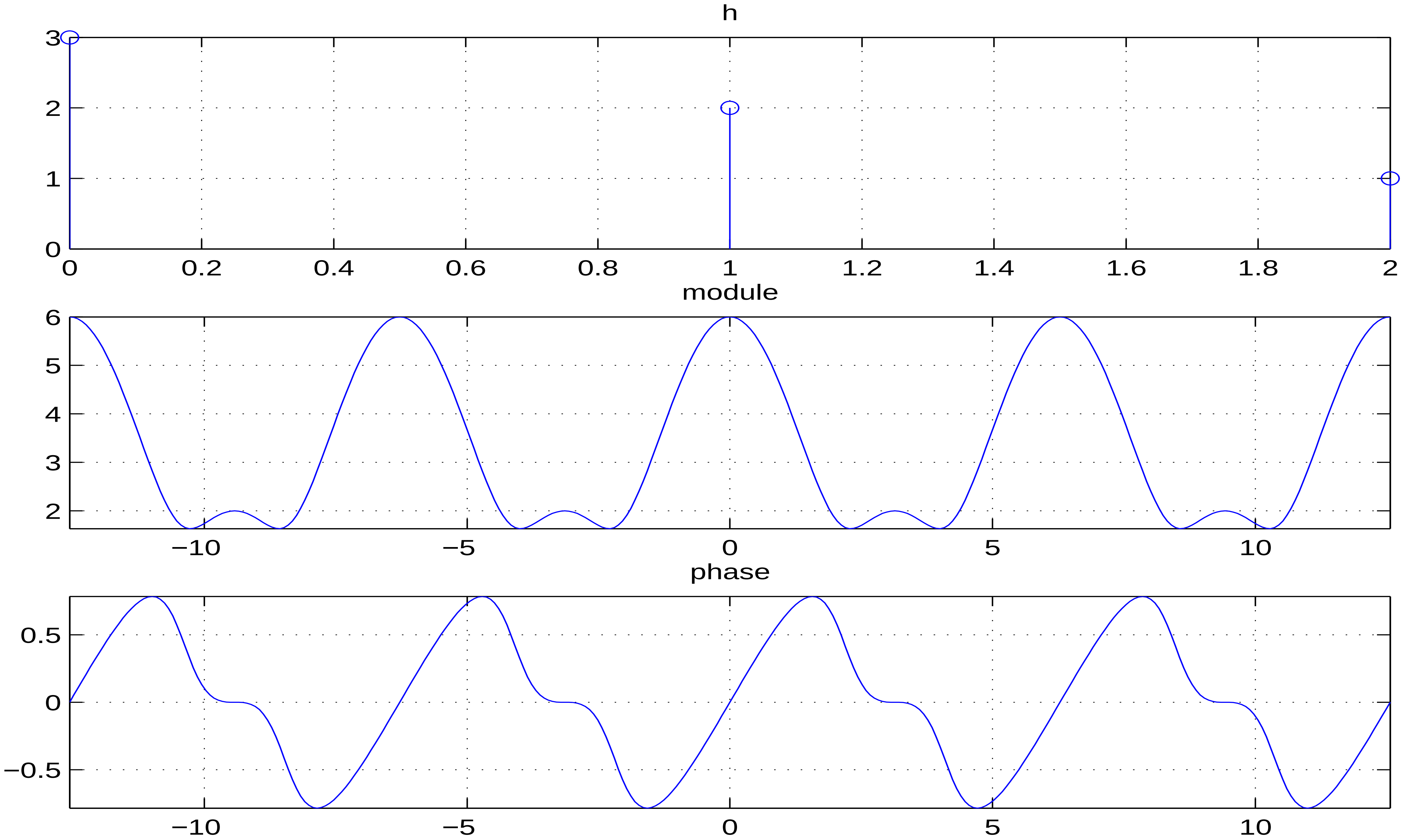
$$h[n] \stackrel{DTFT}{\longrightarrow} H(e^{j\omega})$$

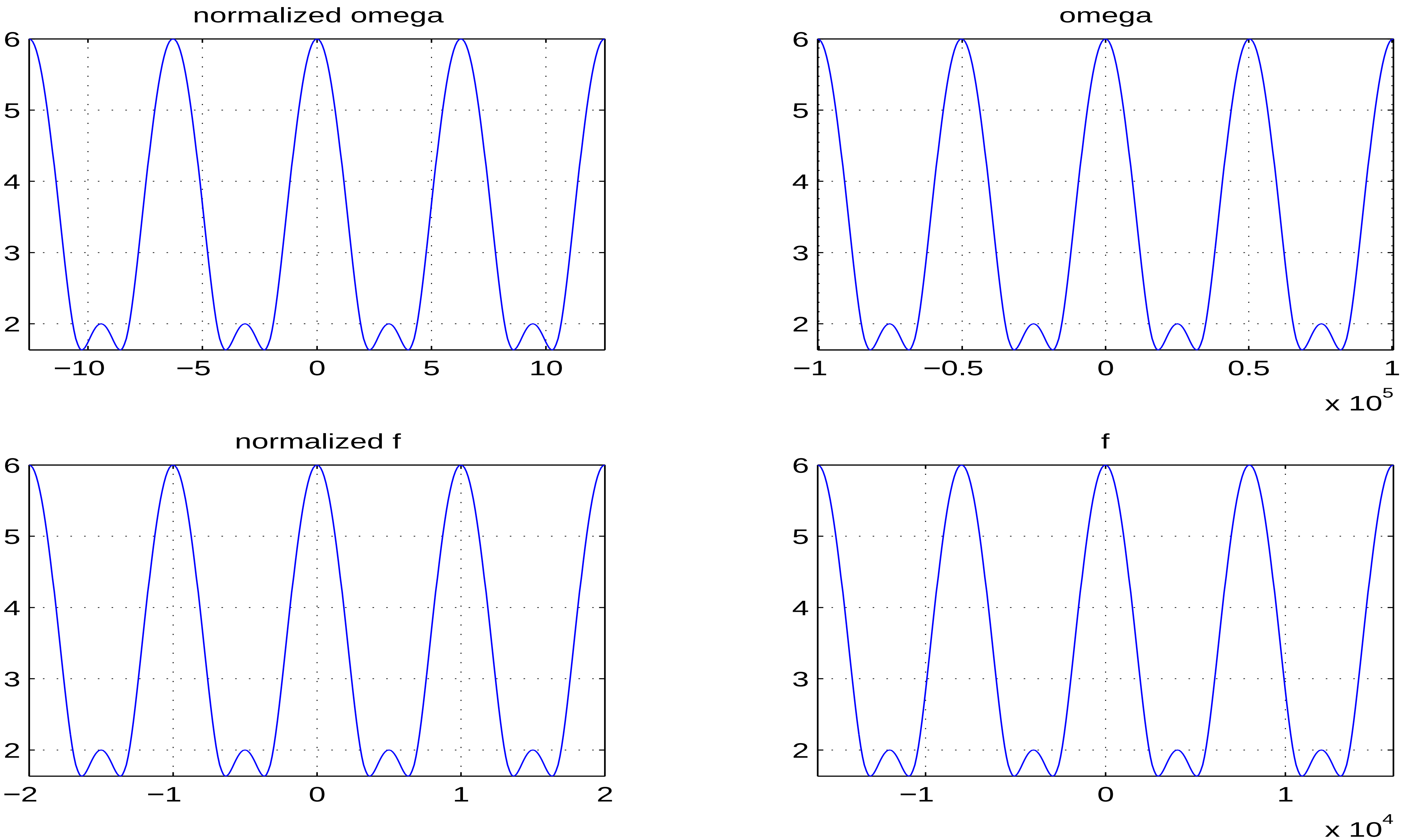
The properties are:

