ISS – Summary

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SIGNALS – FUNDAMENTALS

- continuous time $x(t)$
- discrete time $x[n]$
- time axis modification – shift (delay, advance), $x(t-\tau)$, $x(t+\tau)$, $x[n-m]$, $x[n+m]$
- reflection with shift – careful with the opposite sense of the sign: in $x[-n+m]$ a positive m correspond to the delay of the reflected signal.
- contraction and delatation of the time axis in continuous time domain: $x(mt)$, $x(\frac{t}{m})$.

Energy and power

instantenous power: $p(t) = |x(t)|^2, \quad p[n] = |x[n]|^2$

overall energy (finite/infinite)

$$E_\infty = \lim_{T\to\infty} \int_{-T}^{T} |x(t)|^2 dt = \int_{-\infty}^{+\infty} |x(t)|^2 dt \qquad E_\infty = \lim_{N\to\infty} \sum_{-N}^{N} |x[n]|^2 = \sum_{-\infty}^{\infty} |x[n]|^2$$

everall mean power (finite/infinite)

$$P_\infty = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt \qquad P_\infty = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{-N}^{N} |x[n]|^2$$

$\Rightarrow$ for a periodic signal power is computed over one period.

Periodic signals

repeat after T or N, fundamental period T_1 or N_1, frequency and angular frequency (regular for a continous time signal, normalized for a discrete signal):

$$f_1 = \frac{1}{T_1} \quad \omega_1 = \frac{2\pi}{T_1} \qquad f_1' = \frac{1}{N_1} \quad \omega_1' = \frac{2\pi}{N_1}$$

. . . we don't use apostrophe.

Harmonic signals

are defined by a magnitude, frequency and initial phase:

continuous time:

$$x(t) = C_1 \cos(\omega_1 t + \phi_1) = \frac{C_1}{2} e^{j\phi_1} e^{j\omega_1 t} + \frac{C_1}{2} e^{-j\phi_1} e^{-j\omega_1 t}$$

period $T_1 = \frac{1}{f_1} = \frac{2\pi}{\omega_1}$

discrete time:

$$x[n] = C_1 \cos(\omega_1 n + \phi_1) = \frac{C_1}{2} e^{j\phi_1} e^{j\omega_1 n} + \frac{C_1}{2} e^{-j\phi_1} e^{-j\omega_1 n}$$

period N_1 is computed differently – N_1 has to be an integer number, in some cases we don't succeed.

Important signals

- unit step $\sigma(t),\ \sigma[n]$
- unit impulse $\delta(t),\ \delta[n]$ (in continuous time has peculiar properties, in discrete time is a standard signal)

SIGNALS IN CONTINUOUS TIME DOMAIN– FREQUENCY ANALYSIS

Periodic – Fourier series

signal is periodic $\Rightarrow$ sctrum is discrete (coefficients)

$$c_k = \frac{1}{T_1}\int_{T_1} x(t)e^{-jk\omega_1 t}dt \qquad x(t) = \sum_{k=-\infty}^{+\infty} c_k e^{jk\omega_1 t}$$

c_k are coefficients of Fourier series. Properties:

- $c_k = c_{-k}^{\star}$
- c_0 is a mean value
- are tied with frequencies $k\omega_1$
- pairs of coefficients c_k and c_{-k} with the corresponding exponentials compose a cosine with frequency $k\omega_1$.
- signal shift: $x(t) \rightarrow x(t-\tau)$, $c_k \rightarrow c_k e^{-jk\omega_1\tau}$ – influences only on **arguments** of coefficients

Examples:

- cosine: $x(t) = C_1 \cos(\omega_1 t + \phi_1)$ has only two coefficients: $c_1 = \frac{C_1}{2} e^{j\phi_1}$, $c_{-1} = \frac{C_1}{2} e^{-j\phi_1}$
- periodic series of square signals: using and auxuliary function sinc:

$$\int_{-b}^{b} e^{\pm jxy} dy = 2b \,\mathsf{sinc}(bx)$$

result: $c_k = D\frac{\vartheta}{T_1}\mathsf{sinc}\left(\frac{\vartheta}{2}k\omega_1\right)$. first draw dashed the precomputed auxuliary function, then mark the coefficients on it.

