Fundamentals about complex numbers

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Real numbers, – real axis ($\Re$) . Complex numbers – two components: real and imaginairy ($\Im$) – denoted either as i or j

Special properties of complex unit: $j = \sqrt{-1}, \quad jj = -1, \quad jjj = -j, \quad jjjj = +1.$

Composite shape of complex number:

$$z = a + jb$$

Complex value as a vector

- **magnitude** r of complex number – length of vector
- **phase** ϕ of complex number – angle

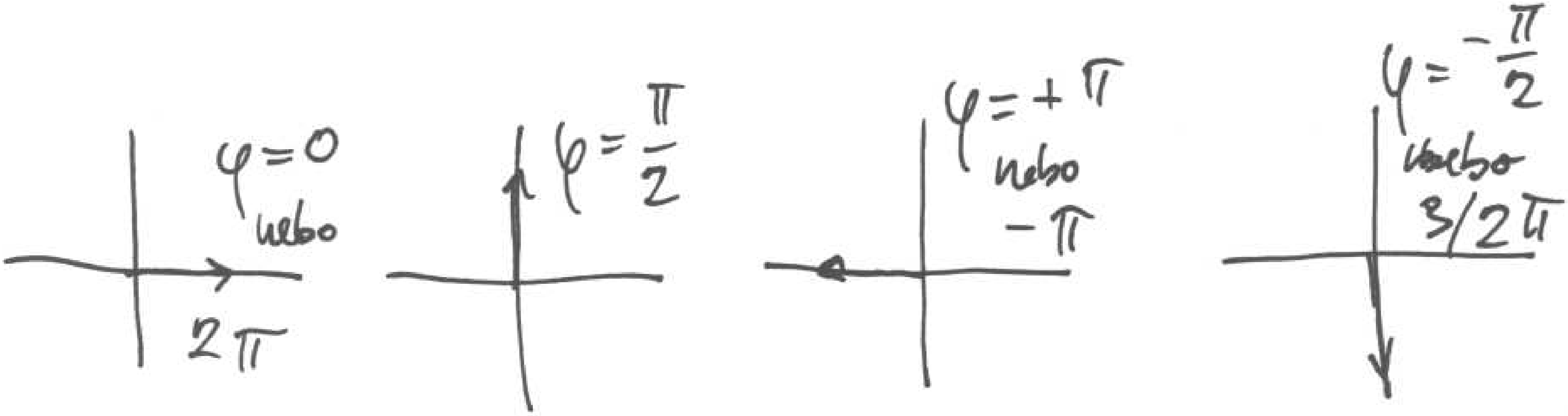
Relationship between a, b and r, ϕ?

$$a = r\cos\phi \qquad b = r\sin\phi$$

$$z = r\cos\phi + jr\sin\phi.$$

$$r = \sqrt{a^2 + b^2} \qquad \phi = \tan^{-1}\frac{b}{a}.$$

Be careful with the phase !!!!



Example 1

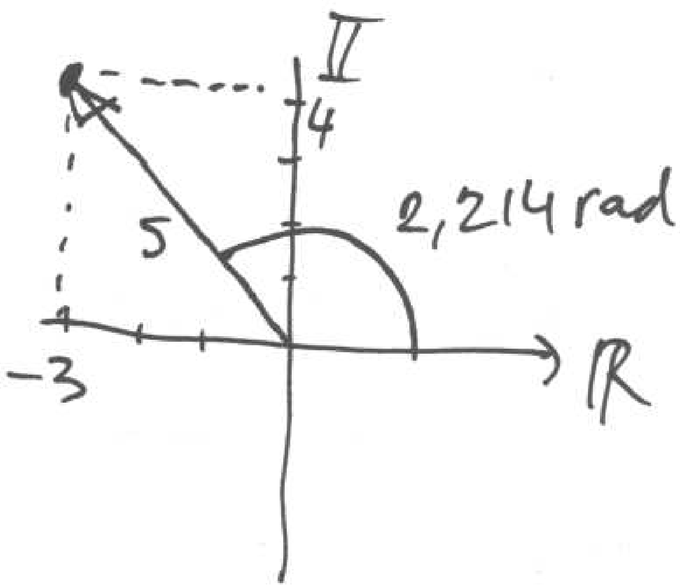
$$z = -3 + j4, \qquad r = \sqrt{3^2 + 4^2} = 5, \qquad \phi = \tan^{-1}\frac{4}{-3} = -0.92$$

