Continuous-time Systems

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- LTI systems – recapitulation.
- frequency characteristics $H(j\omega)$.
- fransfer of a signal thrugh a system with $H(j\omega)$.
- Laplace transform.
- Stability and relationship between LT and $H(j\omega)$.

LTI systems – recapitulation

- linearity: $ax_1(t) + bx_2(t) \longrightarrow ay_1(t) + by_2(t)$. This property is especially important as we often represent a signal as a sum of complex exponentials.
- time invariance: system's properties do not change over time.
- LTI systems are defined by **impulse response**: for the input $\delta(t)$ a system outputs $h(t)$. What $h(t)$ looks like for causal systems?
- to any arbitrary input signal $x(t)$ we compute the output using **convolution**:

$$y(t) = x(t) \star h(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$$

or vice-versa as convolution is communitative. For causal impuls responce we obtain:

$$y(t) = \int_{-\infty}^{t} x(\tau)h(t-\tau)d\tau$$

System's response to the input signal e^{st}

where s is an arbitrary complex value:

$$y(t) = H(s)e^{st}, \quad \text{where} \quad H(s) = \int_{-\infty}^{+\infty} h(t)e^{-st}dt.$$

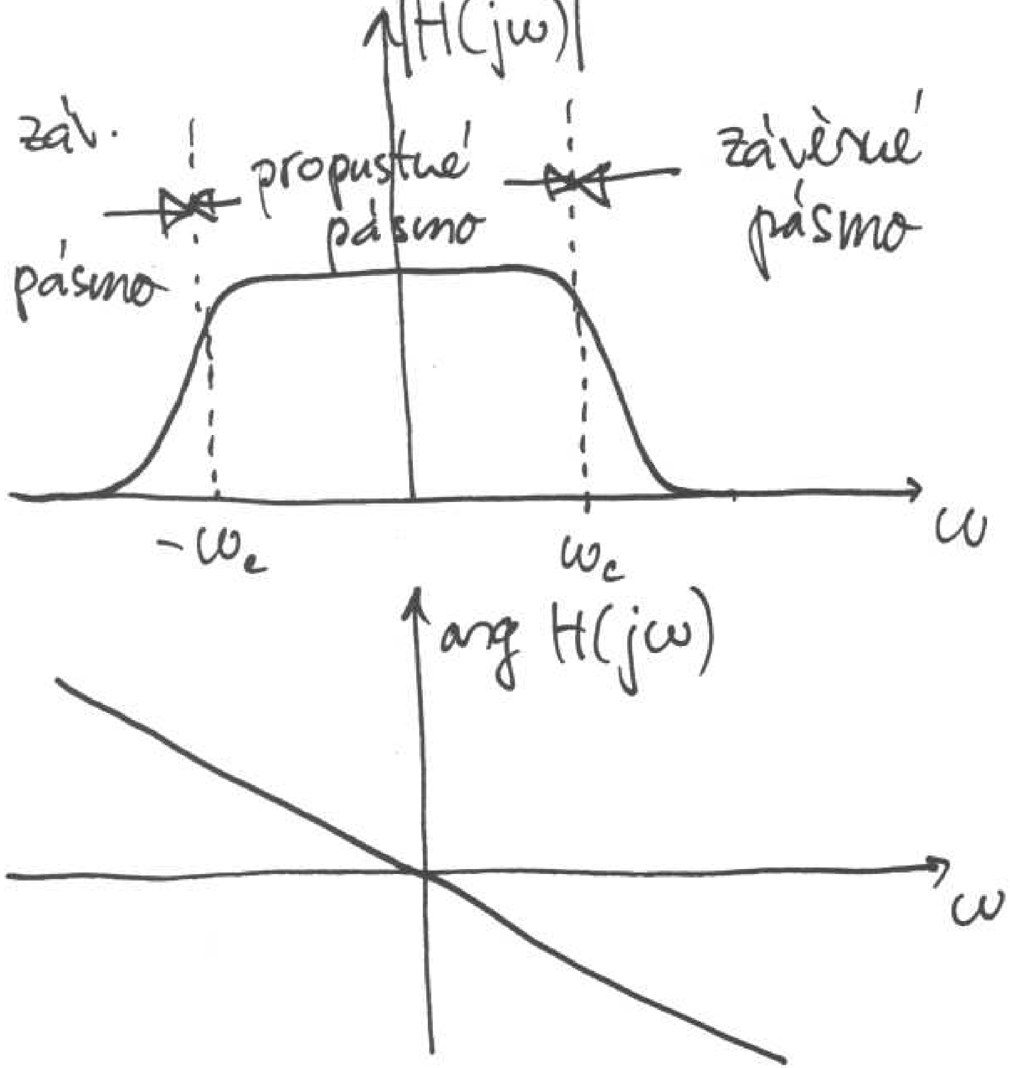
The output is the original signal multiplied by a complex number $H(s)$. We are especially interested in the case when $s = j\omega$, then the input signal is $e^{j\omega t}$.
$y(t) = H(j\omega)e^{j\omega t}, \quad$ where