- periodicity of spectrum (also impulse response is a discrete signal!) – we should have correctly denoted $H(e^{j\omega})$ as $\tilde{H}(e^{j\omega})$:
  - in normalized angular frequencies: 2π rad
  - in regular angular frequencies: $2\pi F_s$ rad/s
  - in normalized frequencies: 1
  - in regular frequencies: F_s Hz
- symmetry:

$$H(e^{j\omega}) = H^{\star}(e^{-j\omega})$$

Example: frequency characteristic for impulse response 3 2 1. $F_s = 8000$ Hz.


h
module
phase



normalized omega
omega
normalized f
f
6
5
4
3
2
−10
−5
0
5
10
−1
−0.5
0.5
1
x 10^5
−2
2
x 10^4

System's response to a harmonic signal

$$x[n] = C_1 \cos(\omega_1 n + \phi_1) = \frac{C_1}{2} e^{j\phi_1} e^{j\omega_1 n} + \frac{C_1}{2} e^{-j\phi_1} e^{-j\omega_1 n}$$

Componennts are multiplied by the complex characteristic values $H(e^{j\omega_1})$ and $H(e^{-j\omega_1})$ that are complex conjugate, thus:

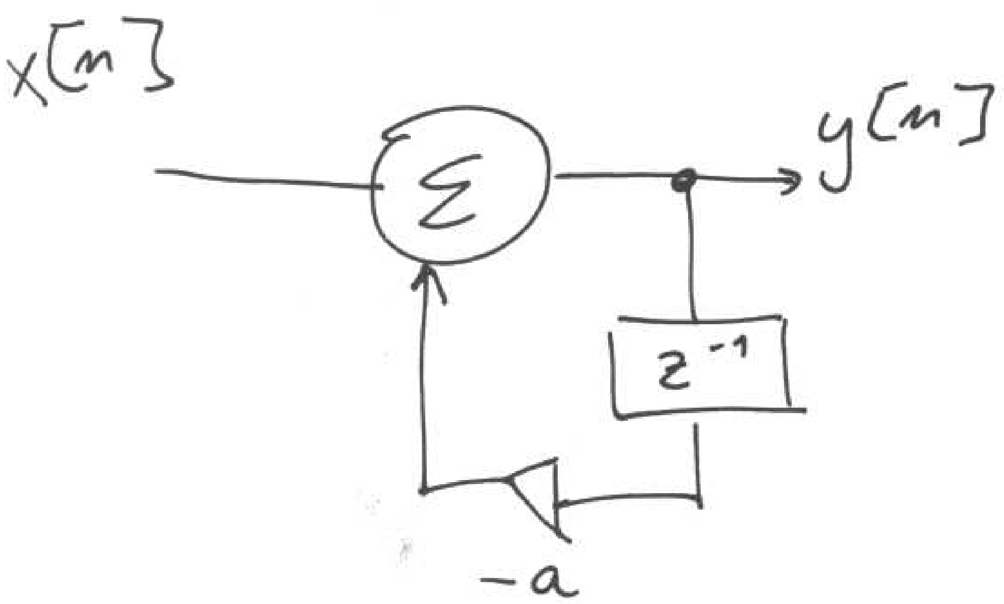
$$y[n] = H(e^{j\omega_1}) \frac{C_1}{2} e^{j\phi_1} e^{j\omega_1 n} + H^\star(e^{j\omega_1}) \frac{C_1}{2} e^{-j\phi_1} e^{-j\omega_1 n} =$$

$$= C_1 |H(e^{j\omega_1})| \cos(\omega_1 n + \phi_1 + \arg H(e^{j\omega_1}))$$

Non-recursive and recursive systems

In the previous example we saw a filter that processes the actual and delayed samples of an input signal. It's impulse response is **finite** - **finite impulse response – FIR** - non-recursive filters.

In **recursive** filters, we take into account also delayed samples of the output (feed-back), e.g.:



Such a filter has the following impulse response:

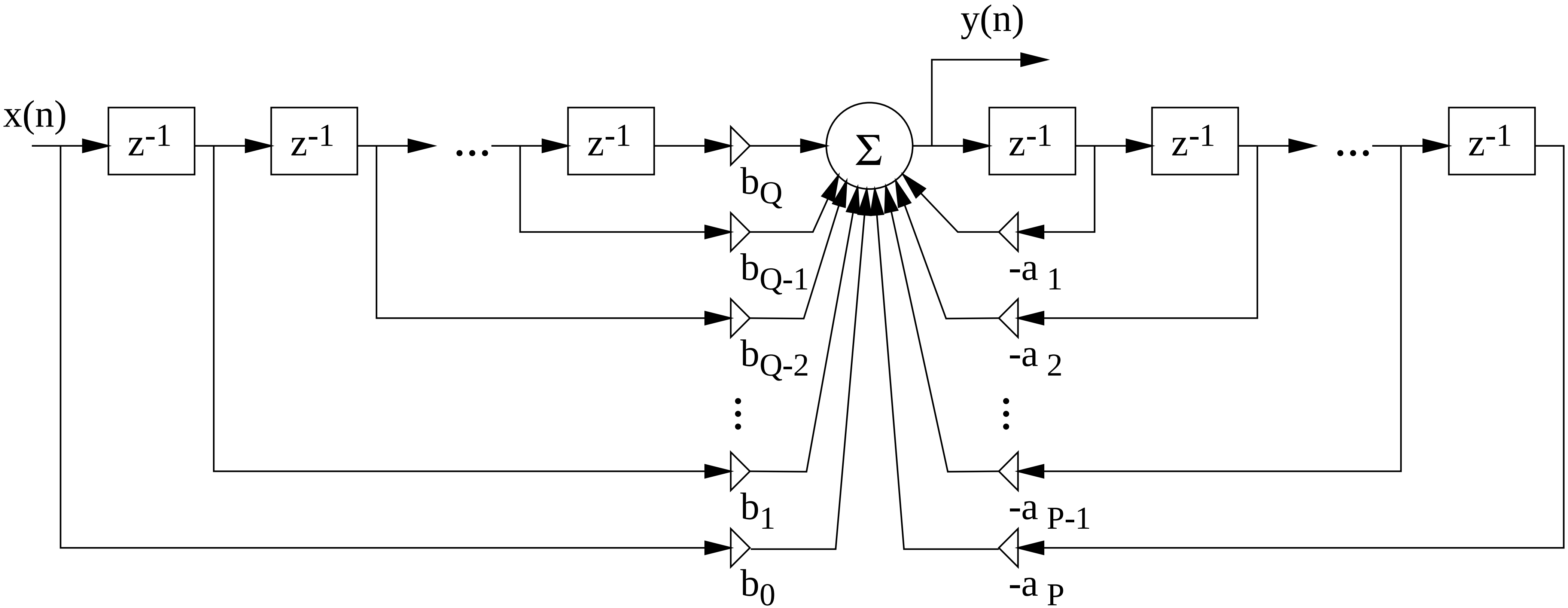
$$h[n] = \begin{cases} 0 & \text{for} \quad n < 0 \\ 1 \quad -a \quad (-a)^2 \quad (-a)^3 \ldots & \text{for} \quad n = 0, 1, 2, 3, \ldots \end{cases}$$