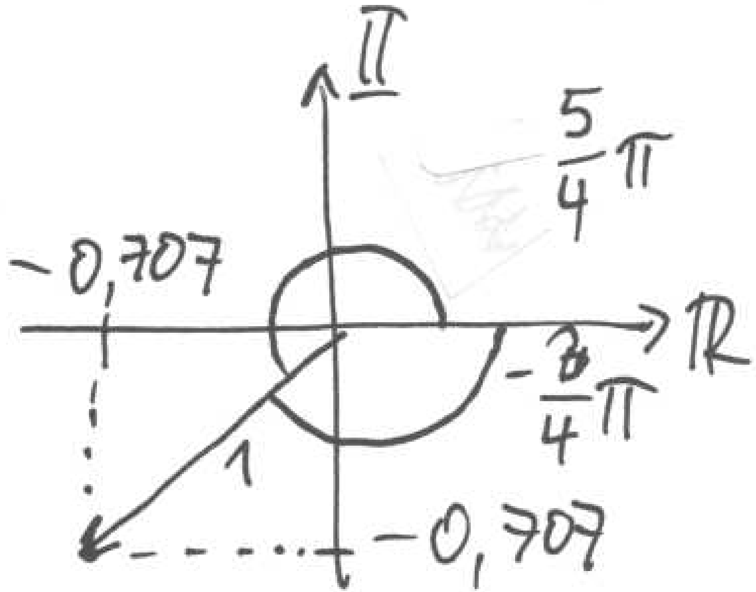
which is **wrong!**. Correct angle is $\phi = \pi - 0.92 = 2.214$ rad.

Example 2

$$z = -\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}, \quad r = \sqrt{\frac{1}{2} + \frac{1}{2}}, \quad \phi = \tan^{-1} 1 = \frac{\pi}{4}$$

which is **wrong!**. Correct angle is $\phi = \frac{5}{4}\pi$ or $\phi = -\frac{3}{4}\pi$.





Exponencial form of a complex number:

$$z = re^{j\phi} \quad \text{nebo} \quad z = r \exp j\phi$$

Operations with complex numbers:

- complex conjugate: $z^\star = a - jb = re^{-j\phi}$.
- addition/subtraction: $z_1 + z_2 = a_1 + a_2 + j(b_1 + b_2)$.
- multiplication/division: $z_1 z_2 = r_1 r_2 e^{j(\phi_1+\phi_2)}$, $\quad \frac{z_1}{z_2} = \frac{r_1}{r_2} e^{j(\phi_1-\phi_2)}$.

The most interesting complex numbers $e^{j\phi}$ lie **on the unit circle**: $r = 1$.

$$\boxed{e^{j\phi} = \cos\phi + j\sin\phi}$$

$$e^{j\phi} + e^{-j\phi} = \cos\phi + j\sin\phi + \cos\phi - j\sin\phi = 2\cos\phi$$

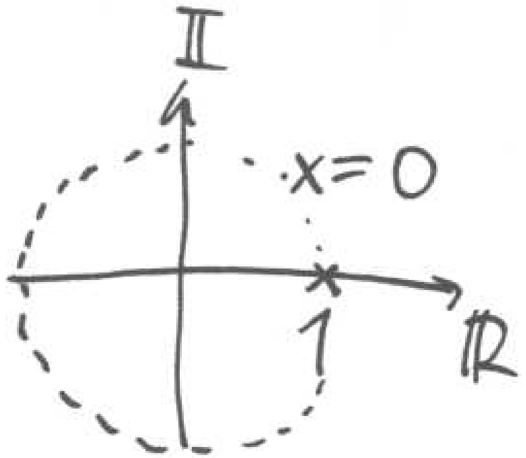
$$\boxed{\cos\phi = \frac{e^{j\phi} + e^{-j\phi}}{2},}$$

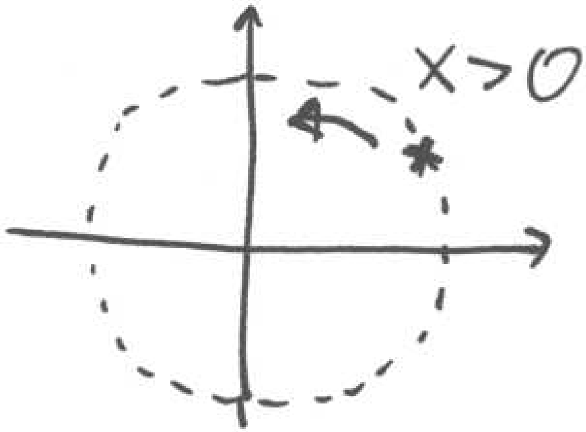
$$e^{j\phi} - e^{-j\phi} = \cos\phi + j\sin\phi - \cos\phi - (-j\sin\phi) = 2j\sin\phi$$

