Fourier transform (FT)

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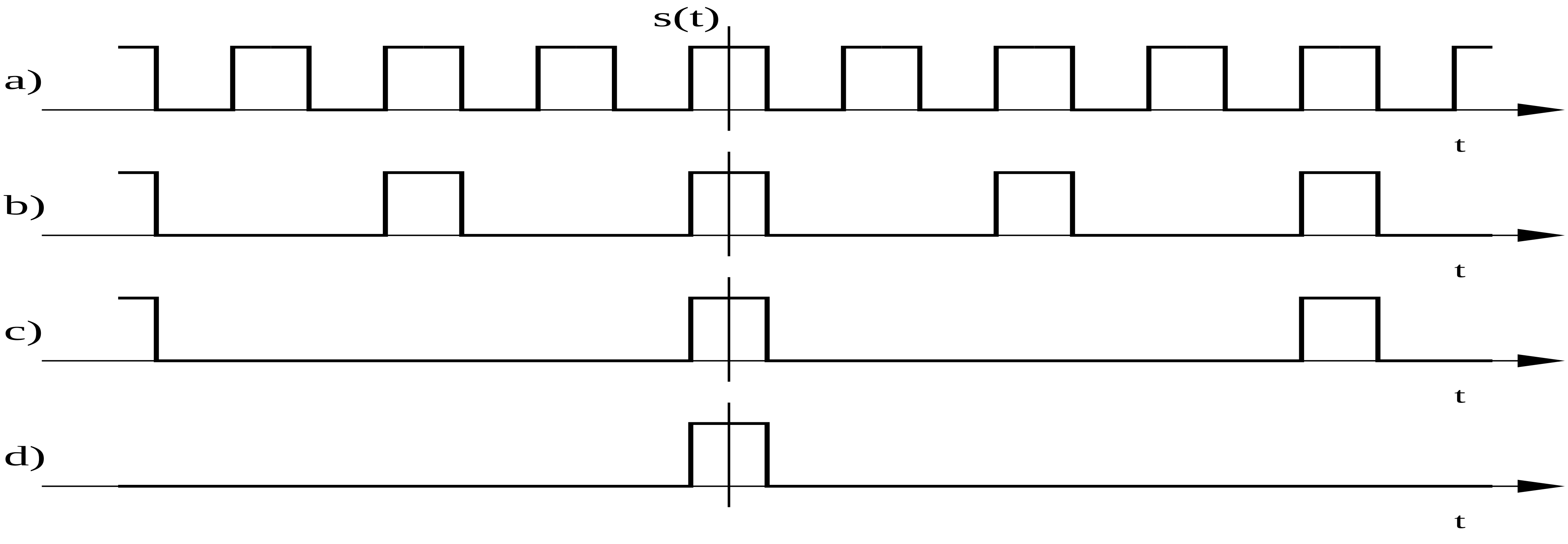
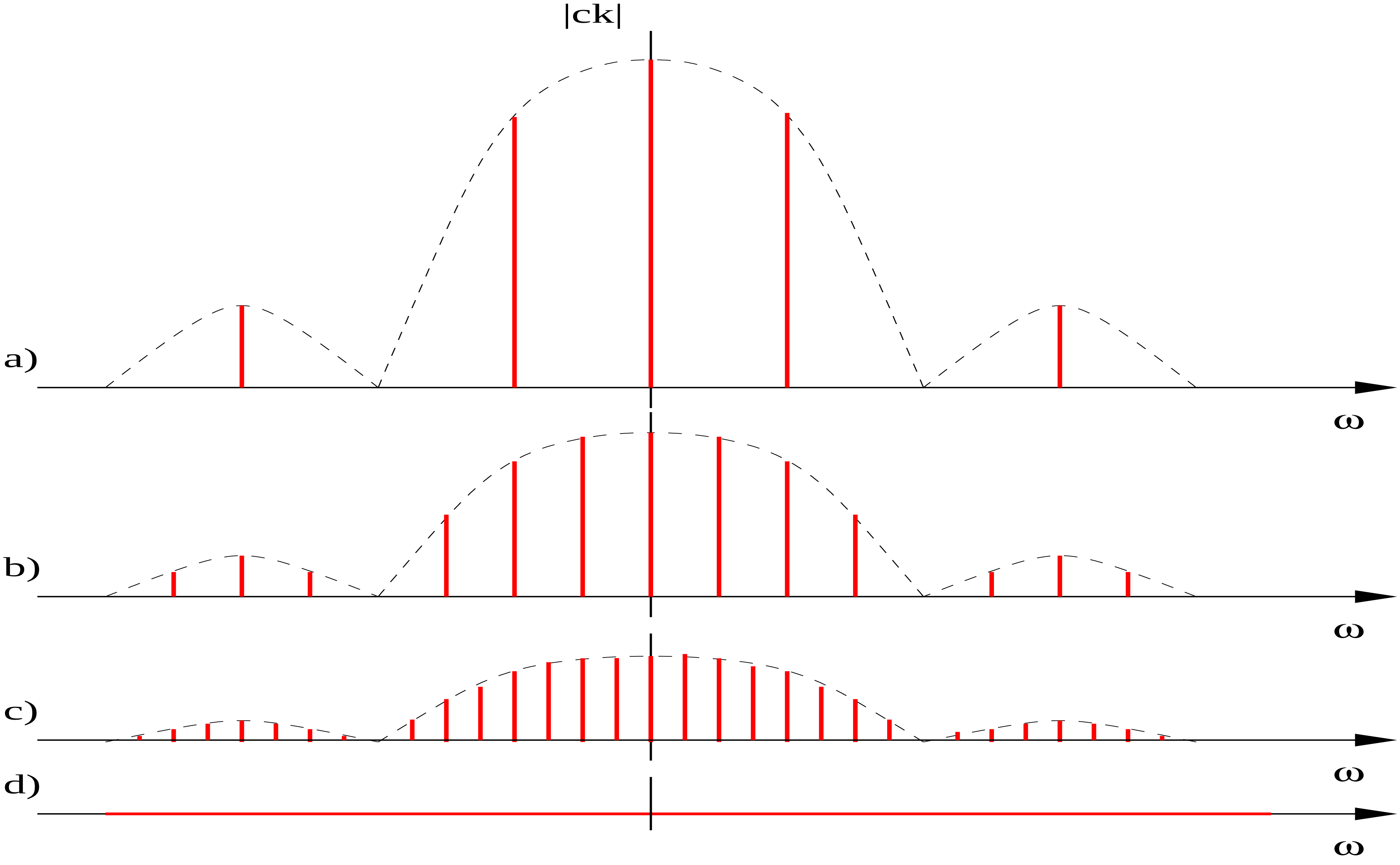
- Fourier transform
- Properties of spectral function
- Spectral function of important signals
- Hints on spectra
- Energy and Parseval theorem.

Reasons for developement of FT:

- We want to do frequency analysis of signals other than periodic
- Nonperiodic signals will be also expressed as a sum of harmonic signals (system responce is nicely calculated for input $e^{j\omega t}$, etc). It will be a little more difficult to imagine as we will obtain an infinite number of components that are infinitelly small.





|ck|
a)
ω
b)
ω
c)
ω
d)
ω

From FS to FT

Coefficients of FS:

$$c_k = \frac{1}{T_1} \int_{-\frac{T_1}{2}}^{+\frac{T_1}{2}} x(t) e^{-jk\omega_1 t} dt.$$

Now, we will be "stretching" the period to infinity

$$T_1 \to \infty, \quad \omega_1 = \frac{2\pi}{T_1} \to d\omega, \quad k\omega_1 \to \omega$$

$$c_k \to dc, \quad \frac{1}{T_1} \to \frac{d\omega}{2\pi},$$

Derive a new equation for the coefficients' calculation:

$$dc = \frac{d\omega}{2\pi} \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt.$$

$2\pi \frac{dc}{d\omega}$ is an infinitelly small coefficient increment on an infinitelly small increment of angular frequency multiplied by 2π. Rather we will introduce term: **Spectral function** $X(j\omega)$.

$$\boxed{X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt,}$$

Function $X(j\omega)$ will be called **Fourier projection/image** or simply **image** of signal $x(t)$. Spectral function $X(j\omega)$ can also be called **spectrum**. Fourier transform is sometimes denoted as $\mathcal{F}$: $x(t) \xrightarrow{\mathcal{F}} X(j\omega)$.