$$\boxed{H(j\omega) = \int_{-\infty}^{+\infty} h(t)e^{-j\omega t}dt.}$$

The complex exponential with frequency ω is now multiplied by a complex number $H(j\omega)$. A value of $H(j\omega)$ is called **transfer** . We can evaluate $H(j\omega)$ for any ω. The function $H(j\omega)$ is called **(complex) frequency characteristic**. Transfer function is a Fourier transform of an impulse response: $H(j\omega) = \mathcal{F}\{h(t)\}$. As $h(t) \in \Re$, then $H(j\omega)$ has the following property:

$$H(j\omega) = H^{\star}(-j\omega).$$

Example: $H(j\omega)$ of a low pass filter:



Transfer of a signal through a system with $H(j\omega)$

Complex exponential $x(t) = c_1 e^{j\omega_1 t}$.

Find a value of $H(j\omega_1)$, decompose it into magnitude and argument:

$$y(t) = H(j\omega_1) c_1 e^{j\omega_1 t} = |H(j\omega_1)||c_1| e^{j(\arg c_1 + \arg H(j\omega_1))} e^{j\omega_1 t}.$$

$\Rightarrow$only magnitude and argument of c_1 change. The period remains unchanged.

Cosine function $x(t) = C_1 \cos(\omega_1 t + \phi_1)$ can be decomposed to the form: $x(t) = \frac{C_1}{2} e^{j\phi_1} e^{j\omega_1 t} + \frac{C_1}{2} e^{-j\phi_1} e^{-j\omega_1 t}$. We work with a **linear system**, that is the exponentials can be processed separatelly and consequently summed up:

$$y(t) = H(j\omega_1) \frac{C_1}{2} e^{j\phi_1} e^{j\omega_1 t} + H(-j\omega_1) \frac{C_1}{2} e^{-j\phi_1} e^{-j\omega_1 t}.$$

We know that $H(j\omega_1)$ and $H(-j\omega_1)$ are complex conjugate, thus $|H(j\omega_1)| = |H(-j\omega_1)|$ and $\arg H(-j\omega_1) = -\arg H(j\omega_1)$.

$y(t) = |H(j\omega_1)|\frac{C_1}{2}e^{j\phi_1 + j\arg H(j\omega_1)}e^{j\omega_1 t} + |H(j\omega_1)|\frac{C_1}{2}e^{-j\phi_1 - j\arg H(j\omega_1)}e^{-j\omega_1 t} =$