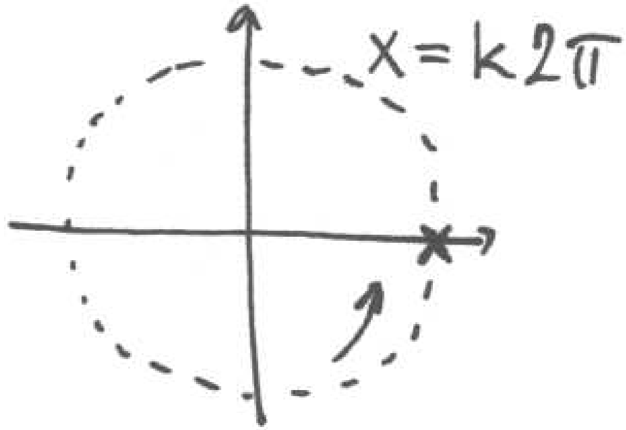
$$\sin\phi = \frac{e^{j\phi} - e^{-j\phi}}{2j}$$

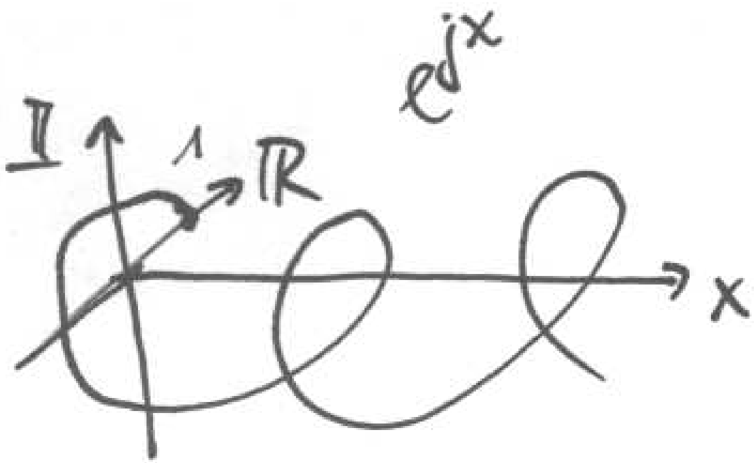
Function e^{jx}

when x is independent:

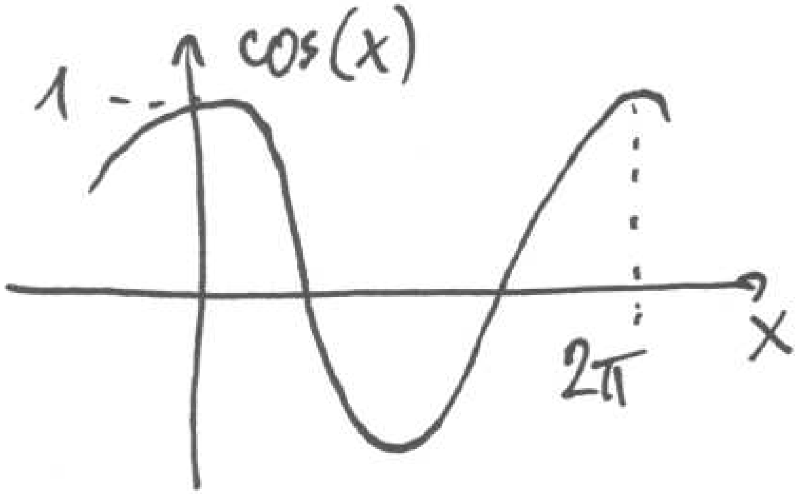


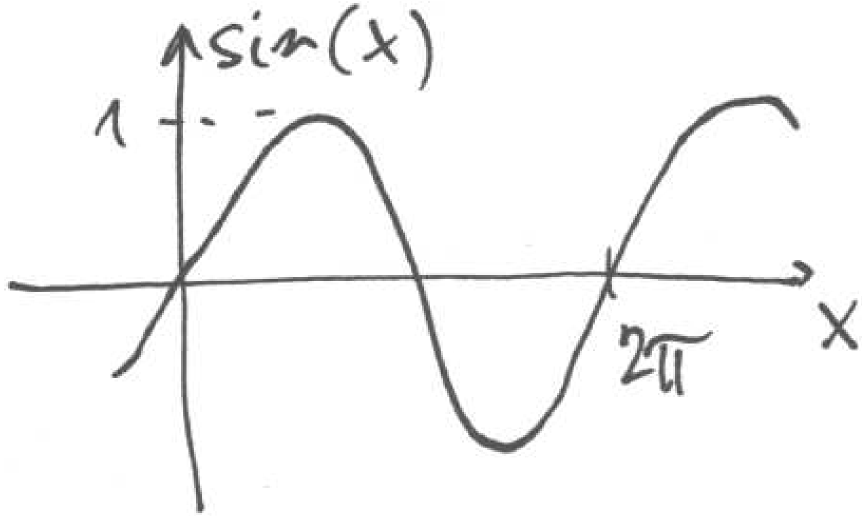






$$\Re(e^{jx}) = \cos x \qquad \Im(e^{jx}) = \sin x$$