All spectra are composed of separatelly plotted modul and argument parts, argument of a positive real number os 0, argument of a negative real number is either π or $-\pi$.

Non-periodic – Fourier transform
signal is non-periodic $\Rightarrow$spectrum is defined by a function

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt \qquad x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{+j\omega t} d\omega$$

$X(j\omega)$ is a spectral function, fundamental properties:

- $X(j\omega) = X^{\star}(-j\omega)$.
- shift of a signal: $x(t) \rightarrow x(t-\tau)$, $X(j\omega) \rightarrow X(j\omega)e^{-j\omega\tau}$ – effect only on the **argument** of a spectral function.

Examples:

- shifted Dirac impulse: $x(t) = \delta(t-\tau), \quad X(j\omega) = e^{-j\omega\tau}$.
- square impuls: $X(j\omega) = D\vartheta \mathrm{sinc}\left(\frac{\vartheta}{2}\omega\right)$, function sinc (decomposition to modul and argument) is the **result**.

SAMPLING

- in time domain, signal is multiplied by a sampling signal: sampling period T_s, sampling frequency $F_s = \frac{1}{T_s}$, angular sampling frequency $\Omega_s = \frac{2\pi}{T_s}$: $x_s(t) = x(t)s(t)$
- in theory, a periodic series of Dirac impulses serves as a sampling signal, the resulting spectral function becomes

$$X_s(j\omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X(\omega - k\omega_1).$$

  if the copies of the spectrum of the original signal overlap $\Rightarrow$ aliasing.
- for signals with bounded spectrum with frequency ω_{max}, sampling theorem (prevent aliasing): $\Omega_s > 2\omega_{max}$ or $F_s > 2f_{max}$
- when ST satisfied, signal $x(t)$ can be idealy reconstructed.
- normalized time: $n = \frac{nT}{T}$ (counter of samples), normalized regular frequency $f' fracfF_s$, normalized angular frequency $\omega' = \frac{\omega}{F_s}$, normalized always by **sampling frequency**.

DISCRETE SIGNAL

basic operations

- sequency of length N – obtained by multiplying with a window

$$R_N[n] = \begin{cases} 1 & \text{for } \; n \in [0, N-1] \\ 0 & \text{otherwise} \end{cases}$$

- periodization: $\tilde{x}[n] = x[\mod_N n]$
- periodic shift: $x[n] \longrightarrow x[\mod_N (n-m)]$
- circular shift: $x[n] \longrightarrow R_N[n] x[\mod_N (n-m)]$

convolution

- linear: $x[n] \star y[n] = \sum_{k=-\infty}^{\infty} x[k]y[n-k]$ (for a sequence of length N results in a sequence of length $2N-1$),
- periodic: $x[n]\tilde{\star}y[n] = \sum_{k=0}^{N-1} x[k]y[\mathrm{mod}_N(n-k)]$ (has samples along the whole axis).
- circular $x[n] \circledN y[n] = R_N[n] \sum_{k=0}^{N-1} x[k]y[\mathrm{mod}_N(n-k)]$, results in a sequence of the same length as the original signal N.

Spectral analysis of discrete signals

DTFT Discrete Time Fourier Transform

Signal is sampled (discrete), spectrum is periodic

$$\tilde{X}(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \qquad x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \tilde{X}(e^{j\omega})e^{+j\omega n}d\omega$$

Properties

- is periodic as the signal is discrete.
- is a function defined for all ω, as the signal is arbitrary.
- can be displayed on different frequenxy axes.

Discrete Fourier Series of a discrete signal with period N
Signal is sampled (discrete), spectrum is periodic. Signal is periodic thus spectrum is discrete (coefficients, not function)

$$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n] e^{-j\frac{2\pi}{N}kn} \qquad \tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1} \tilde{X}[k] e^{j\frac{2\pi}{N}kn}$$

Properties of DFS coefficients:

- coefficients are tied to normalized angular frequencies : $k\omega_1$, where $\omega_1 = \frac{2\pi}{N}$.
- coefficients possess standard properties: $\tilde{X}[k] = \tilde{X}^{\star}[-k]$, and moreover are **periodic** : $\tilde{X}[k] = \tilde{X}[k+gN]$

Example: DFS of a harmonic signal with period N: $x[n] = C_1 \cos(\frac{2\pi}{N}n + \phi_1)$:

$$|\tilde{X}[1]| = |\tilde{X}[N-1]| = \frac{NC_1}{2} \qquad \arg \tilde{X}[1] = -\arg \tilde{X}[N-1] = \phi$$

Discrete Fourier transform converts a signal sequence of length N to a spectrum sequence of length N.