$= |H(j\omega_1)|C_1 \cos\left[\omega_1 t + \phi_1 + \arg H(j\omega_1)\right].$

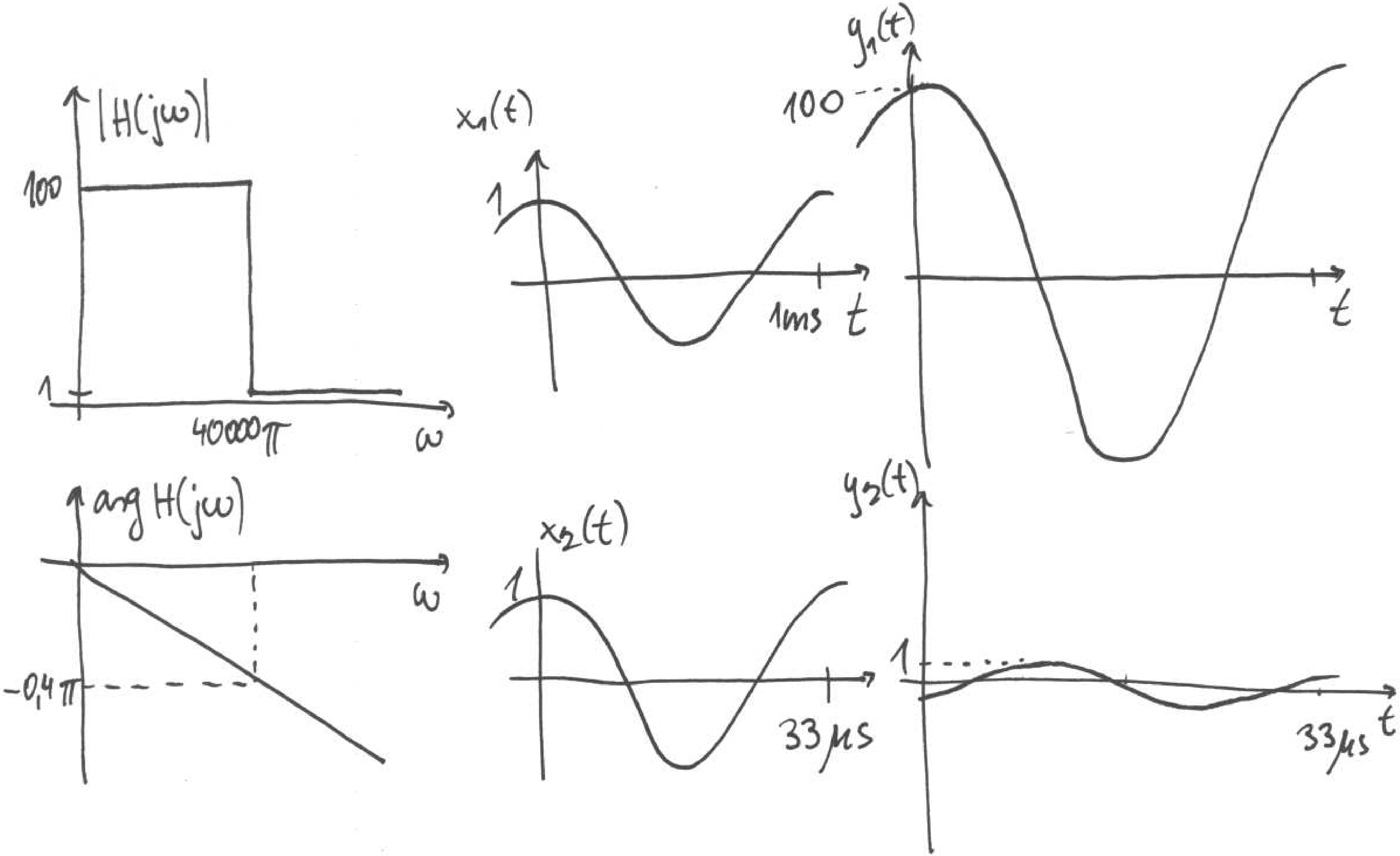
$\Rightarrow$the resulting cosine has changed magnitude and phase.

Example: Ideal Hi-Fi amplifier amplifies from 0 to 20 kHz:

$$|H(j\omega)| = \begin{cases} 100 & \text{for } 0 \le |\omega| \le 40000\pi \\ 1 & \text{for } |\omega| > 40000\pi \end{cases} \qquad \arg H(j\omega) = -\frac{\omega}{100000}.$$

How will it react on a cosine signal with magnitude 1 V and frequencies $f_1 = 1\ kHz$ and $f_2 = 30\ kHz$?

$x_1(t) = \cos(2000\pi t), \quad \omega_1 = 2000\pi, \quad H(j\omega_1) = 100e^{-j0.02\pi}$
$y_1(t) = 100\cos(2000\pi t - 0.02\pi).$
$x_2(t) = \cos(60000\pi t), \quad \omega_2 = 60000\pi, \quad H(j\omega_1) = 1e^{-j0.6\pi}$
$y_1(t) = 1\cos(60000\pi t - 0.6\pi).$


|H(jω)|
100
1
40000π
ω
arg H(jω)
ω
-0,4π
x₁(t)
1
1ms
t
y₁(t)
100
t
x₂(t)
1
33μs
y₂(t)
1
33μs
t

Arbitrary periodic signal can be decomposed into FS:

$$x(t) = \sum_{k=-\infty}^{+\infty} c_k e^{jk\omega_1 t}$$

$H(j\omega)$ will alter every coefficient with respect to the frequency it is tied with:

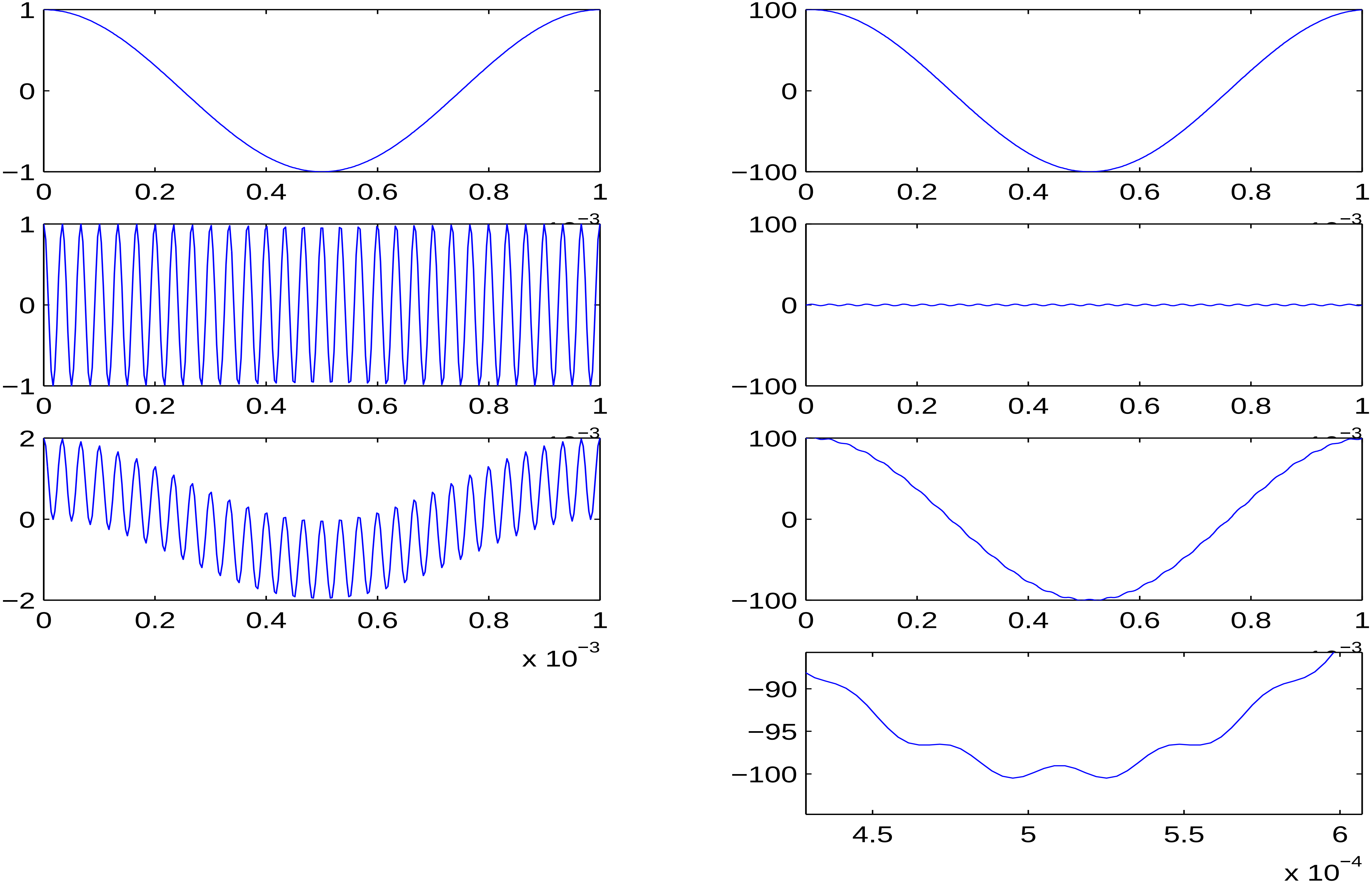
$$y(t) = \sum_{k=-\infty}^{+\infty} H(jk\omega_1) c_k e^{jk\omega_1 t},$$

Again, we do simple multiplication of the FS coefficients (no convolution).

FS coefficient computation, multiplication and signal restoration can be faster than convolution!

Example: A mixture of the signals from the previous example
$x(t) = \cos(2000\pi t) + \cos(60000\pi t) = c_1 e^{j\omega_1 t} + c_{-1} e^{-j\omega_1 t} + c_{30} e^{j30\omega_1 t} + c_{-30} e^{-j30\omega_1 t}$,
where $c_1 = c_{-1} = c_{30} = c_{-30} = \frac{1}{2}$.