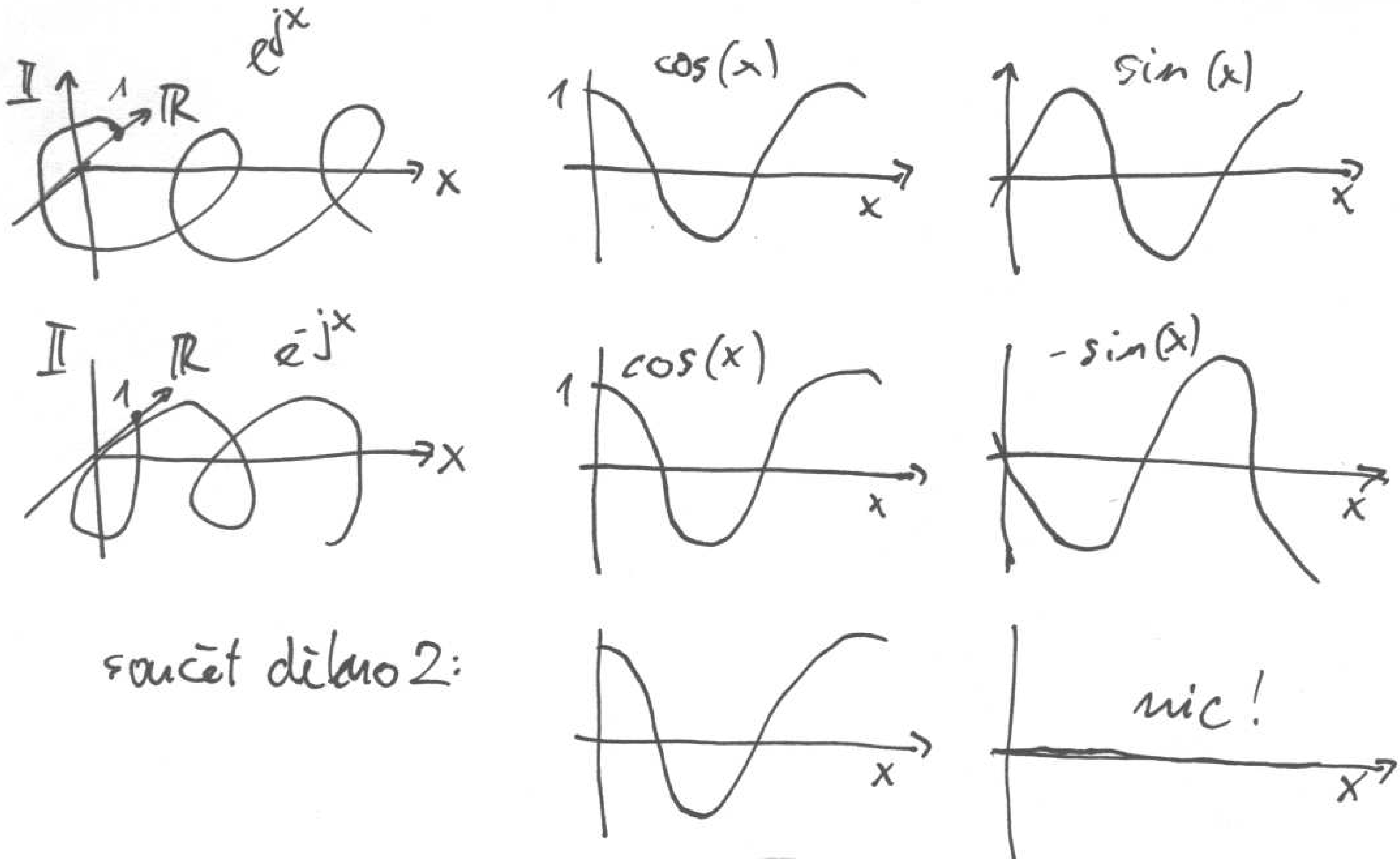




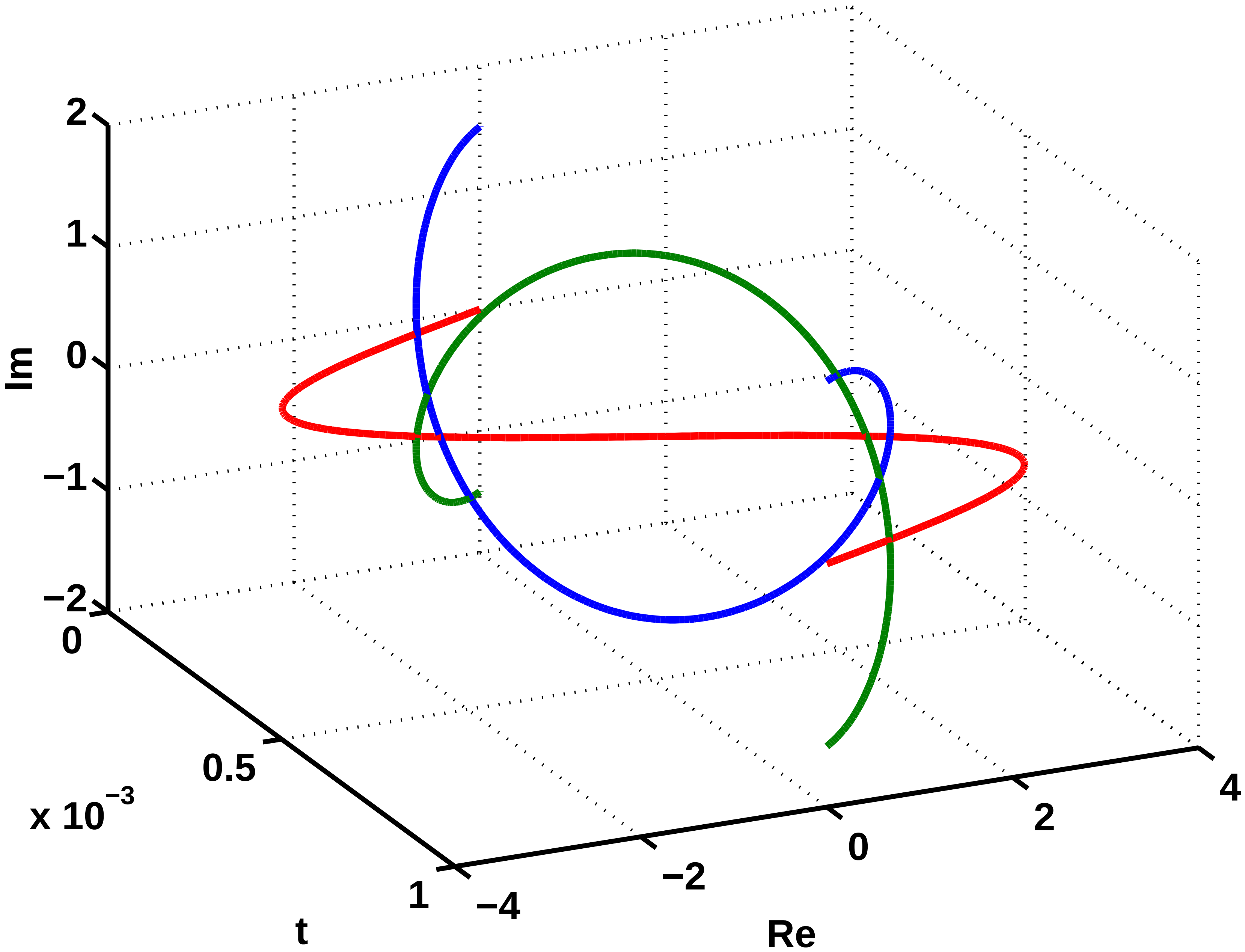
projection of point e^{jx} into $\Re$ and $\Im$ axis!