Fundamental properties of spectral function

If projection exists, then:

$$X(j\omega) = X^{\star}(-j\omega)$$

which follows from,

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\cos(\omega t)dt - j\int_{-\infty}^{+\infty} x(t)\sin(\omega t)dt.$$

Other properties are special cases. Even signal has only real spectrum:

$$x(t) = x(-t) \Rightarrow X(j\omega) = \Re\{X(j\omega)\}$$

Odd signal has only imaginary spectrum:

$$x(t) = -x(-t) \Rightarrow X(j\omega) = j\Im\{X(j\omega)\}$$

Inverse Fourier transform

Signal synthesis from FS coefficients:

$$x(t) = \sum_{k=-\infty}^{+\infty} c_k e^{jk\omega_1 t} = \frac{1}{2\pi} \sum_{k=-\infty}^{+\infty} 2\pi \frac{c_k}{\omega_1} e^{jk\omega_1 t} \omega_1.$$

By transition $T_1 \to \infty$, we obtain:

$$\boxed{x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{+j\omega t} d\omega.}$$

Convergence of FT

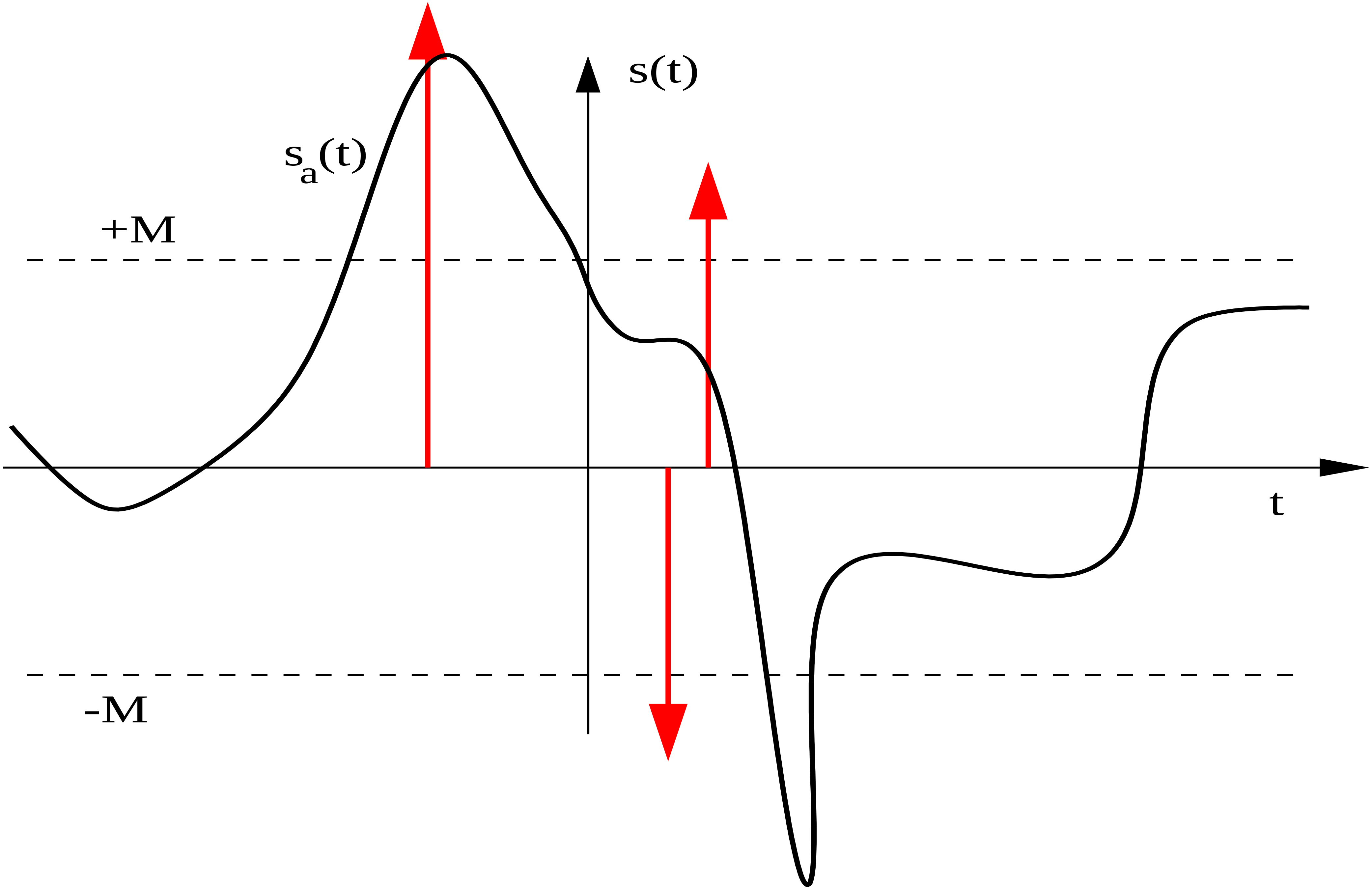
If we assume that spectral function will consist of impulse signals, for FT to converge it only needs to satisfy the following restrictions:

- the signal has to be bounded with a constant $M < \infty$,

  $$|x(t)| < M$$

  Its energy thus does not have to be finite. It allows us to calculate FT of a direct current of a periodic signal!

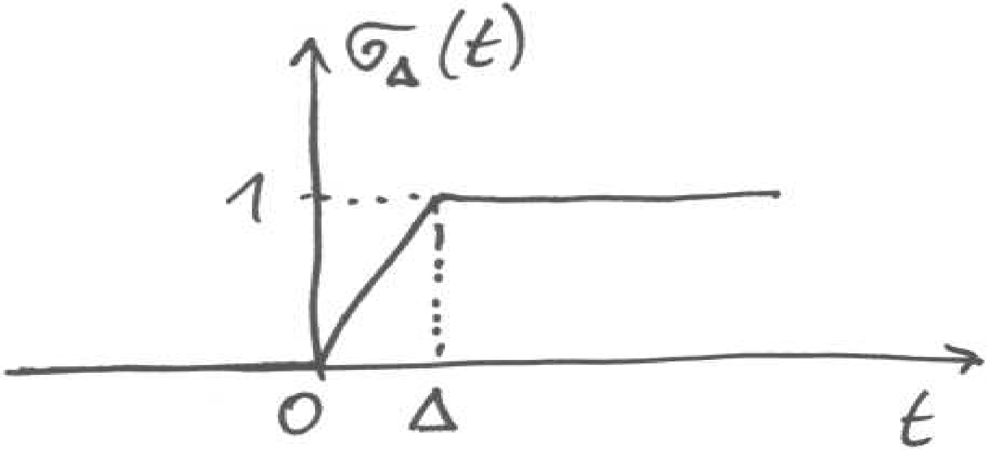
- Signal can be composed from unit impulses - shifted and scaled. This property will help us during sampling, where we will be looking for a spectrum of a periodic sequence of unit impulses.

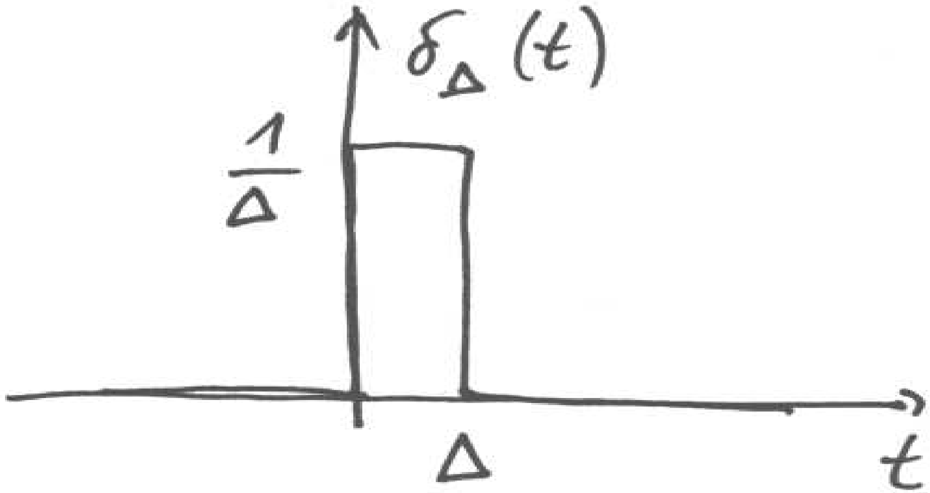

s(t)
$s_a(t)$
+M
t
-M

SPECTRAL FUNCTION OF IMPORTANT SIGNALS

Unit impuls

Unit impuls is an infinitely sharp peak bounding unit area, that is a derivation of the unit step function: $\delta_\Delta(t) = \frac{d\sigma_\Delta(t)}{dt}$, where Δ goes to 0:

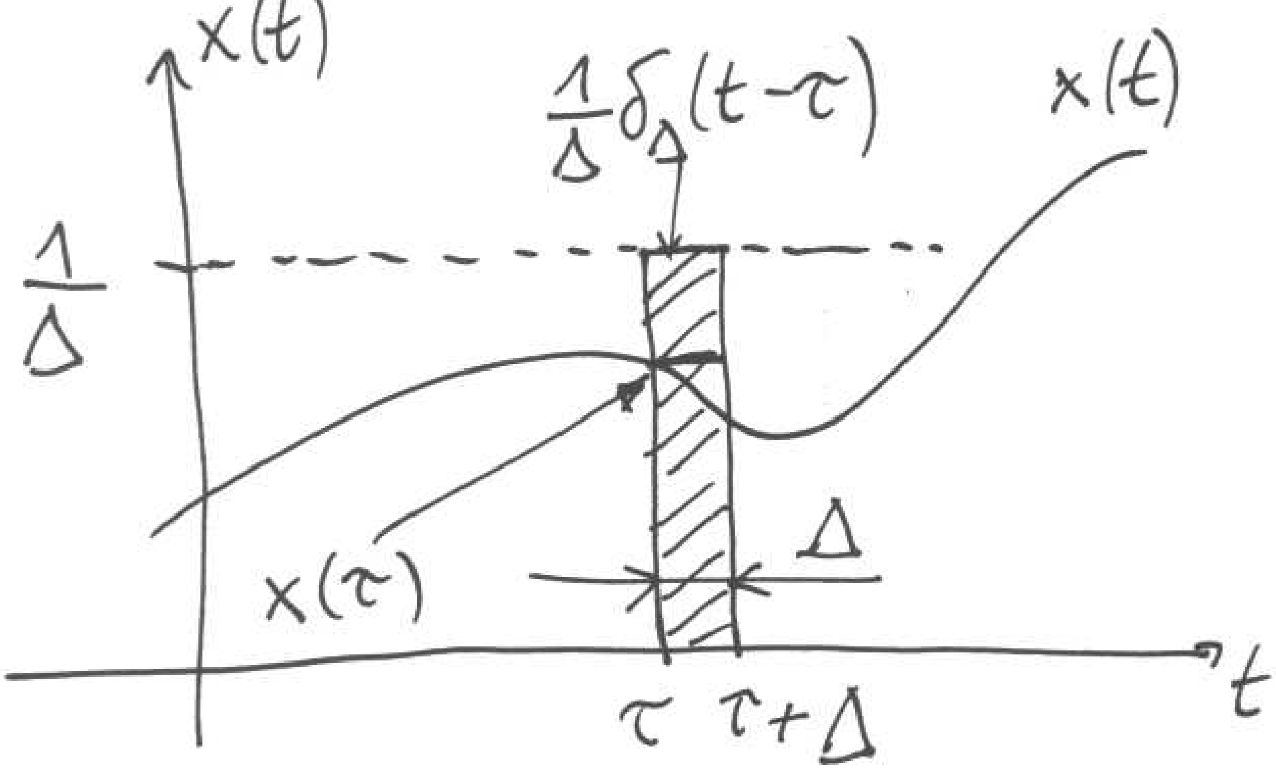




Unit impulse can be used in sampling, it satisfies:

$$\int_{-\infty}^{+\infty} x(t)\delta(t-\tau)dt = x(\tau)$$

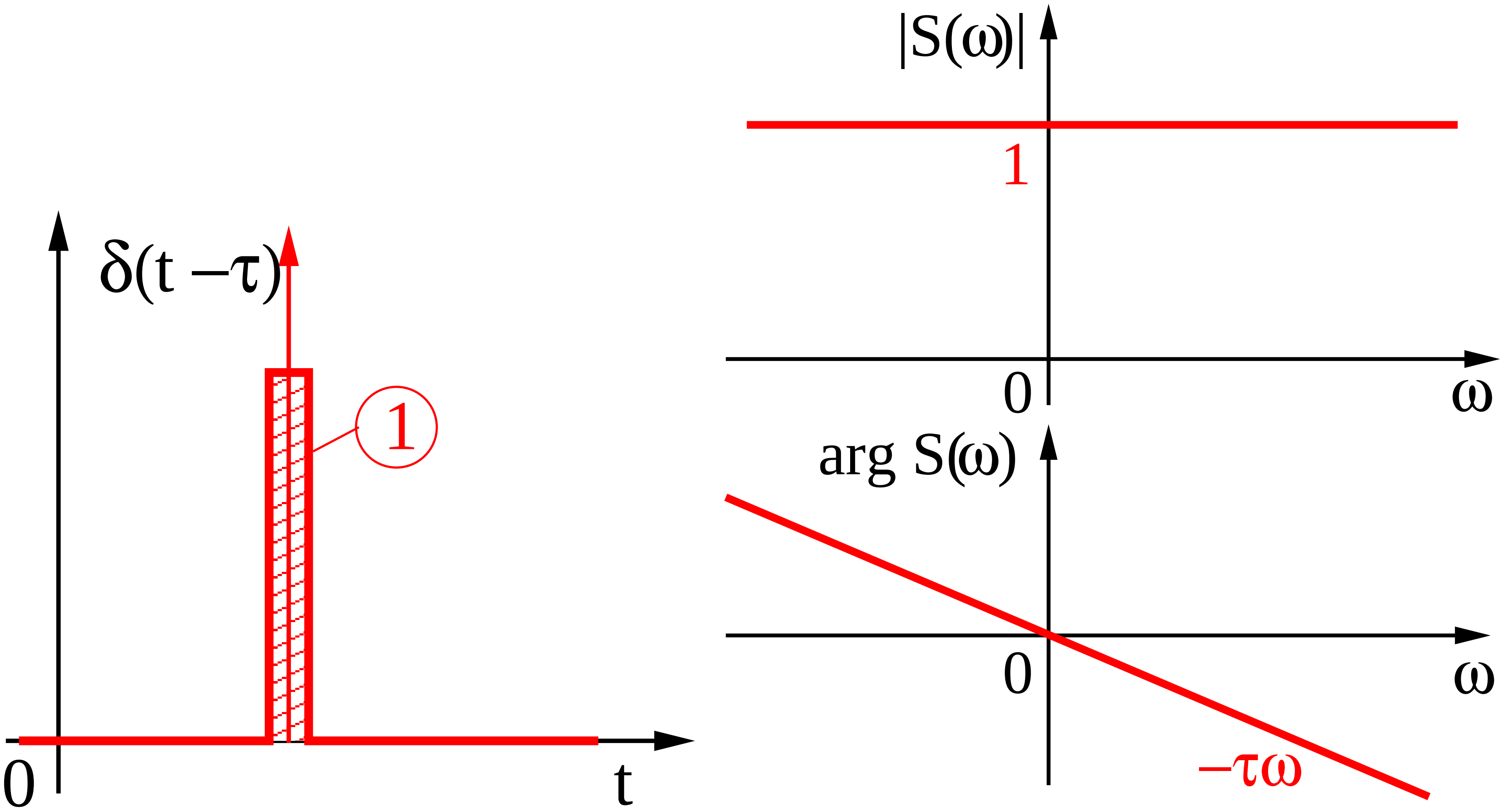
Why? If we multiply signal x by $\delta_\Delta(t-\tau)$ and integrate, we obtain an area of the size $\Delta\frac{1}{\Delta}x(\tau) = x(\tau)$. If Δ is sufficiently short, we can assume the signal to be a constant within the area it bounds. Thus the equation holds for $\Delta \to 0$.



$$\int_{\infty}^{+\infty} x(t)\delta_\Delta(t-\tau)dt = \int_{\tau}^{\tau+\Delta} x(t)\delta_\Delta(t-\tau)dt \approx \int_{\tau}^{\tau+\Delta} x(\tau)\delta_\Delta(t-\tau)dt = \Delta\frac{1}{\Delta}x(\tau) = x(\tau)$$

Spectral function of unit impuls

$$X(j\omega) = \int_{-\infty}^{+\infty} \delta(t-\tau) e^{-j\omega t} dt = e^{-j\omega \tau}$$