After transfer throught the amplifier, new coefficients are:
$c_{1,y} = \frac{1}{2} 100 e^{-j0.02\pi}, \quad c_{-1,y} = \frac{1}{2} 100 e^{j0.02\pi}, \quad c_{30,y} = \frac{1}{2} 1 e^{-j0.6\pi}, \quad c_{-30,y} = \frac{1}{2} 1 e^{j0.6\pi}$.
$y(t) = 100 \cos(2000\pi t - 0.02\pi) + 1 \cos(60000\pi t - 0.6\pi)$.


1
0
−1
0 0.2 0.4 0.6 0.8 1
100
0
−100
2
−2
−90
−95
−100
4.5 5 5.5 6
x 10−3
x 10−4

Transfer of a continuous-time signal through a system with $H(j\omega)$

An arbitrary signal is decomposed to an infinite number of infinitelly small exponentials using FT:

$$X(j\omega) = \int_{-\infty}^{+\infty} s(t)e^{-j\omega t}dt$$

where $X(j\omega)$ is a spectral function of input signal and is defined as:

$$X(j\omega) = 2\pi\frac{dc_x}{d\omega}.$$

Likewise, we can define a spectral function of output signal:

$$Y(j\omega) = 2\pi\frac{dc_y}{d\omega}.$$

For some ω_1 at which we find $dc_{x,1}$ and $dc_{y,1}$, the following holds:

$$dc_{y,1} = H(j\omega_1)dc_{x,1}.$$

And:

$$Y(j\omega_1)\frac{d\omega}{2\pi} = H(j\omega_1)X(j\omega_1)\frac{d\omega}{2\pi}.$$

The above definitions hold for all ω_1, thus:

$$\boxed{Y(j\omega) = H(j\omega)X(j\omega)}$$

Can be proved also by:

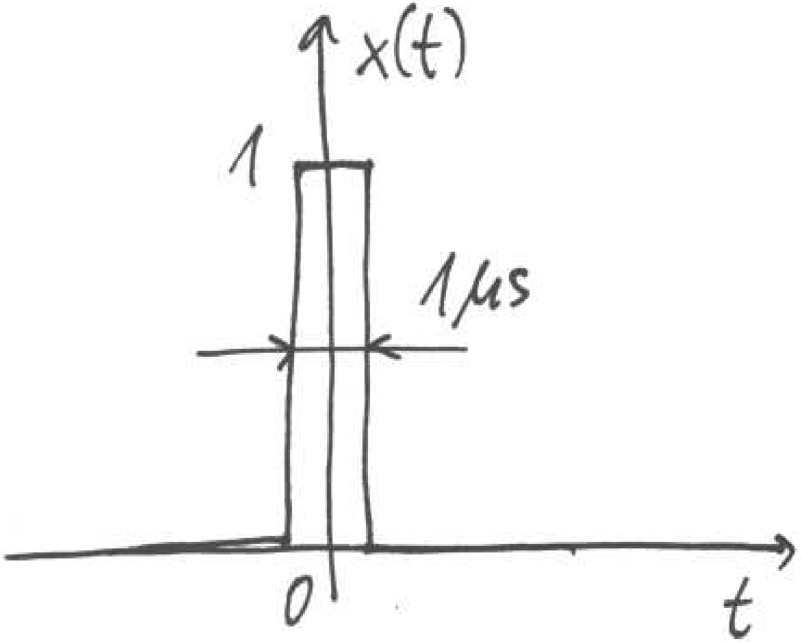
$$Y(j\omega) = \mathcal{F}\{y(t)\} = \int_{-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau\right] e^{-j\omega t}dt = \dots$$