Cos function

$$\cos x = \frac{e^{jx} + e^{-jx}}{2}$$



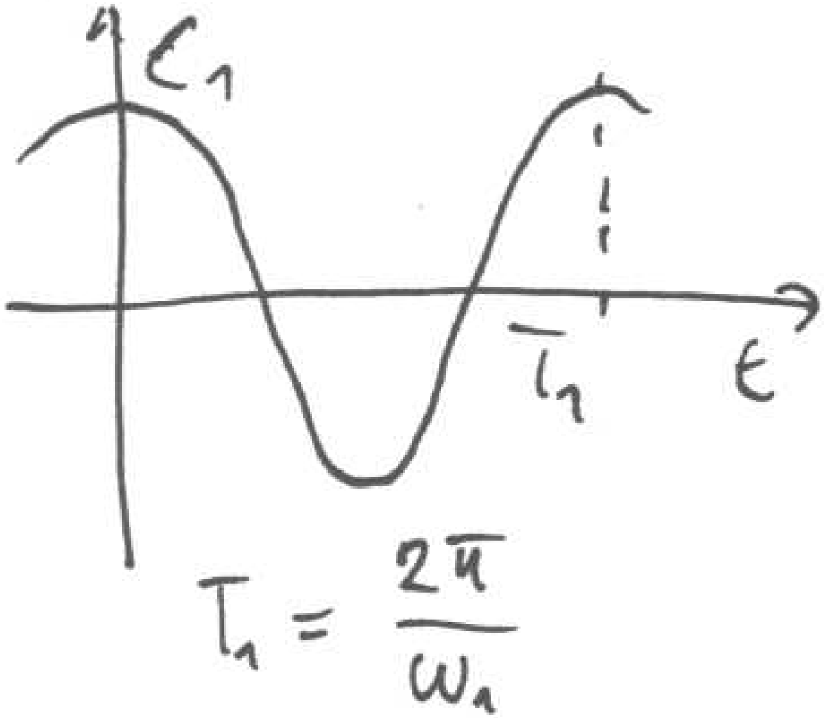
e^{jx}
I
R
1
x
cos(x)
sin(x)
e^{-jx}
cos(x)
-sin(x)
součet děleno 2:
nic!

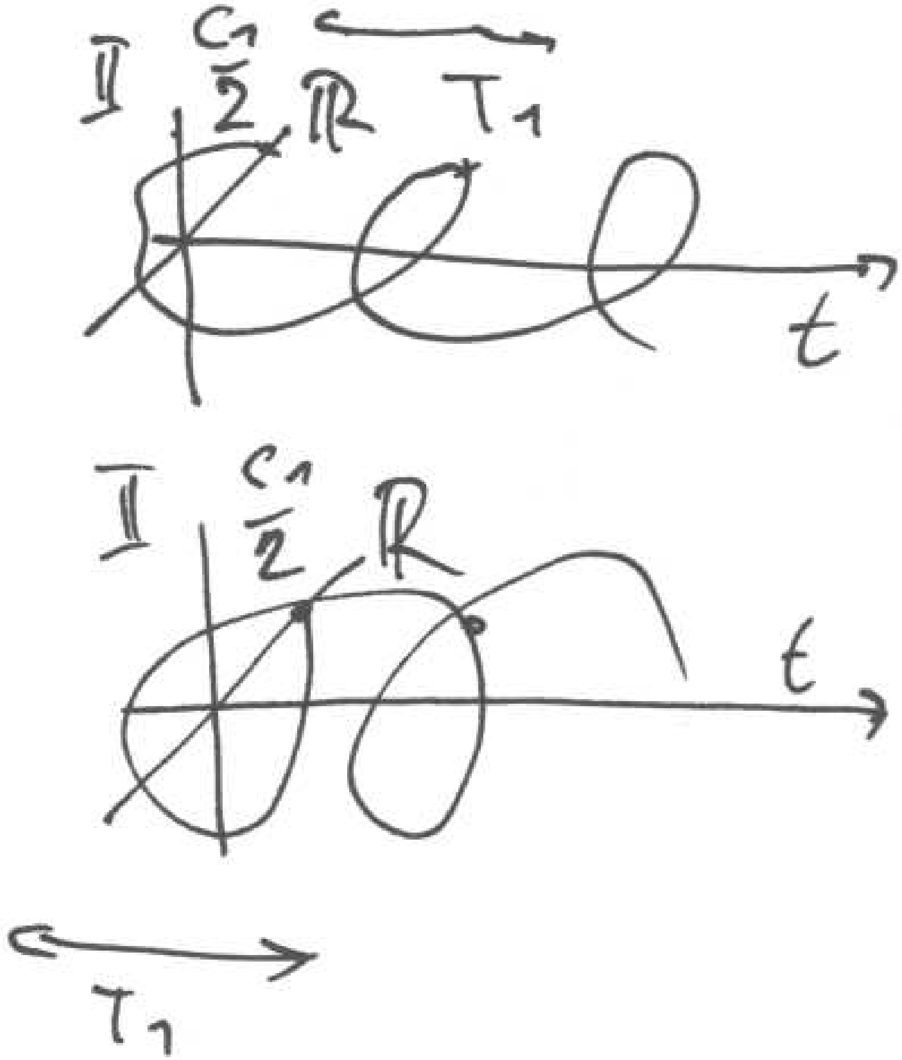


2
1
0
−1
−2
Im
0
0.5
1
x 10−3
t
−4
−2
0
2
4
Re

General cos function without initial phase

$$C_1 \cos(\omega_1 t) = \frac{C_1}{2} e^{j\omega_1 t} + \frac{C_1}{2} e^{-j\omega_1 t}.$$