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn} \qquad x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k] e^{+j\frac{2\pi}{N}kn}, \quad \text{only for } n, k = 0 \ldots N-1$$

Connection of coefficients $X[k]$ with frequency:

- normalized regular frequency $\frac{k}{N}$ to $\frac{N-1}{N}$.
- normalized angular frequency $2\pi\frac{k}{N}$ to $2\pi\frac{N-1}{N}$
- regular frequency $\frac{k}{N}F_s$ to $\frac{N-1}{N}F_s$
- regular angular frequency $\frac{k}{N}2\pi F_s$ to $\frac{N-1}{N}2\pi F_s$

Properties:

- $X[k] = X^{\star}[N-k]$
- circular shift: $x[n] \longrightarrow R_N x[\mathrm{mod}_N(n-m)]$, $X[k] \longrightarrow X[k]e^{-j\frac{2\pi}{N}mk}$
- calculation using FFT.
- convinient for calculating FS and FT with discrete time, however with restrictions !

SYSTEM

- causal, time invariant, linear, stable.
- described by impulse response: in continuous time excited by Dirac impulse $\delta(t)$ (only theory), we obtain $h(t)$. In discrete time excited by unit impulse $\delta[n]$ (also in practise), we obtain $h[n]$.
- response to arbitrary input: convolution with impulse response:

$$y[n] = x[n] \star h[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k] \qquad y(t) = x(t) \star h(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau.$$

(symmetry).

- impulse response of a causal system

$$h[n] = 0 \text{ for } n < 0 \qquad h(t) = 0 \text{ for } t < 0$$

- convolution computation using "paper" method: **discrete : write numbers, multiply and sum up. continuous**: drow functions, multiply and integrate – area computation.

Systems in continuous time domain

- **complex frequency characteristic** is a FT of impulse response:

$$H(j\omega) = \int_{-\infty}^{+\infty} h(t) e^{-j\omega t} dt.$$

  has the same properties as a FT of a continuous non-periodic signal: non-periodic nor discrete. $H(j\omega) = H^\star(-j\omega)$.

- Transfer of a cosine:
  $x(t) = C_1 \cos(\omega_1 t + \phi_1) \quad y(t) = |H(j\omega_1)| C_1 \cos\left[\omega_1 t + \phi_1 + \arg H(j\omega_1)\right].\Rightarrow$ the output cosine has a modified magnitude and phase
- Transfer of a periodic signal: multiply coefficients c_k by $H(jk\omega_1)$
- Transger of a general signal: **spectral image of convolution is multiplication**

$$x(t) \xrightarrow{\mathcal{F}} X(j\omega) \quad Y(j\omega) = H(j\omega)X(j\omega) \quad Y(j\omega) \xrightarrow{\mathcal{F}^{-1}} y(t).$$

System definition by a **difference equation:**

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

Laplac transform:

$$X(s) = \int_{-\infty}^{+\infty} x(t) e^{-st)} dt,$$

we are interested in LT derication: $\dfrac{dx(t)}{dt} \longrightarrow sX(s)$.

Transfer function:

$$H(s) = \frac{Y(s)}{X(s)} = \frac{\displaystyle\sum_{k=0}^{M} b_k s^k}{\displaystyle\sum_{k=0}^{N} a_k s^k},$$

- frequency characteristic: $H(j\omega) = H(s)|_{s=j\omega}$
- nulls and poles of a transfer function (polynom roots in numerator and denominator):

$$H(s) = \frac{b_M}{a_N} \frac{\prod_{k=1}^{M} (s - n_k)}{\prod_{k=1}^{N} (s - p_k)}.$$

We can estimate behaviour of frequency characteristic.

- stability: all polls are found in the left half-plane of the complex plane. $\Re\{p_k\} < 0$.