or:

$$h[n] = \begin{cases} 0 & \text{pro} \quad n < 0 \\ (-a)^n & \text{pro} \quad n \geq 0 \end{cases}$$

The impulse response is **infinite - infinite impulse response – IIR**. The example filter is pure recursive.

General recursive system



The output can be written by a difference equation:

$$y[n] = \sum_{k=0}^{Q} b_k x[n-k] - \sum_{k=1}^{P} a_k y[n-k], \tag{1}$$

where $x[n-k]$ are actual and delayed samples of the input and $y[n-k]$ are delayed samples of the output (note coeff. at sums).

types of filters again:

- **FIR** – non-recursive: only $b_0 \dots b_Q$ are non-zero. The impulse response is given directly by the coefficients of the filter:

$$h[n] = \begin{cases} 0 & \text{for } \ n < 0 \ \text{ and for } \ n > Q \\ b_n & \text{for } \ 0 \leq n \leq Q \end{cases}$$

- **IIR** – pure recursive: only $b_0, a_1 \dots a_P$ are non-zero values.
- **IIR** – generally recursive: a_i and b_i are non-zero values.

z-TRANSFORM

similarly as Laplace transform in continuous domain, it helps us to describe discrete signals and systems using complex variable z. z-transform is defined as:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n},$$

where z is a complex variable. We denote:

$$x[n] \xrightarrow{\mathcal{Z}} X(z)$$

and the inverse transform:

$$X(z) \xrightarrow{\mathcal{Z}^{-1}} x[n]$$

(we will not use the inverse trqansform)

We are not interested in computing the z-transform but rather in the following 3 properties:

1. **Linearity**:

$$x_1[n] \longrightarrow X_1(z)$$

$$x_2[n] \longrightarrow X_2(z)$$

$$ax_1[n] + bx_2[n] \longrightarrow aX_1(z) + bX_2(z)$$

2. **Delay of a signal**:

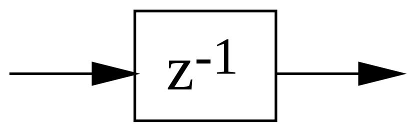
$$x[n] \longrightarrow X(z)$$

$$x[n-k] \longrightarrow \sum_{n=-\infty}^{\infty} x[n-k]z^{-n} = \sum_{n=-\infty}^{\infty} x[n]z^{-n-k} = z^{-k} \sum_{n=-\infty}^{\infty} x[n]z^{-n} = z^{-k}X(z)$$

The most relevant is a 1 sample delay:

$$x[n-1] \longrightarrow z^{-1}X(z)$$

This is why we represent it as



3. **Relationship to DTFT**: Fourier transform with discrete time computes a spectrum of a signal with discrete time:

$$\tilde{X}(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

it resembles ZT, if from the whole complex plane z we use only $e^{j\omega}$:

$$\tilde{X}(e^{j\omega}) = X(z)|_{z=e^{j\omega}},$$

where ω is a normalized angular frequency. It can be understood that DTFT is ZT on the uit circle. One period of the unit circle is 2π, which is an additional proof that DTFT is periodic...

Transfer function of a recursive system

For a system

we define a transfer function as:

$$H(z) = \frac{Y(z)}{X(z)}$$

Transfer function of a system is obtained by z-transforming the difference equation. ZT is linear and for a delayed signal $x[n-k]$ we write $X(z)z^{-k}$:

$$y[n] = \sum_{k=0}^{Q} b_k x[n-k] - \sum_{k=1}^{P} a_k y[n-k] \longrightarrow Y(z) = \sum_{k=0}^{Q} b_k X(z) z^{-k} - \sum_{k=1}^{P} a_k Y(z) z^{-k}$$

After a re-arrangement of components:

$$Y(z) + \sum_{k=1}^{P} a_k Y(z) z^{-k} = \sum_{k=0}^{Q} b_k X(z) z^{-k}$$

and we get the transfer function:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{Q} b_k z^{-k}}{1 + \sum_{k=1}^{P} a_k z^{-k}} = \frac{B(z)}{A(z)},$$