General cos function with initial phase

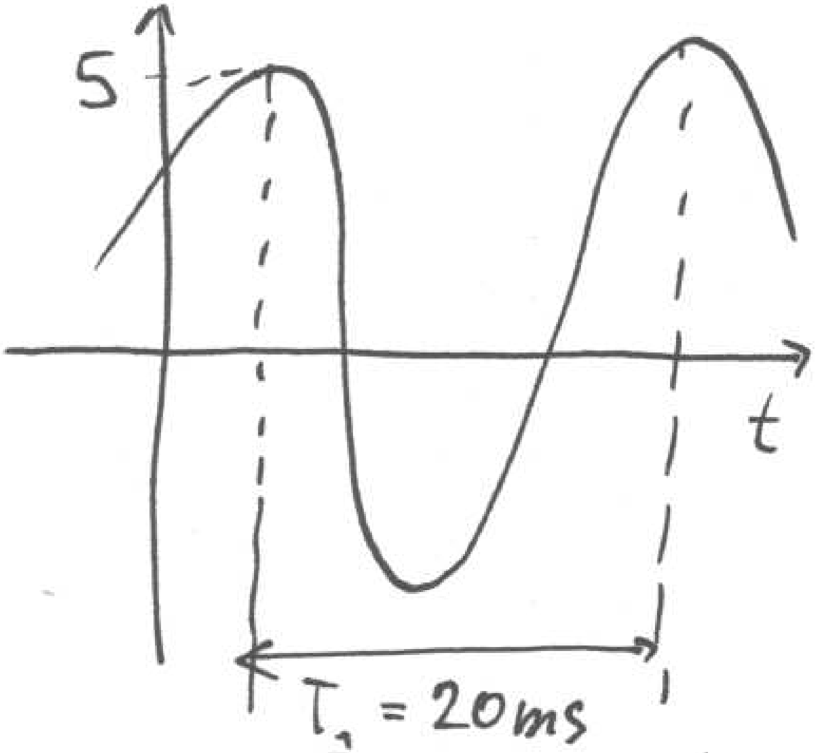
$$C_1\cos(\omega_1 t+\phi_1)=\frac{C_1}{2}e^{j(\omega_1 t+\phi_1)}+\frac{C_1}{2}e^{-j(\omega_1 t+\phi_1)}=\frac{C_1}{2}e^{j\phi_1}e^{j\omega_1 t}+\frac{C_1}{2}e^{-j\phi_1}e^{-j\omega_1 t}$$