Systems in discrete time domain

- fundamental blocks: delay, constant multiplication, addition.
- **complex frequency characteristic** is a DTFT of an impulse response:

$$H(e^{j\omega}) = \sum_{k=0}^{\infty} h[k]e^{-j\omega k}$$

has the same properties as a DTFT of a discrete signal: periodic and $H(e^{j\omega}) = H^\star(e^{-j\omega})$

Transfer of a cosine: $x[n] = C_1 \cos(\omega_1 n + \phi_1)$,
$y[n] = C_1 |H(e^{j\omega_1})| \cos(\omega_1 n + \phi_1 + \arg H(e^{j\omega_1}))$

Systen definition by a difference equation

$$y[n] = \sum_{k=0}^{Q} b_k x[n-k] - \sum_{k=1}^{P} a_k y[n-k],$$

- **FIR** – non-recursive: only $b_0 \dots b_Q$ are defined (non-zero). Impulse response is given by the filter coefficients:

$$h[n] = \begin{cases} 0 & \text{for } \ n < 0 \ \text{ and for } \ n > Q \\ b_n & \text{for } \ 0 \leq n \leq Q \end{cases}$$

- **IIR** – pure recursive: only $b_0, a_1 \ldots a_P$ are non-zero.
- **IIR** – general recursive: a_i i b_i are non-zero.

z – **transform**

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n},$$

of the most interest is ZT delay : $x[n-k] \longrightarrow z^{-k} X(z)$

Transfer function

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{Q} b_k z^{-k}}{1 + \sum_{k=1}^{P} a_k z^{-k}} = \frac{B(z)}{A(z)},$$

- frequency characteristic: $H(e^{j\omega}) = H(z)|_{z=e^{j\omega}}$

- nulls and poles:

$$H(z) = \frac{B(z)}{A(z)} = b_0 z^{P-Q} \frac{\prod_{k=1}^{Q}(z - n_k)}{\prod_{k=1}^{P}(z - p_k)},$$

  we can estimate behaviour of the frequency characeristic.

- Stability: system is stable when all poles are found *within unit circle* $|p_k| < 1$

RANDOM SIGNALS

- continuous time: system $\{\xi_t\}$ of random values is defined for all $t \in \Re$ and is called a random process $\xi(t)$.
- discrete time: systen $\{\xi_n\}$ of random values is defined for all $n \in N$ and is called a random process $\xi[n]$.

Set of realizations (with size Ω): denote $\xi_\omega(t)$, and $\xi_\omega[n]$. Estimation on a realization set : **corpora estimation**

Functions to descrive random process

- Distribution function

$$F(x,t) = \mathcal{P}\{\xi(t) < x\}, \quad F(x,t) = \mathcal{P}\{\xi[n] < x\},$$

  set estimation. For each x, count how many values less or equal to x and devide by Ω.

- Probability density function

$$p(x,t) = \frac{\delta F(x,t)}{\delta x}, \quad p(x,n) = \frac{\delta F(x,n)}{\delta x}$$

set estimation using a histogram – dont forget to divide by Ω and Δ, to ensure that $\int_{-\infty}^{+\infty} p(x,t)dx = 1$.

- probability

$$\mathcal{P}\{a < \xi(t) < b\} = F(b,t) - F(a,t) = \int_a^b p(x,t)dx$$

moments

- mean value:

$$a(t) = E\{\xi(t)\} = \int_{-\infty}^{+\infty} xp(x,t)dx \quad a[n] = E\{\xi[n]\} = \int_{-\infty}^{+\infty} xp(x,n)dx$$

set estimation is simply an average over all realizations.

- variance, standard deviation:

$$D(t) = E\{[\xi(t) - a(t)]^2\} = \int_{-\infty}^{+\infty} [x - a(t)]^2 p(x,t)dx$$

$$D[n] = E\{[\xi[n] - a[n]]^2\} = \int_{-\infty}^{+\infty} [x - a[n]]^2 p(x,n)dx$$

$$\sigma(t) = \sqrt{D(t)} \qquad \sigma[n] = \sqrt{D[n]}$$

set estimation:

$$\hat{D}(t) = \frac{1}{\Omega}\sum_{\omega=1}^{\Omega}[\xi_\omega(t) - \hat{a}(t)]^2, \quad \hat{\sigma}(t) = \sqrt{\hat{D}(t)}, \quad \hat{D}[n], \sigma[n] = \ldots$$

- (auto-)correlation function (continous time)

$$R(t_1, t_2) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x_1 x_2 p(x_1, x_2, t_1, t_2) dx_1 dx_2,$$

(auto-)correlation coefficients (discrete time):

$$R(n_1, n_2) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x_1 x_2 p(x_1, x_2, n_1, n_2) dx_1 dx_2,$$

2-dimensional probability disctribution function can be estimated from a set by a 2D histogram (dont forget to divide by Ω and Δ^2).