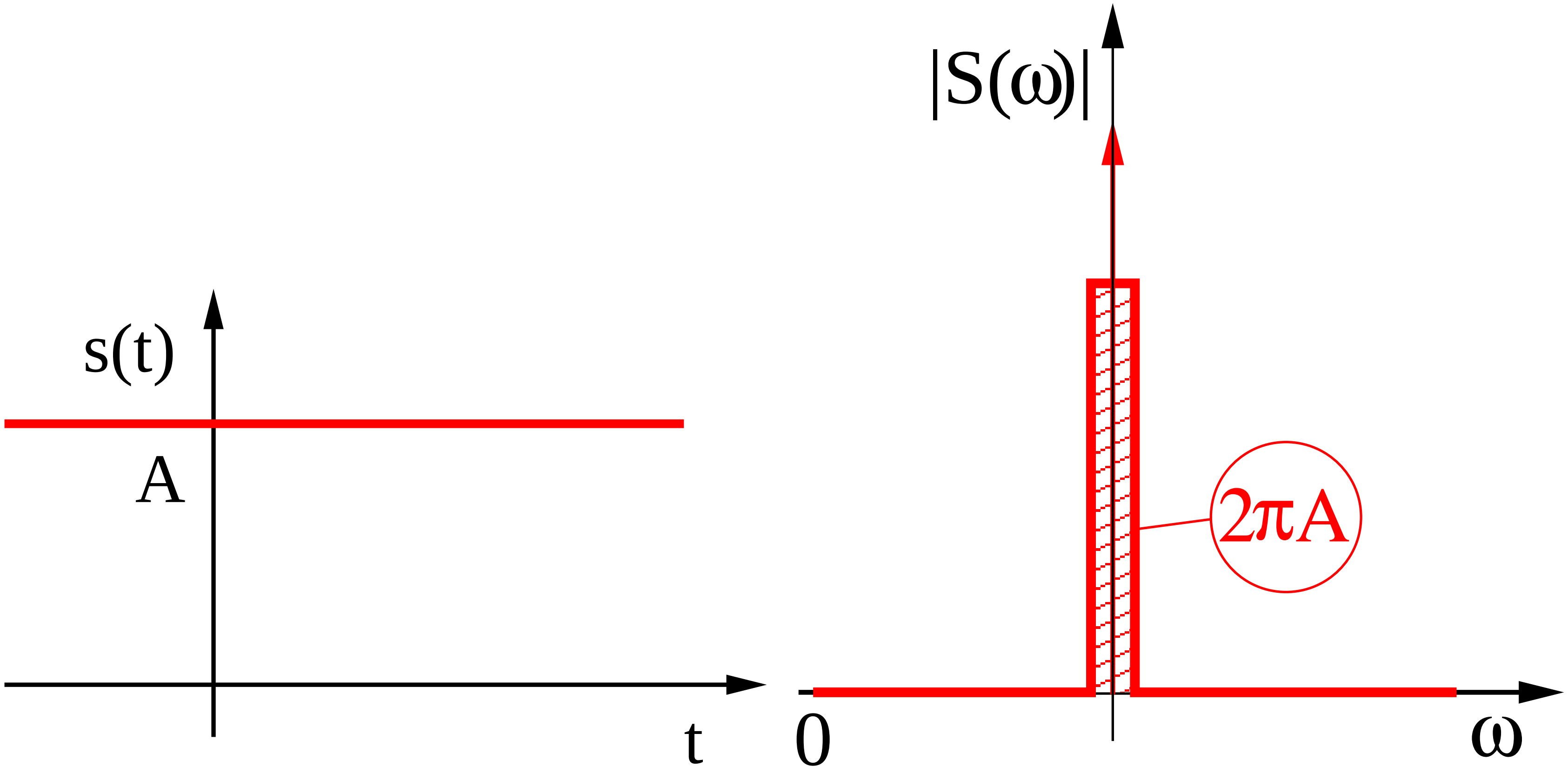


Direct signal

$$X(j\omega) = 2\pi A\delta(\omega)$$

Prove by inverse FT:

$$x(t) = \frac{1}{2\pi}2\pi A\int_{-\infty}^{+\infty}\delta(\omega)e^{j\omega t}d\omega = A.$$



Periodic signal described using FS

$$x(t) = \sum_{k=-\infty}^{+\infty} c_k e^{jk\omega_1 t}$$

First, lets see what a signal with the following spectral function looks like:

$$X(j\omega) = 2\pi\delta(\omega - \omega_0)$$

$$x(t) = \frac{1}{2\pi} 2\pi \int_{-\infty}^{+\infty} \delta(\omega - \omega_0) e^{j\omega t} d\omega = \ldots$$

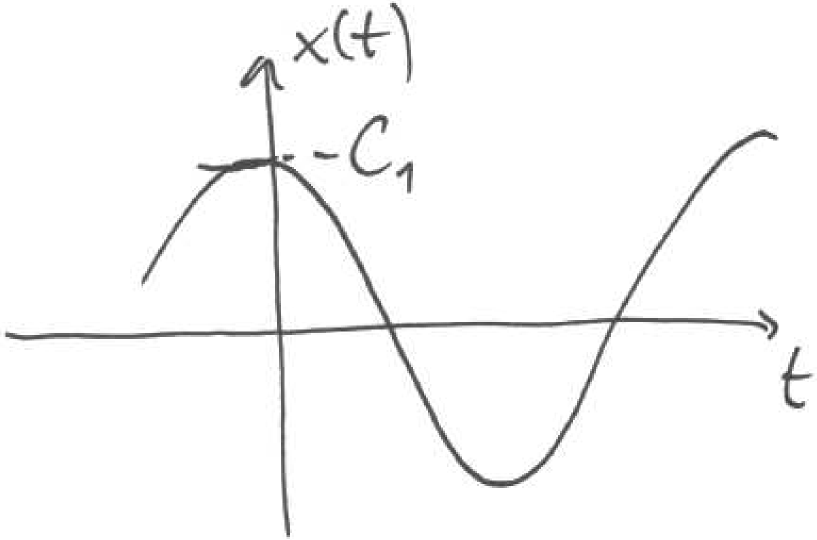
after variable substitution:

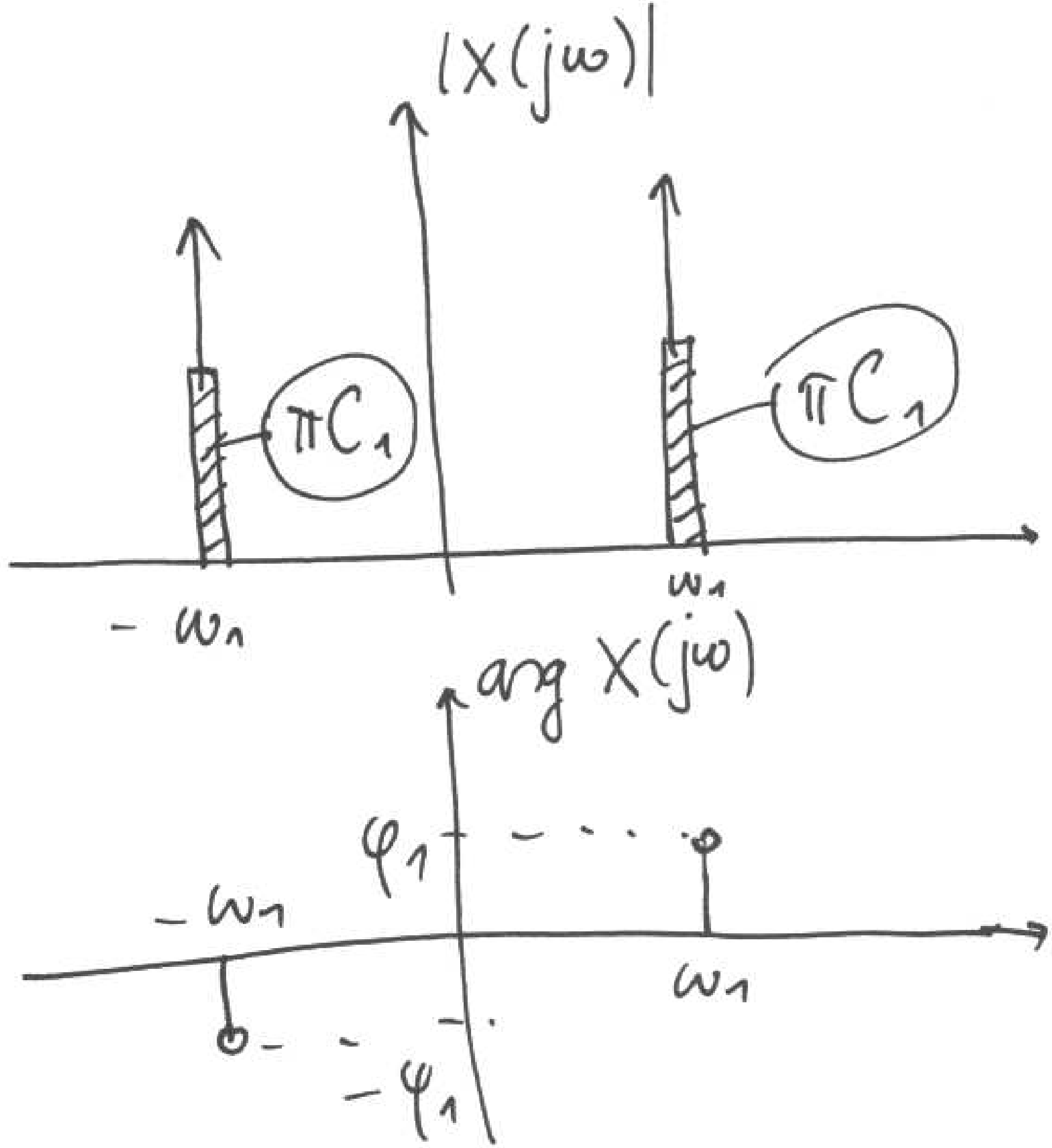
$$x(t) = \int_{-\infty}^{+\infty} \delta(r) e^{j(r+\omega_0)t} dr = e^{j\omega_0 t} \int_{-\infty}^{+\infty} \delta(r) e^{jrt} dr = e^{j\omega_0 t}$$