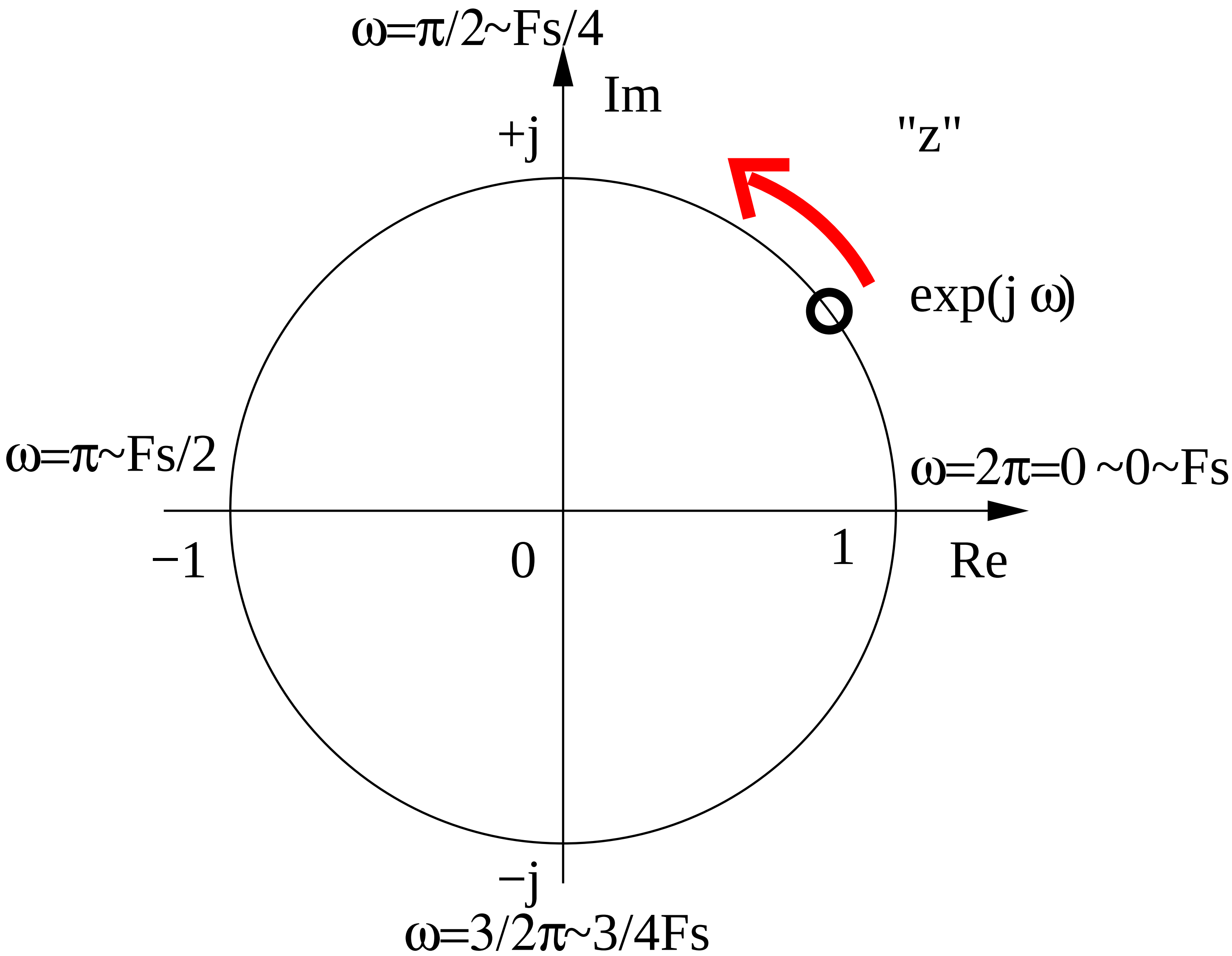
where $A(z)$ and $B(z)$ are two polynomials. The coefficient a_0 has to be equal to 1 eventhough it does not accure physically in the filter. It is a matimatical trick to denote that the filter has an output.

Frequency characteristics of a filter

we substitute z by $e^{j\omega}$ and calculate the tranfrom for ω in the interval we are interested in – mostly from 0 to π (half of the sampling frequency):

$$H(e^{j\omega}) = H(z)|_{z=e^{j\omega}} = \frac{\sum_{k=0}^{Q} b_k e^{-j\omega k}}{1 + \sum_{k=1}^{P} a_k e^{-j\omega k}}$$

The equation looks difficult but we can easily evaluate it in Matlab using function `freqz(b,a,N)`, where vectors `a` and `b` contain polynoms' coefficients and `N` indicate the number of samples (from 0 to the half of the sampling frequency).


ω=π/2~Fs/4
Im
"z"
+j
exp(j ω)
ω=π~Fs/2
ω=2π=0 ~0~Fs
−1
0
1
Re
−j
ω=3/2π~3/4Fs

Nulls and poles of $H(z)$ function and what with them...

Transfer function can be also defined as a product:

$$H(z) = \frac{B(z)}{A(z)} = \frac{b_0 + b_1 z^{-1} + \ldots + b_Q z^{-Q}}{1 + a_1 z^{-1} + \ldots + a_P z^{-P}} = \frac{z^{-Q}(b_0 z^Q + b_1 z^{Q-1} + \ldots + b_Q)}{z^{-P}(z^P + a_1 z^{P-1} + \ldots + a_P)} =$$

$$= b_0 \frac{z^{-Q}}{z^{-P}} \frac{\prod_{k=1}^{Q}(z - n_k)}{\prod_{k=1}^{P}(z - p_k)} = b_0 z^{P-Q} \frac{\prod_{k=1}^{Q}(z - n_k)}{\prod_{k=1}^{P}(z - p_k)},$$

- n_k are called **nulls**. These are the points in the plane z for which holds $B(z) = 0$ and therefore $H(z) = 0$.
- p_k are called **poles**. These are the points in the plane z for which $A(z) = 0$ and therefore $H(z) = \infty$.

If $a_k, b_k \in \Re$, then poles p_k and nulls n_k are either real or complex conjugate. If the orders of numerator and denominator are different, then z^{P-Q} is responsible for

- $(P-Q)$-fold null in the origin, if P is greater than Q.
- $(Q-P)$-fold pole in the origin, if P is less than Q.

Stability

System is stable, if all poles lie *within unity circle:*

$$|p_k| < 1$$

Frequency characteristic from nulls and poles

Similarly as for continuous time sytems, we can estimate frequncy characteristic $H(e^{j\omega})$ from the nulls and poles of a system:

$$H(e^{j\omega}) = b_0 z^{-(Q-P)} \frac{\prod_{k=1}^{Q}(z-n_k)}{\prod_{k=1}^{P}(z-p_k)},\Big|_{z=e^{j\omega}} = b_0 e^{j\omega(P-Q)} \frac{\prod_{k=1}^{Q}(e^{j\omega}-n_k)}{\prod_{k=1}^{P}(e^{j\omega}-p_k)},$$

For a given $e^{j\omega}$ each braces is a complex number that we can comprehend as a vector from the null or pole point to the point $e^{j\omega}$. To obtain a value of a complex characteristic for a given frequency ω we :

- multiply modules of all the numbers from the numerator and sum up arguments.
- divide mudules of all the numbers from the denominator and subtract arguments.

Term $e^{j\omega(P-Q)}$ is other than 1 only in case when the order of the polynoms differs – nulls or poles in the origin – the magnitude of the complex characteristics remains unmodified, the change occurs in the argument only.