Stacionarity

same characteristics (values of the functions) for all t, n; moments , autocorrelation functions (continuous) depend only on $\tau = t_2 - t_1$, autocorrelation coefficients (discrete) depend only on $k = n_2 - n_1$.

Ergodic process

function value estimation from one realization $x(t)$ of length T (continuous) or $x[n]$ of length N (discrete).

Time estimation

- $F(x)$, $p(x)$ – histograms estimated over one realization.
- mean value, variance, standard deviation:

$$\hat{a} = \frac{1}{T}\int_0^T x(t)dt \quad \hat{D} = \frac{1}{T}\int_0^T [x(t)-\hat{a}]^2 dt \quad \hat{\sigma} = \sqrt{\hat{D}}$$

$$\hat{a} = \frac{1}{N}\sum_{n=0}^{N-1} x[n] \quad \hat{D} = \frac{1}{N}\sum_{n=0}^{N-1} [x[n]-\hat{a}]^2 \quad \hat{\sigma} = \sqrt{\hat{D}}$$

- autocorrelation function $\hat{R}(\tau) = \frac{1}{T} \int_0^T x(t)x(t+\tau)dt$
- autocorrelation coefficients:
  $\hat{R}(k) = \frac{1}{N} \sum_{n=0}^{N-1} x[n]x[n+k]$, , skewed, but reliable estimation
  $\hat{R}(k) = \frac{1}{N-|k|} \sum_{n=0}^{N-1} x[n]x[n+k]$, no skewed, but nonreliable estimation at margins.

Power Spectral Density – PSD

- continuous time:

$$G(j\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} R(\tau) e^{-j\omega\tau} d\tau \quad R(\tau) = \int_{-\infty}^{+\infty} G(j\omega) e^{+j\omega\tau} d\omega$$

- discrete time:

$$G(e^{j\omega}) = \sum_{k=-\infty}^{\infty} R[k] e^{-j\omega k} \quad R[k] = \frac{1}{2\pi} \int_{-\pi}^{\pi} G(e^{j\omega}) e^{+j\omega k} d\omega$$

- both G share properties with FT and DTFT (e.g. periodicity in $G(e^{j\omega})$, etc.)

- estimation of PSD using DFT: for $\omega_k = \frac{2\pi}{N} k$:

$$\hat{G}(e^{j\omega_k}) = \frac{1}{N} |X[k]|^2.$$

in some cases not reliable, averaging over several segments (Welch method).

Transfer of a random signal through a linear system

$$G_y(j\omega) = |H(j\omega)|^2 G_x(j\omega)$$

$$G_y(e^{j\omega}) = |H(e^{j\omega})|^2 G_x(e^{j\omega})$$

QUANTIATION

raunding to given quantization levels. $L = 2^b$ levels from x_{min} to x_{max}, quantization step

$$\Delta = \frac{x_{max} - x_{min}}{L-1} \approx \frac{x_{max} - x_{min}}{L}.$$

for every $x[n]$ choose the closest level: $x[n] \rightarrow x_q[n]$. Quantization error: $e[n] = x[n] - x_q[n]$.

Influence of the error to signal quality: signal to noise ratio:

$$SNR = 10 \log_{10} \frac{P_s}{P_e} \quad \text{[dB]}.$$

For a cosine with magnitude A and correct setting of x_{min} and x_{max}:

$$SNR = 10 \log_{10} \frac{3}{2}(2^b)^2 = 10 \log_{10} \frac{3}{2} + 10 \log_{10} 2^{2b} = 1.76 + 20b \log_{10} 2 = 1.76 + 6\,b \ \text{ dB}.$$