It is a complex exponential rotating on frequency ω_0. FT of a periodic signal defined by means of FS is thus:

$$X(j\omega) = \sum_{k=-\infty}^{+\infty} 2\pi c_k \delta(\omega - k\omega_1).$$

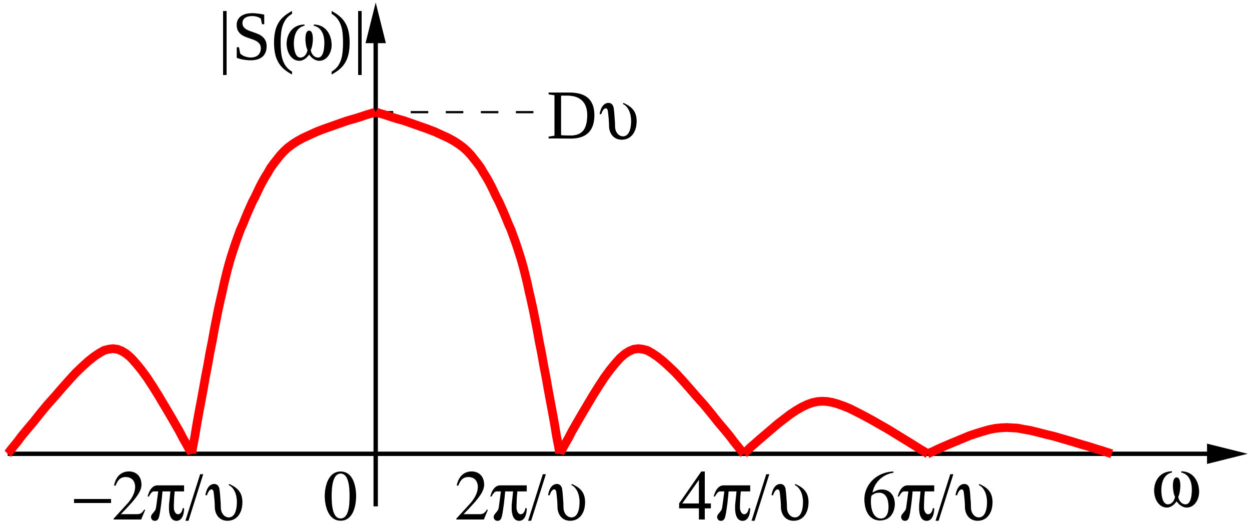
Example of a harmonic signal $x(t) = C_1 \cos(\omega_1 t + \phi_1) = c_{-1} e^{-j\omega_1 t} + c_1 e^{j\omega_1 t}$

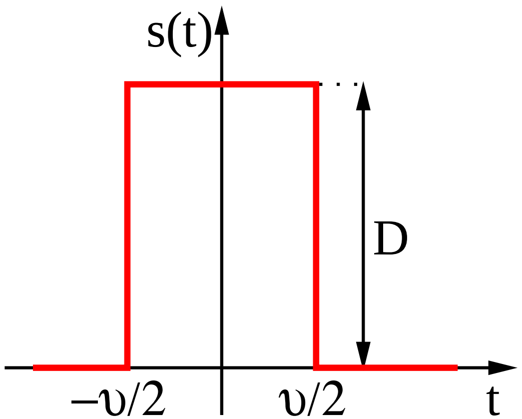


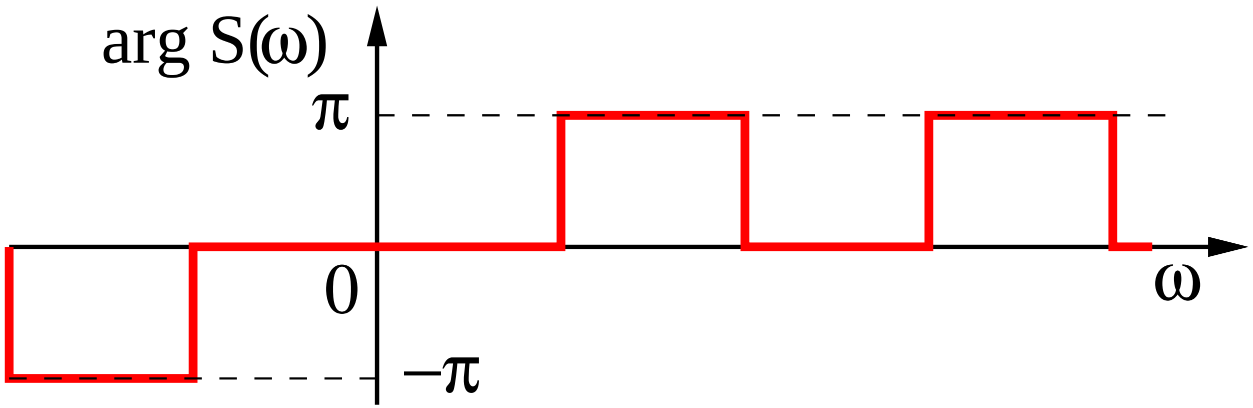


Square impuls We know: $\int_{-b}^{b} e^{\pm jxy} dy = 2b \text{ sinc}(bx)$. Let's define $b = \frac{\vartheta}{2}, \quad y = t, \quad x = \omega$, we obtain:

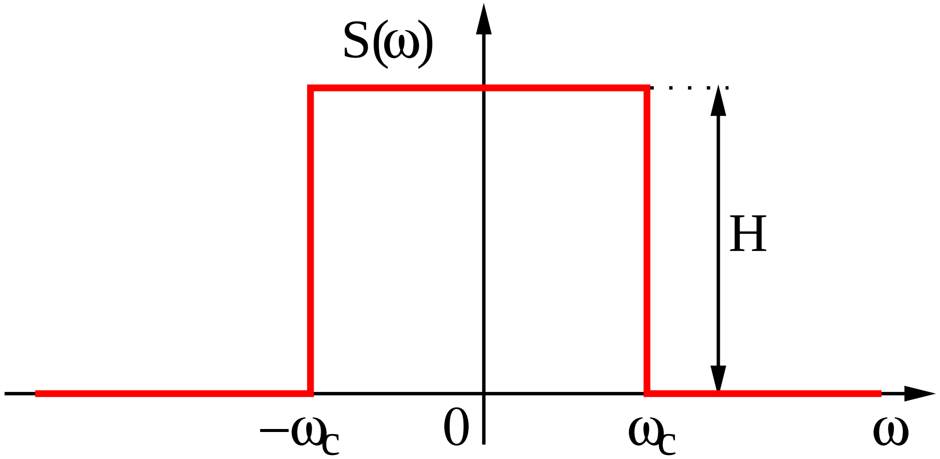
$$X(j\omega) = D \int_{-\frac{\vartheta}{2}}^{+\frac{\vartheta}{2}} e^{-j\omega t} dt = D2\frac{\vartheta}{2} \text{sinc}\left(\frac{\vartheta}{2}\omega\right) = D\vartheta \text{sinc}\left(\frac{\vartheta}{2}\omega\right)$$