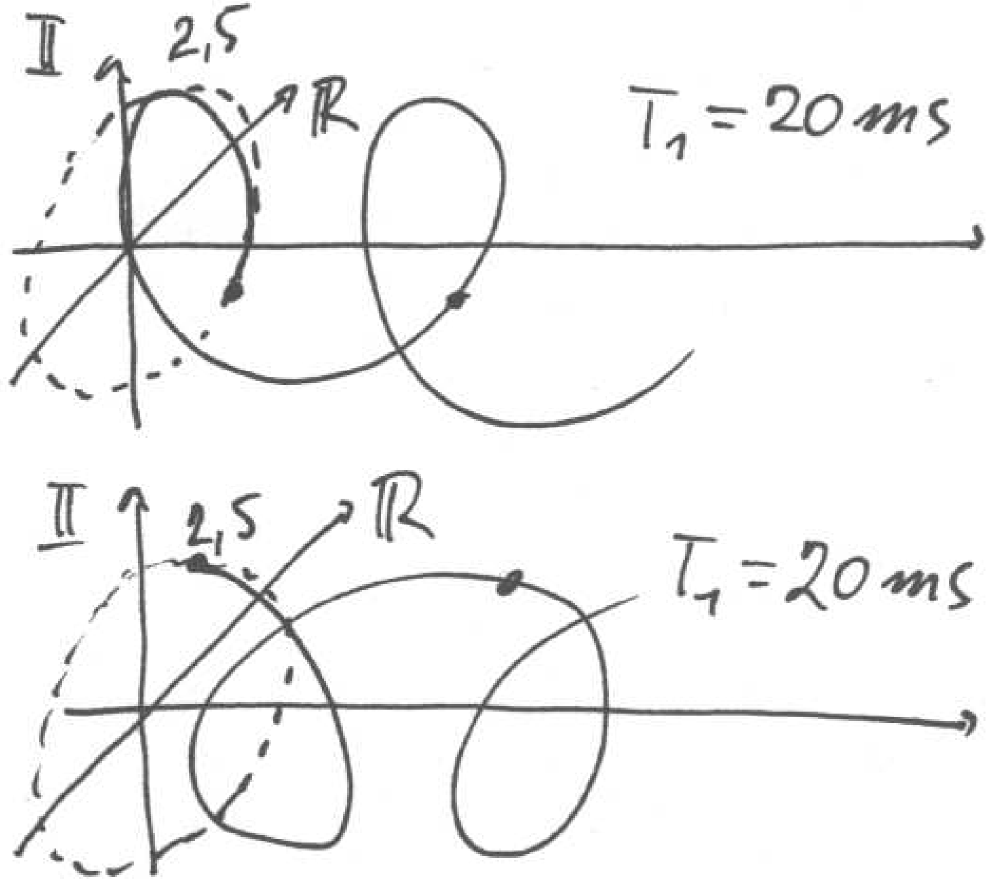
$c_1=\frac{C_1}{2}e^{j\phi_1}$ and $c_{-1}=\frac{C_1}{2}e^{-j\phi_1}$ are complex constants varying in time. $\frac{C_1}{2}e^{j(\omega_1 t+\phi_1)}$ and $\frac{C_1}{2}e^{-j(\omega_1 t+\phi_1)}$ for $t=0$ and for $t=kT_1$....

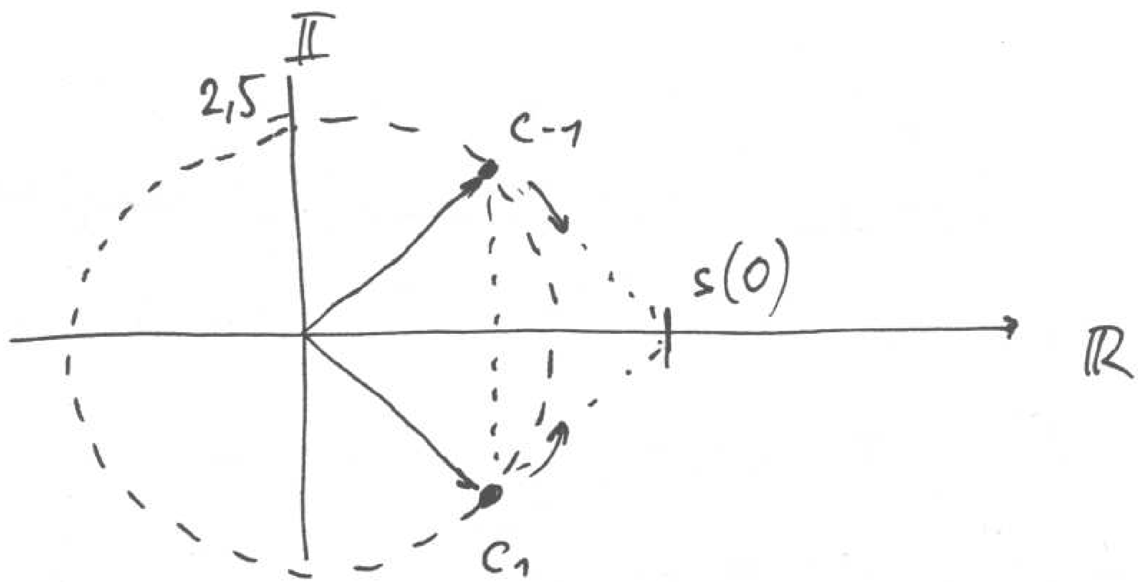
Example:

$$x(t)=5\cos(100\pi t-\frac{\pi}{4})=2.5e^{-j\frac{\pi}{4}}e^{j100\pi t}+2.5e^{+j\frac{\pi}{4}}e^{-j100\pi t}$$

Coefficients: $c_1=2.5e^{-j\frac{\pi}{4}}$, $c_{-1}=2.5e^{+j\frac{\pi}{4}}$.


5
t
T₁ = 20ms


I
2,5
R
T₁ = 20ms
I
2,5
R
T₁ = 20ms



I
2,5
c-1
s(0)
R
c1