Examples

Example 1. Non-recursive filter is defined by a difference equation:

$$y[n] = x[n] + 0.5x[n-1]$$

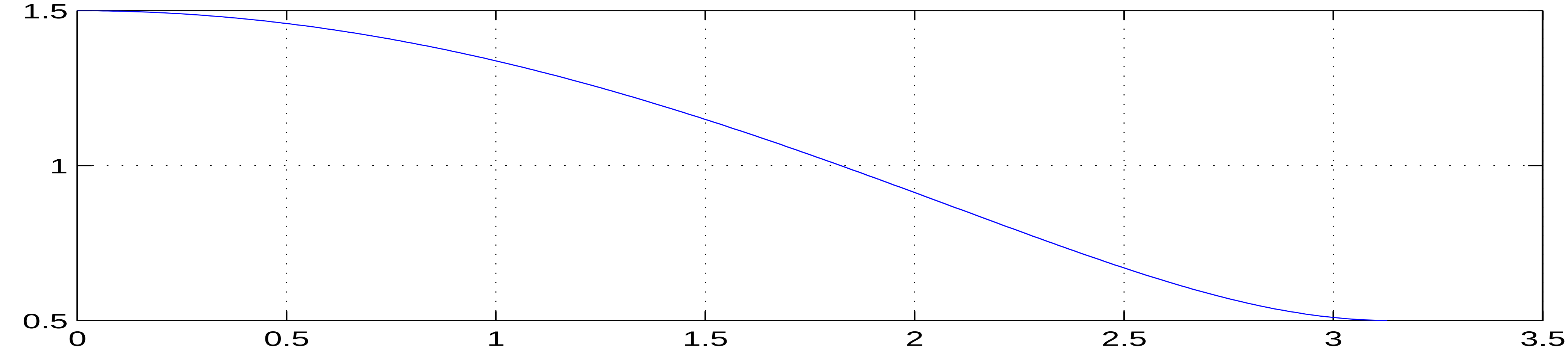
1. What is its impulse response?
2. Find parameters of the transfer function (coef a, b).?
3. Compute frequency characteristic?
4. Is the filter stable?
5. Find freq. charfacteristic from nulls and poles.

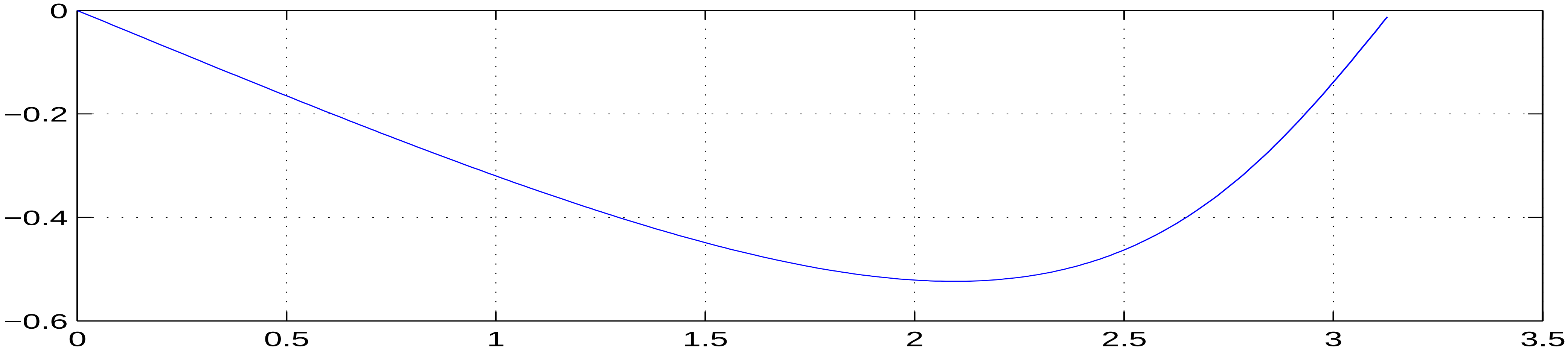
Solution

1. $h[n] = 1,\ 0.5$ for $n = 0, 1$, and zero elsewhere.
2. $Y(z) = X(z) + 0.5X(z)z^{-1} \qquad Y(z) = X(z)[1 + 0.5z^{-1}]$
   $H(z) = 1 + 0.5z^{-1} = \frac{1+0.5z^{-1}}{1}$, thus $b_0 = 1$, $b_1 = 0.5$, $a_0 = 1$.

3. We could use substitution $z = e^{j\omega}$, but we rather call a Matlab function:

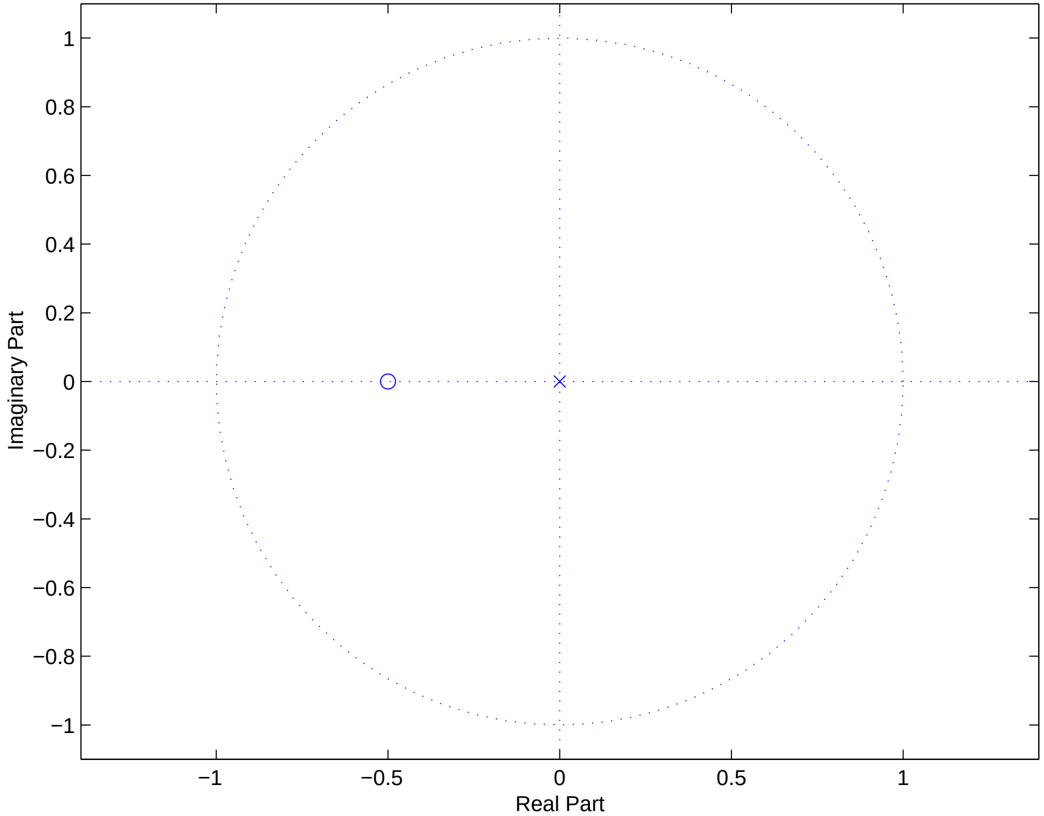
```
H=freqz([1 0.5],[1],256); om=(0:255)/256 * pi;
subplot(211); plot(om,abs(H)); grid
subplot(212); plot(om,angle(H)); grid
```