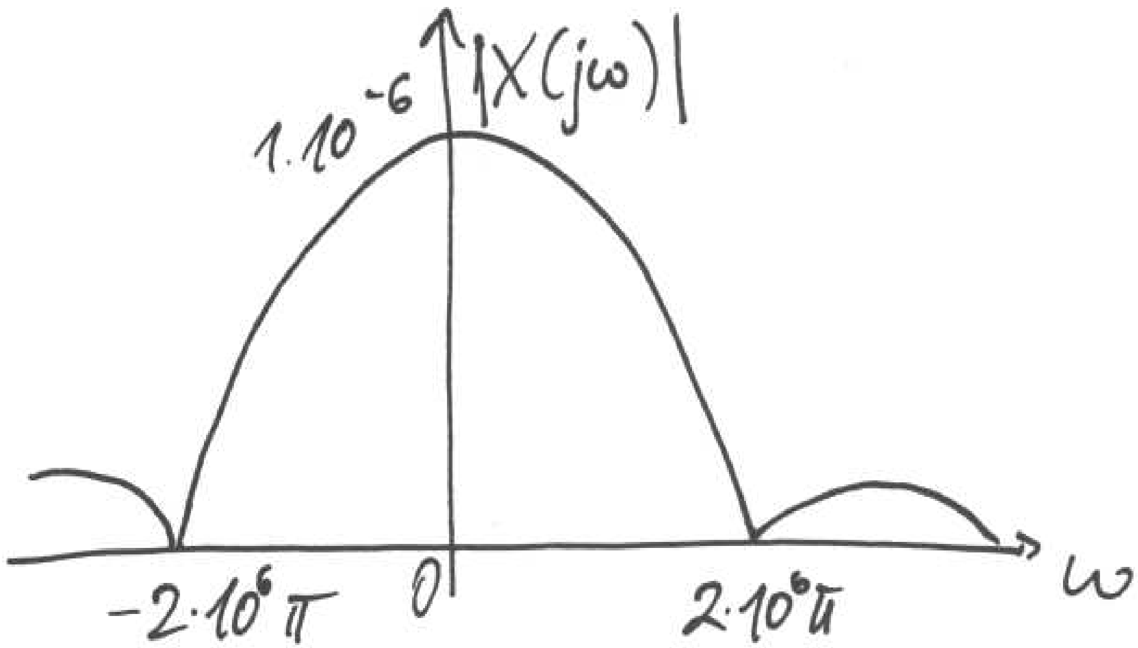
Example: The input signal into Hi-Fi amplifier is a square signal defined by $\vartheta = 1\ \mu$s, $D = 1$ V. Calculate the output $y(t)$

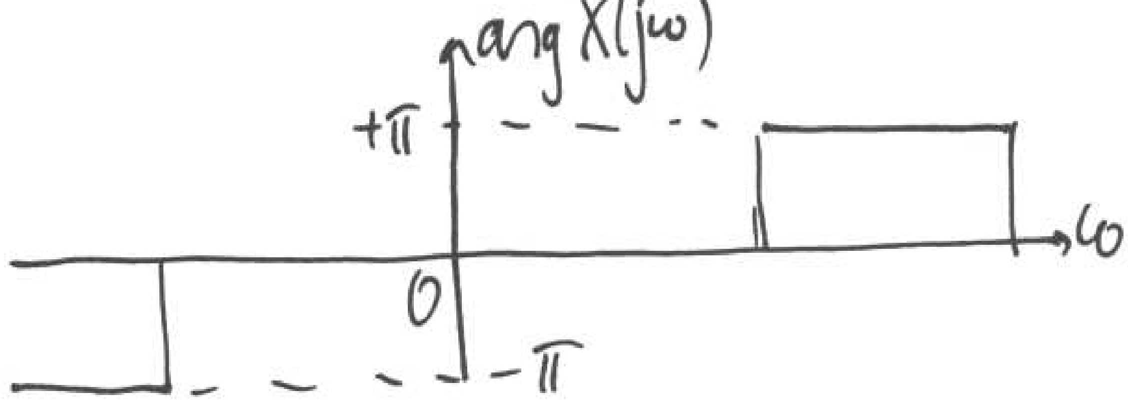
We start off with:

$$x(t) \xrightarrow{\mathcal{F}} X(j\omega) \quad Y(j\omega) = H(j\omega)X(j\omega) \quad Y(j\omega) \xrightarrow{\mathcal{F}^{-1}} y(t).$$