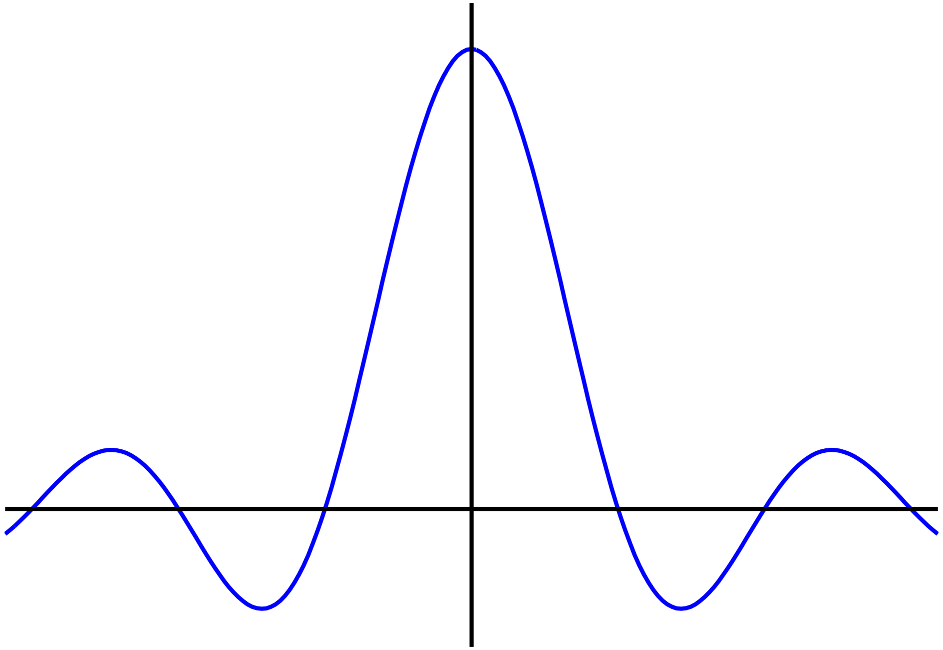






Inverse projection of a square spectral function





$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{+j\omega t}d\omega = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} He^{+j\omega t}d\omega = \frac{H}{2\pi}\int_{-\omega_c}^{\omega_c} e^{+j\omega t}d\omega =$$

$$= \frac{H}{2\pi}2\omega_c \mathsf{sinc}(\omega_c t) = \frac{H\omega_c}{\pi}\mathsf{sinc}(\omega_c t)$$

What is the maximum of the function? Points where the function crosses time axis?

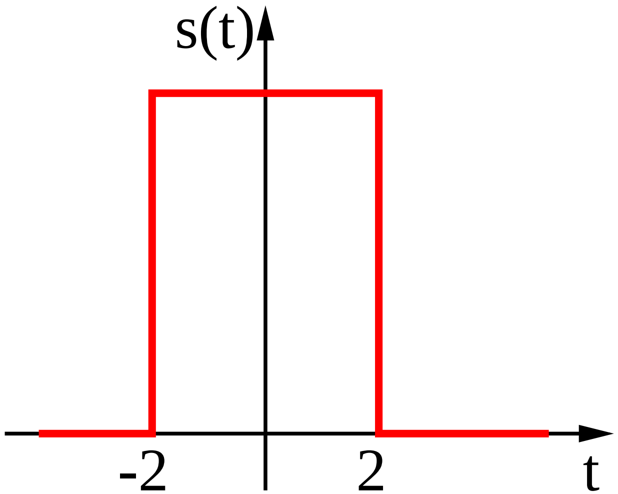
Hints to the spetcra of non periodic signals

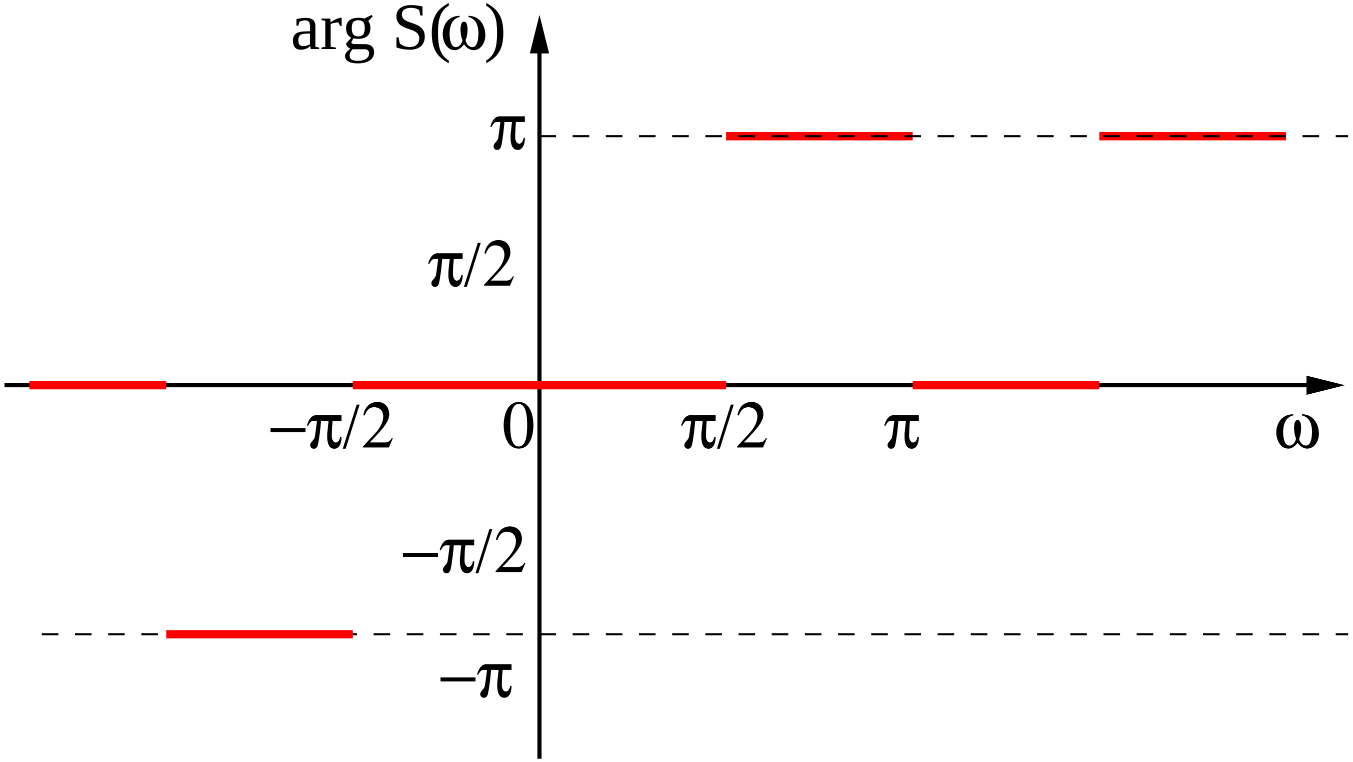
	$x(t)$	$X(j\omega)$
linearity	$ax_a(t) + bx_b(t)$	$aX_a(j\omega) + bX_b(j\omega)$
shift in time	$x(t-\tau)$	$X(j\omega)e^{-j\omega\tau}$
change of scale	$s(mt) \quad m > 0$	$\frac{1}{m}X\left(\frac{\omega}{m}\right)$
convolution	$x_1(t) \star x_2(t) = \int_{-\infty}^{\infty} x_1(\tau)x_2(t-\tau)d\tau$	$X_1(j\omega)X_2(j\omega)$

Time-shift

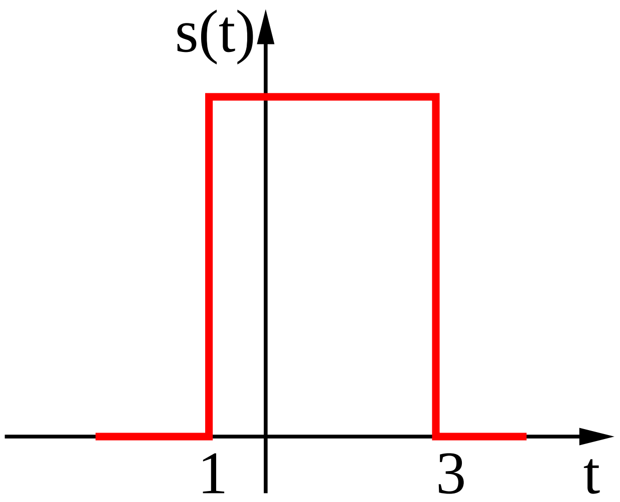
Same as for FS, only argument/phase of spectral function will be changed, by: $-\omega\tau$.