$\Rightarrow$The filter is a low pass filter:

4. nulls and poles: $H(z) = \frac{1+0.5z^{-1}}{1} = \frac{z(1+0.5z^{-1})}{z} = \frac{z+0.5}{z}$ Numerator is zero for $z = -0.5$, thus, filter
   has 1 null point $n_1 = -0.5$. Denominator becomes zero for $z = 0$, thus one pole: $p_1 = 0$



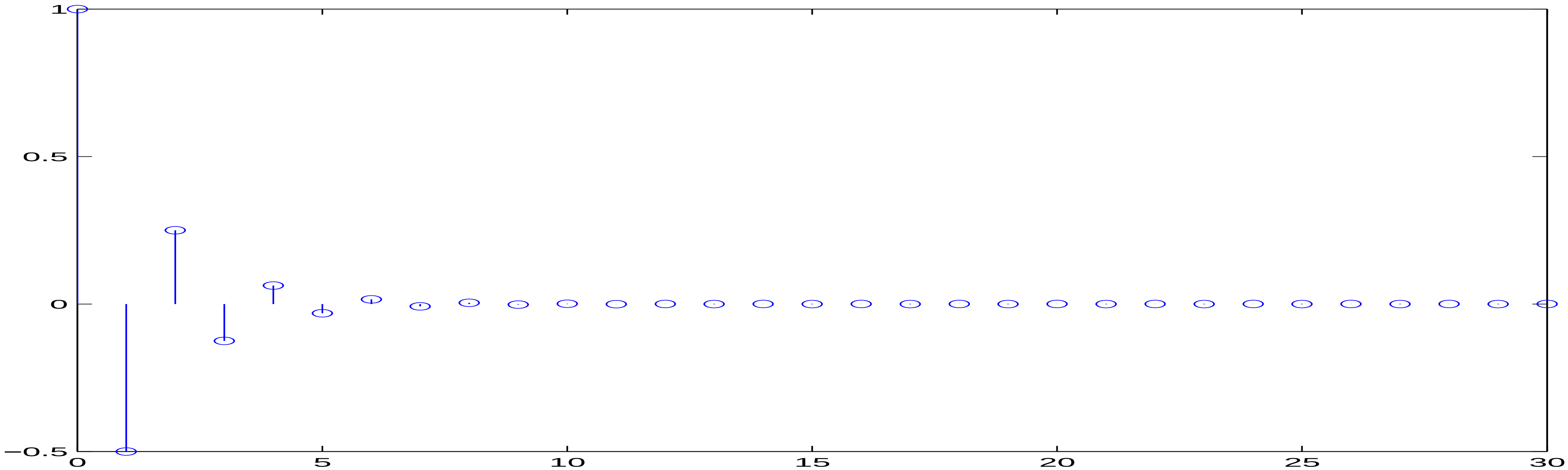
   $\Rightarrow$the filter is stable.

5. frequency characteristics from nulls and poles:
   $H(z) = \frac{z-(-0.5)}{z-0}$ $\quad H(e^{j\omega}) = \frac{e^{j\omega}-(-0.5)}{e^{j\omega}-0}$

Example 2. Recursive filter is defined by a difference equation:

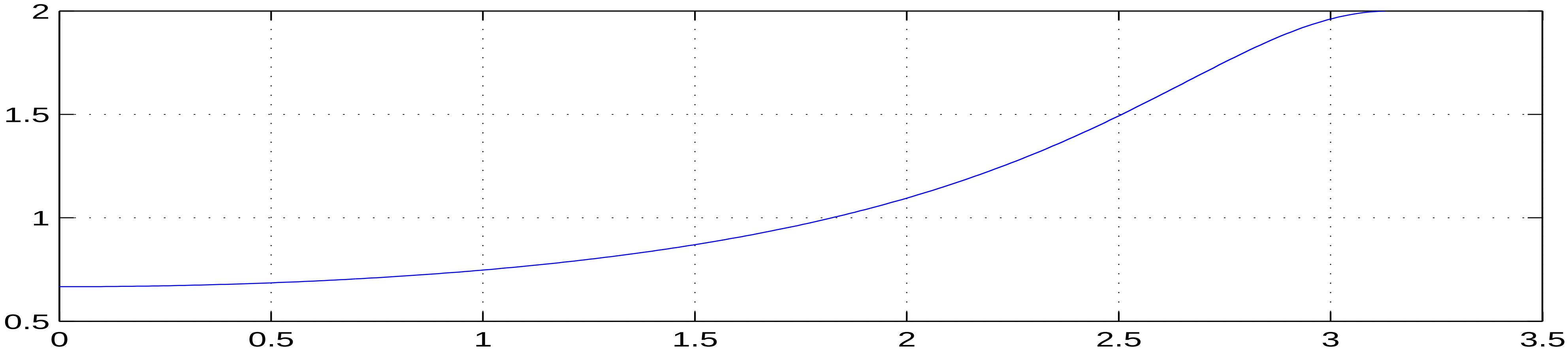
$$y[n] = x[n] - 0.5y[n-1]$$

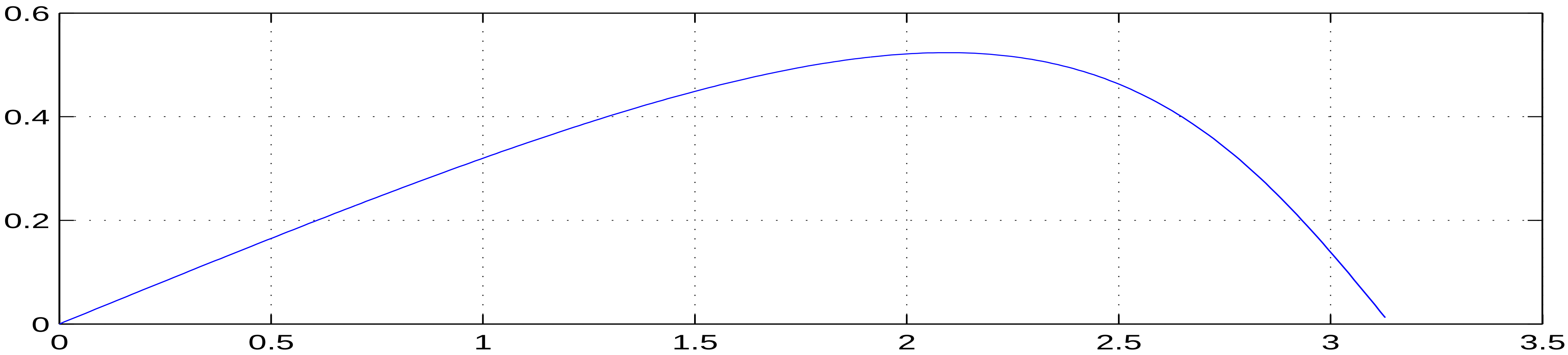
1. infinite impulse response:



2. $Y(z) = X(z) - 0.5Y(z)z^{-1}$ $\quad Y(z)[1 + 0.5z^{-1}] = X(z)$ $\quad H(z) = \frac{1}{1+0.5z^{-1}}$, thus $b_0 = 1$, $a_0 = 1$, $a_1 = 0.5$.