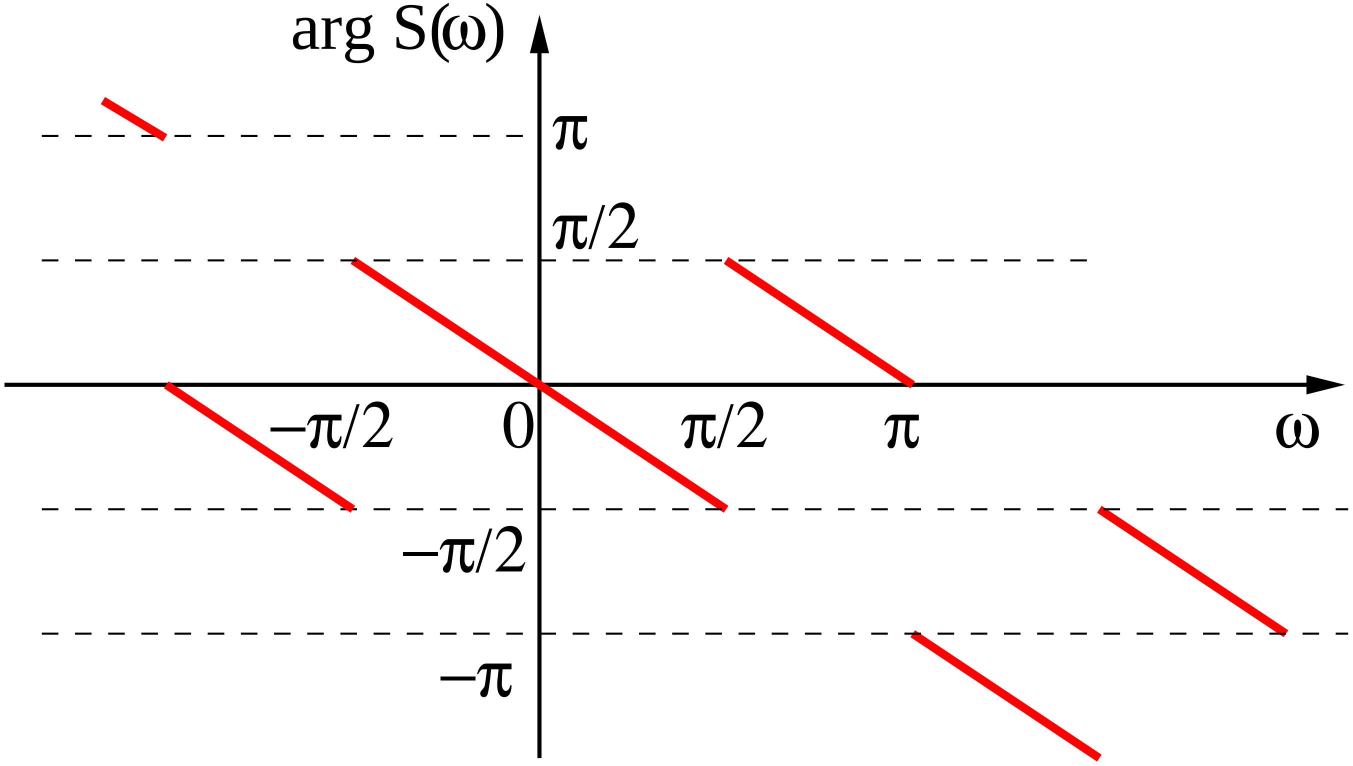
Example:



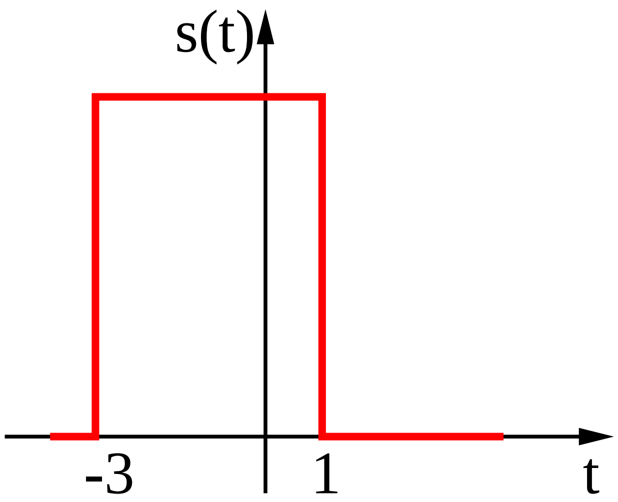


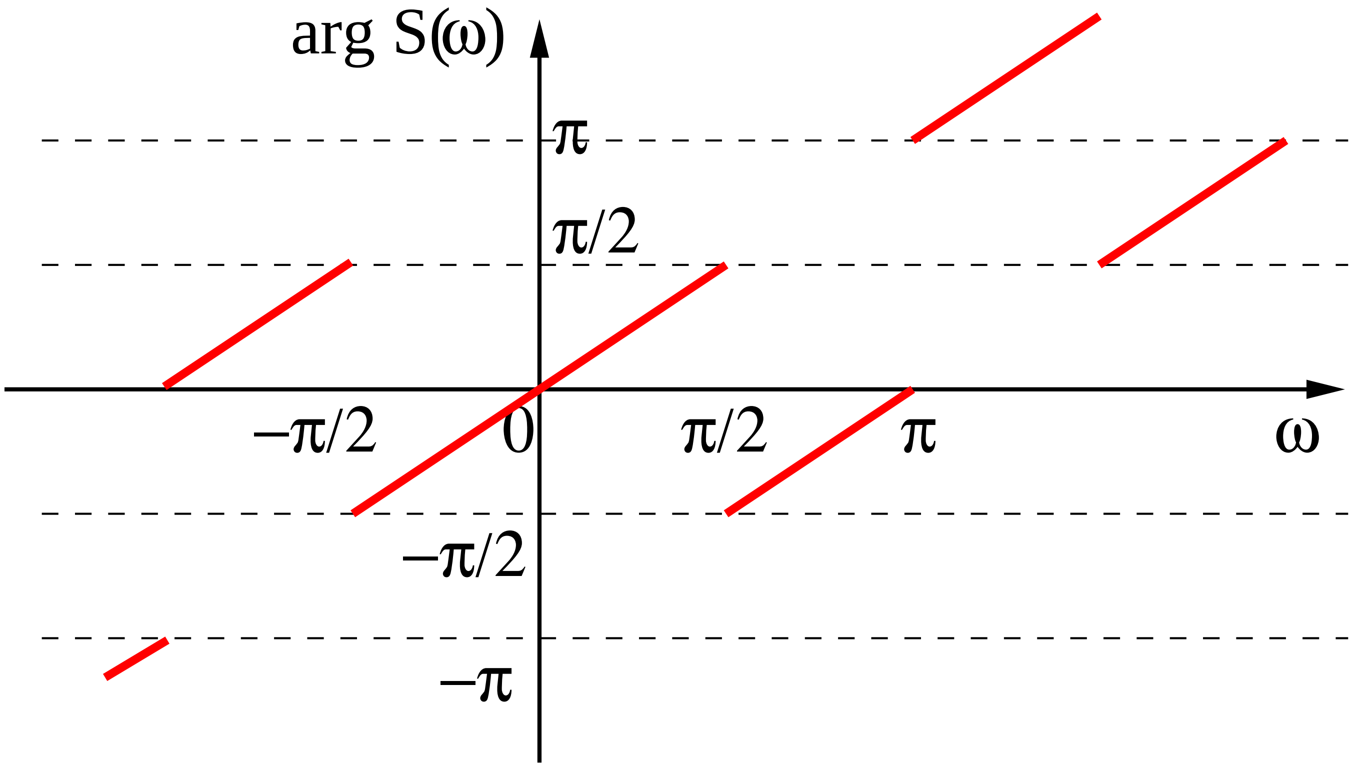
1 s delay: $X(j\omega)$ multiplied by function $e^{-j\omega}$, thus ω will be subtracted from the original argument:



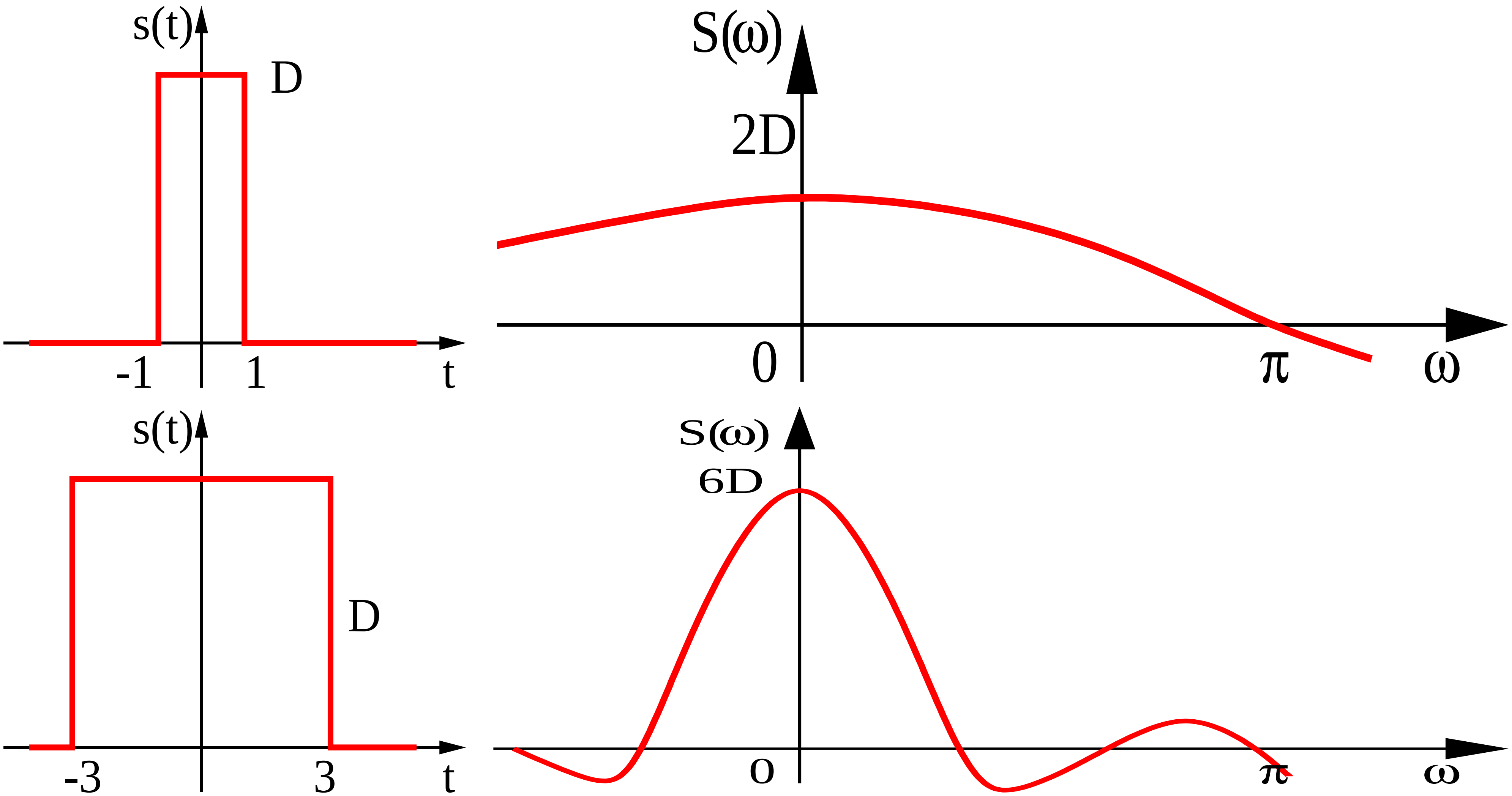


1 s advance: $X(j\omega)$ multiplied by function $e^{j\omega}$, thus ω will be added to the original argument:





Examples of time axis scale

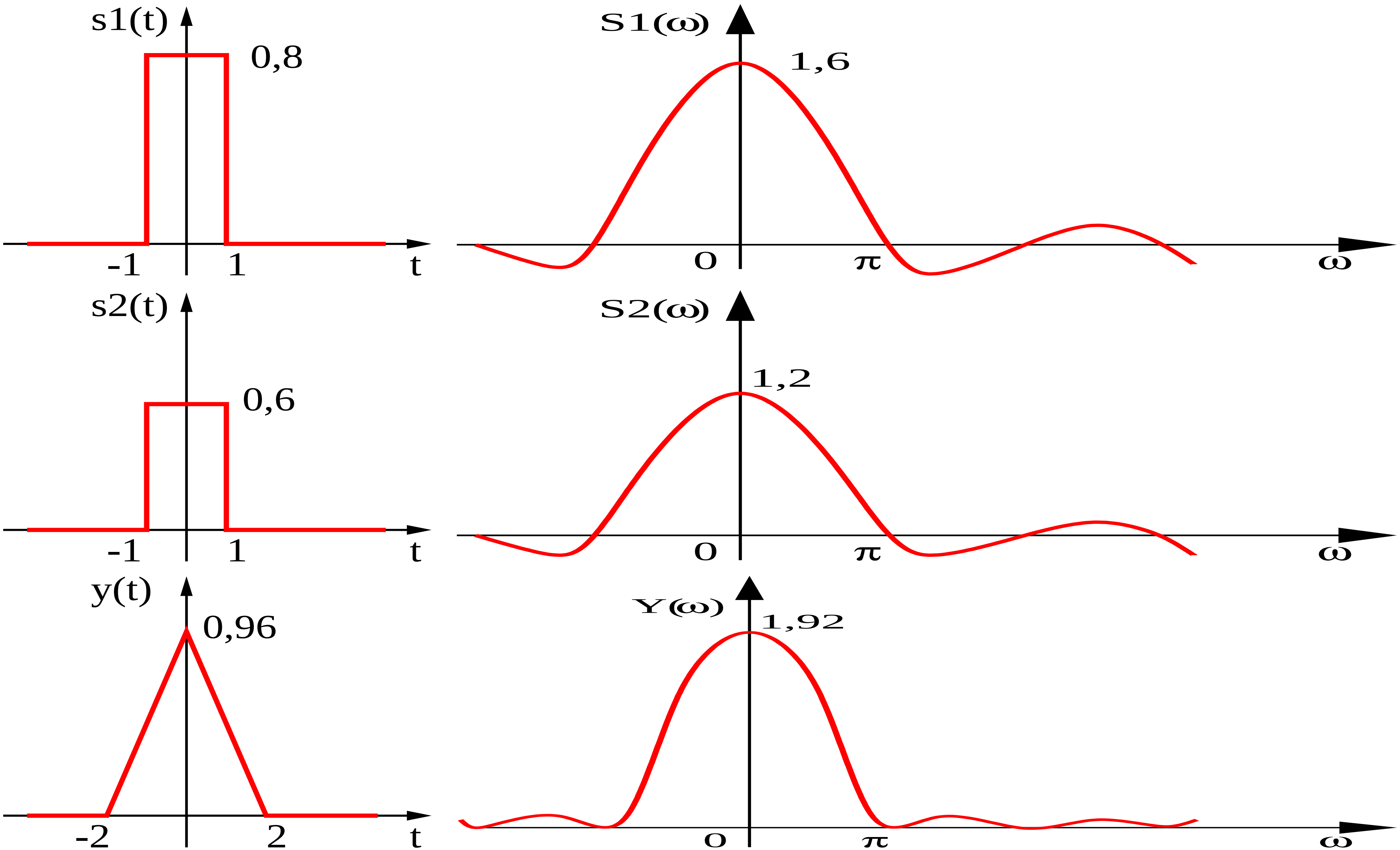


Spectrum of convolution

$$\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} x_1(\tau)x_2(t-\tau)d\tau\right]e^{-j\omega t}dt = \int_{-\infty}^{\infty} x_1(\tau)\left[\int_{-\infty}^{\infty} x_2(t-\tau)e^{-j\omega t}dt\right]d\tau =$$

$$= \int_{-\infty}^{\infty} x_1(\tau)\left[X_2(j\omega)e^{-j\omega\tau}\right]d\tau = X_2(j\omega)\int_{-\infty}^{\infty} x_1(\tau)e^{-j\omega\tau}d\tau = X_1(\omega)X_2(\omega)$$

Example about spectrum of convolution



Parseval theorem - absolute energy of signal using spectral function

$$\int_{-\infty}^{\infty} s^2(t)dt = \int_{-\infty}^{\infty} x(t)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega\right]dt =$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\underbrace{\left[\int_{-\infty}^{\infty} x(t)e^{j\omega t}dt\right]}_{X(-j\omega)}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)X(-j\omega)d\omega = \int_{-\infty}^{\infty} L_d(\omega)d\omega$$

$L_d(\omega)$ is called **(double sided) spectral density function of energy**

$$L_d(\omega) = \frac{|X(j\omega)|^2}{2\pi}$$