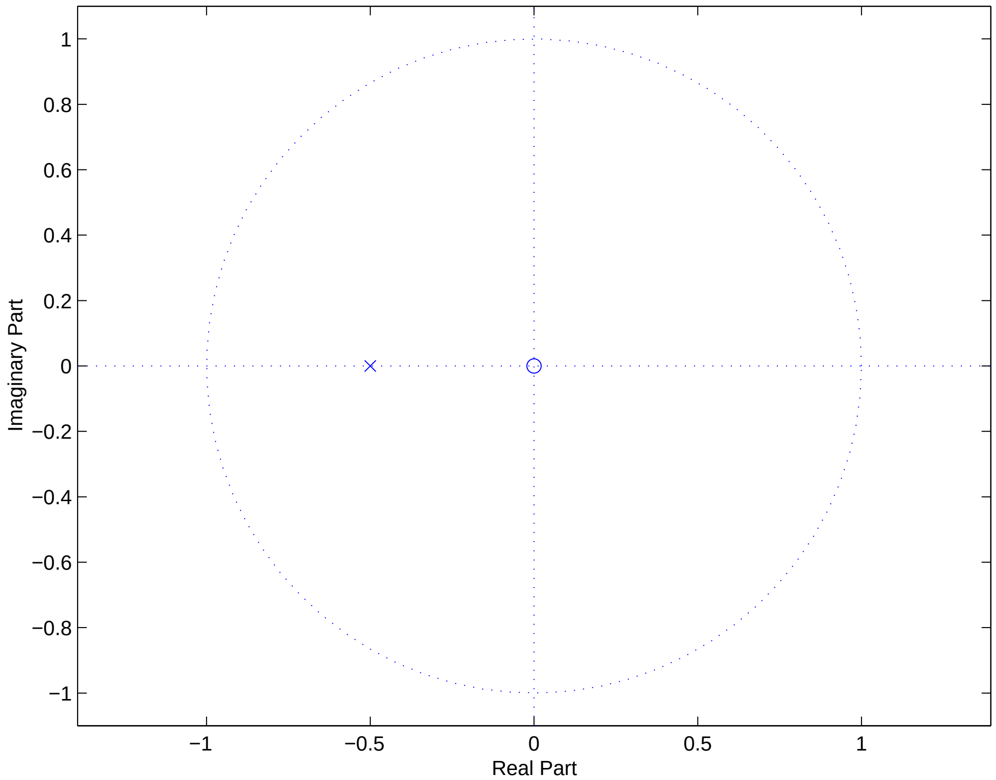
3.
```
H=freqz([1],[1 0.5],256); om=(0:255)/256 * pi;
subplot(211); plot(om,abs(H)); grid
subplot(212); plot(om,angle(H)); grid
```





$\Rightarrow$High pass filter

4. null and poles: $H(z) = \frac{1}{1+0.5z^{-1}} = \frac{z}{z(1+0.5z^{-1})} = \frac{z}{z+0.5}$ Numerator is zero for $z = 0$, one zero: $n_1 = 0$. Denominator is zero for $z = -0.5$, one pole: $p_1 = -0.5$



⇒filter is stable.

5. frequency characteristic: $H(z) = \frac{z-0}{z-(-0.5)}$ $\qquad H(e^{j\omega}) = \frac{e^{j\omega}-0}{e^{j\omega}-(-0.5)}$

Excercise 3. Real filter: In matlab we have found the parameters of a low-pass filter:

- sampling freq 16000 Hz.
- end of pass band 3000 Hz.
- begin of stop band 3500 Hz.
- max. attenuation in pass band 3 dB
- min. attenuation in stop band 40 dB.

How did we do it?

```
Fs = 16000; Wp = 3000/8000; Ws = 3500/8000;
Rp = 3; Rs = 40;
[N, Wn] = ellipord(Wp, Ws, Rp, Rs)
[B,A] = ellip(N,Rp,Rs,Wn)
```

We obtained a filter with order 5. The coefficients in the numerator are:

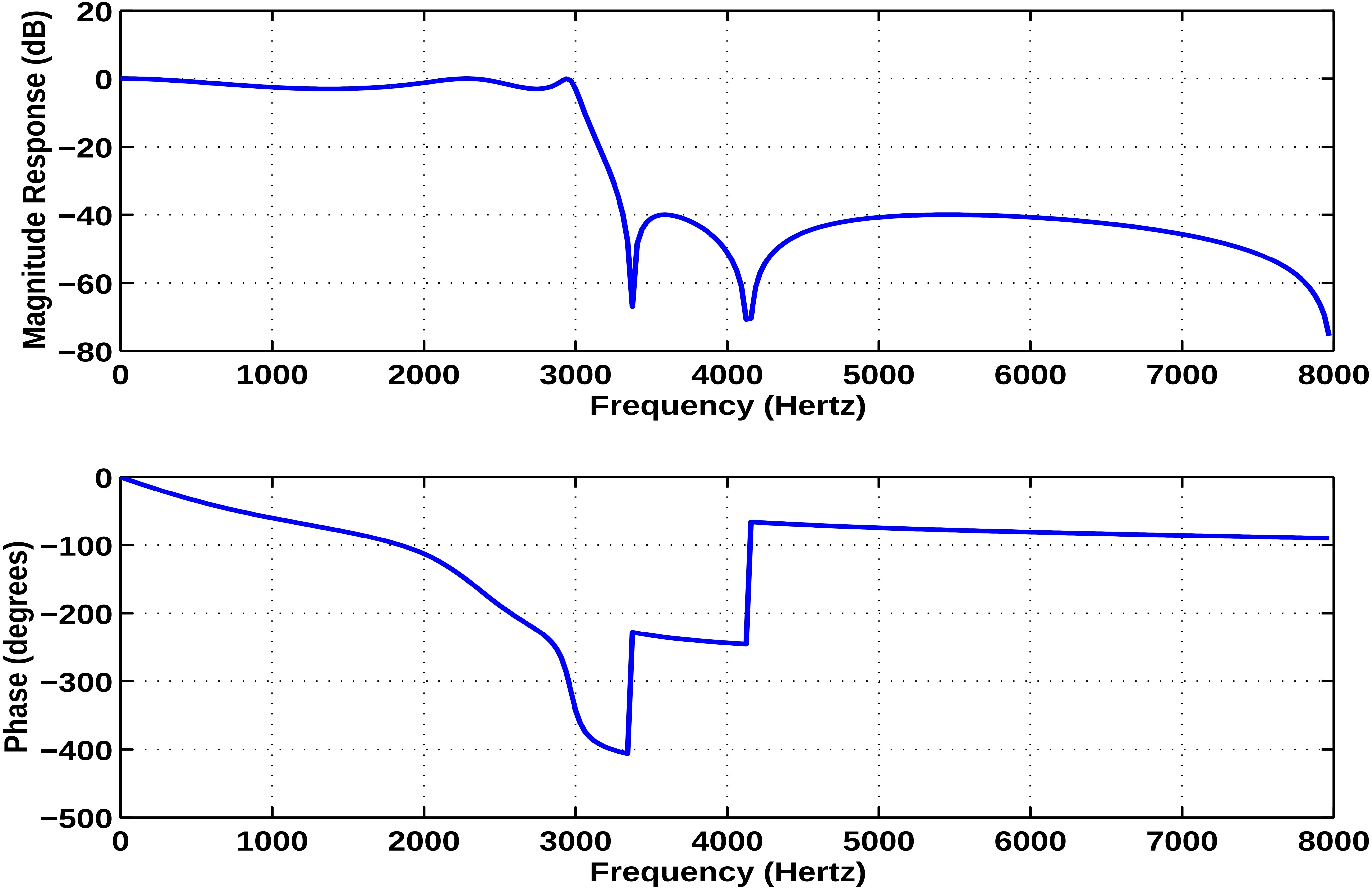
$b_0 = 0.0378$, $b_1 = 0.0235$, $b_2 = 0.0592$, $b_3 = 0.0592$, $b_4 = 0.0235$, $b_5 = 0.0378$

The denominator coefficients are:

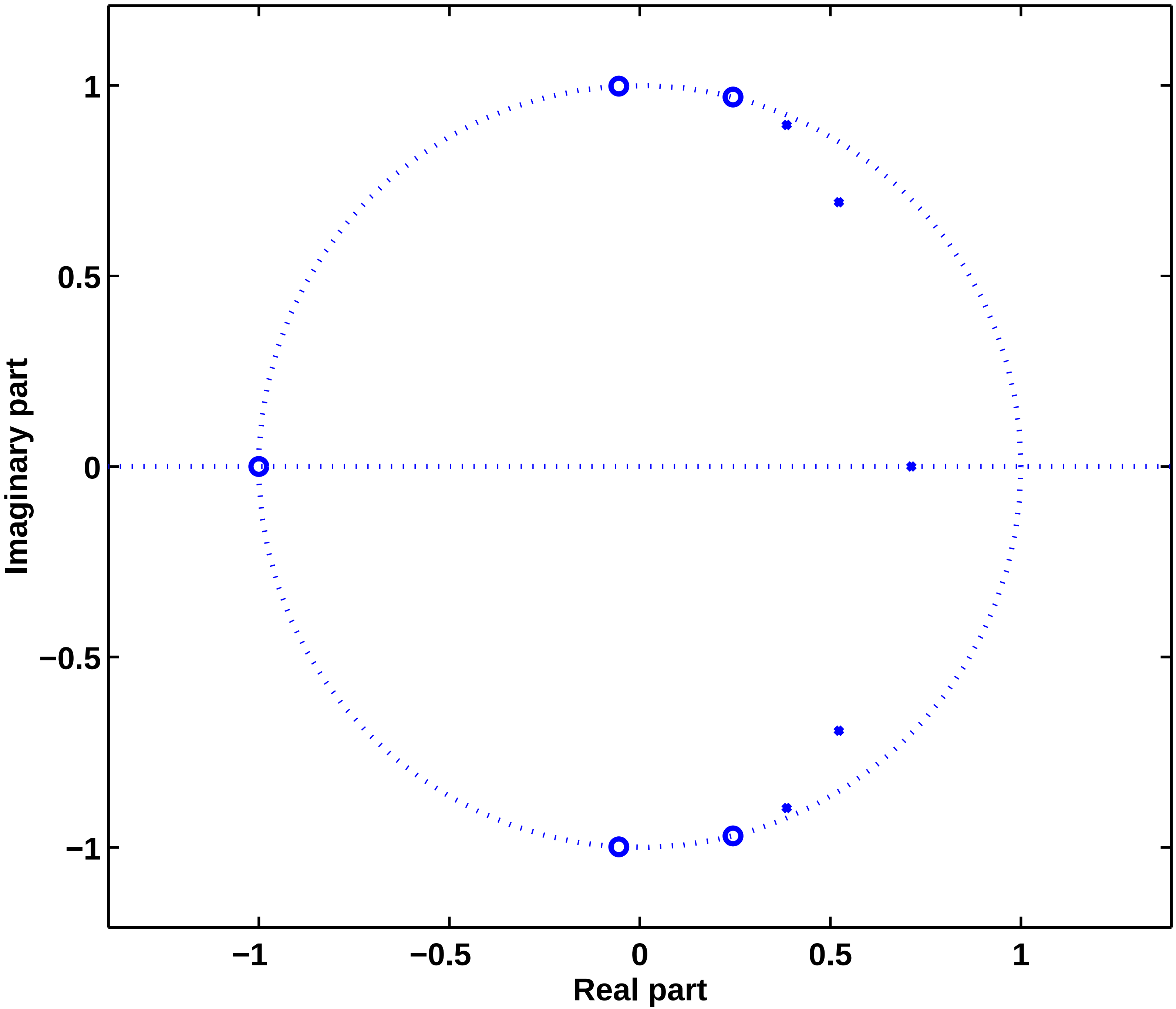
$a_0 = 1$, $a_1 = -2.5271$, $a_2 = 3.8031$, $a_3 = -3.3632$, $a_4 = 1.8395$, $a_5 = -0.5112$.

Frequency characteristics, poles and nulls:

```
figure(1); freqz (B,A,256,16000);
figure(2); zplane(roots(B), roots(A));
```



Magnitude Response (dB)
20
0
−20
−40
−60
−80
0
1000
2000
3000
4000
5000
6000
7000
8000
Frequency (Hertz)
Phase (degrees)
0
−100
−200
−300
−400
−500
0
1000
2000
3000
4000
5000
6000
7000
8000
Frequency (Hertz)


1
0.5
0
−0.5
−1
Imaginary part
−1
−0.5
0
0.5
1
Real part

$\Rightarrow$ The filter is just stable but due to possible numerical error it can become unstable.

$\Rightarrow$ If a pole is close to the unit circle, it determines the maximum of the filter

$\Rightarrow$ If a null is close to the unit circle, it determines the minimum of the filter