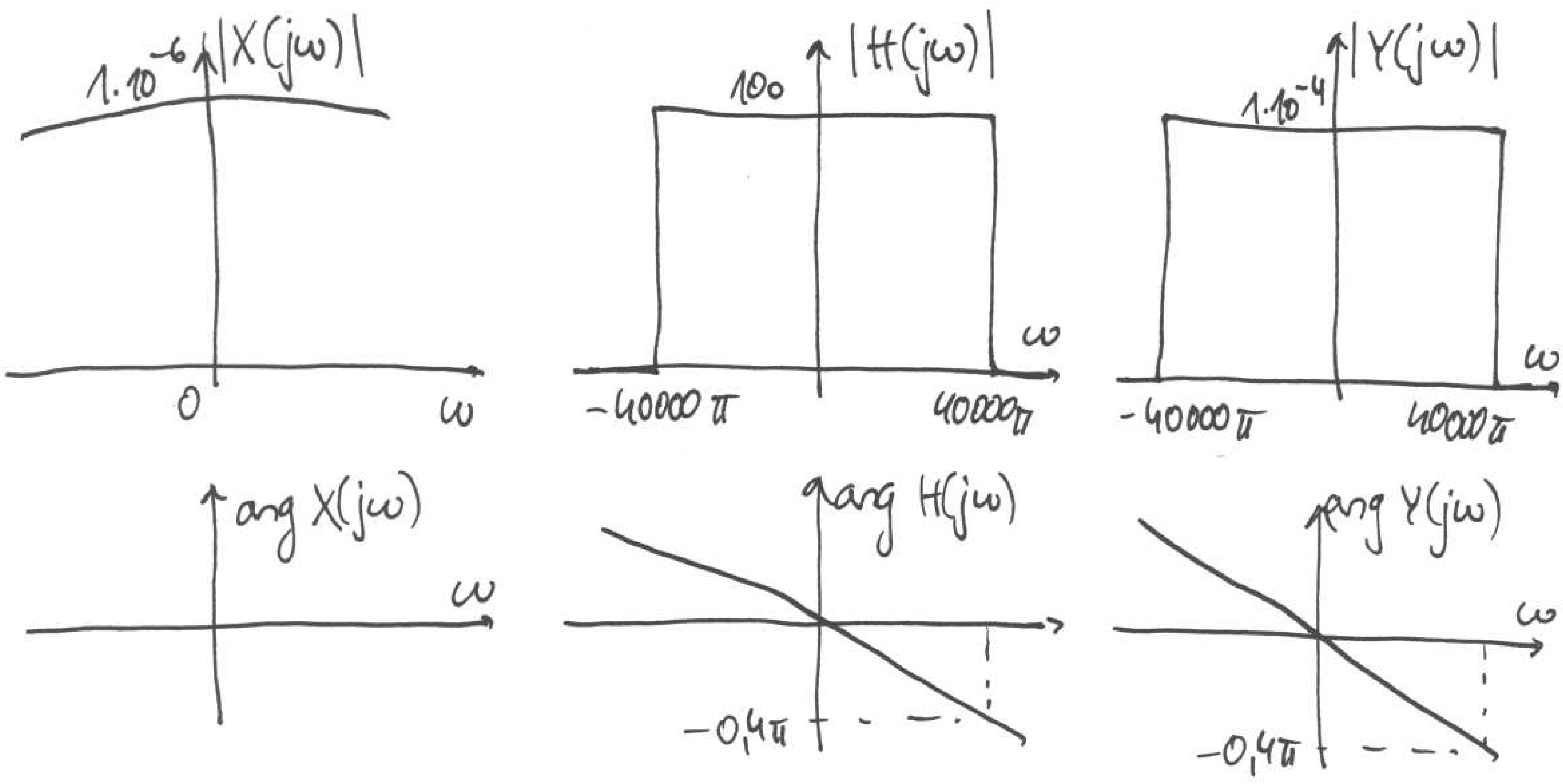
Direct FT: $X(j\omega) = D\vartheta \text{sinc}\left(\frac{\vartheta}{2}\omega\right) = 1 \times 1 \times 10^{-6}\text{sinc}(0.5 \times 10^{-6}\omega)$, the function touchs the ω axis at $\frac{\vartheta}{2}\omega_a = \pi$, $\omega_a = \frac{\pi}{0.5\times 10^{-6}} = 2M\pi$.







The function is multiplied by $H(j\omega)$, that is much 'thinner'



The result is a square signal with linear phase. As the last step, an inverse Fourier transform has to be applied.

First, lets assume the phase is zero ("zp"–zero phase). The signal with square shaped spectrum is:

$$y_{zp}(t) = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} He^{+j\omega t} = \frac{H\omega_c}{\pi}\text{sinc}(\omega_c t) = 1\times 10^{-4}\frac{40000\pi}{\pi}\text{sinc}(40000\pi t) =$$

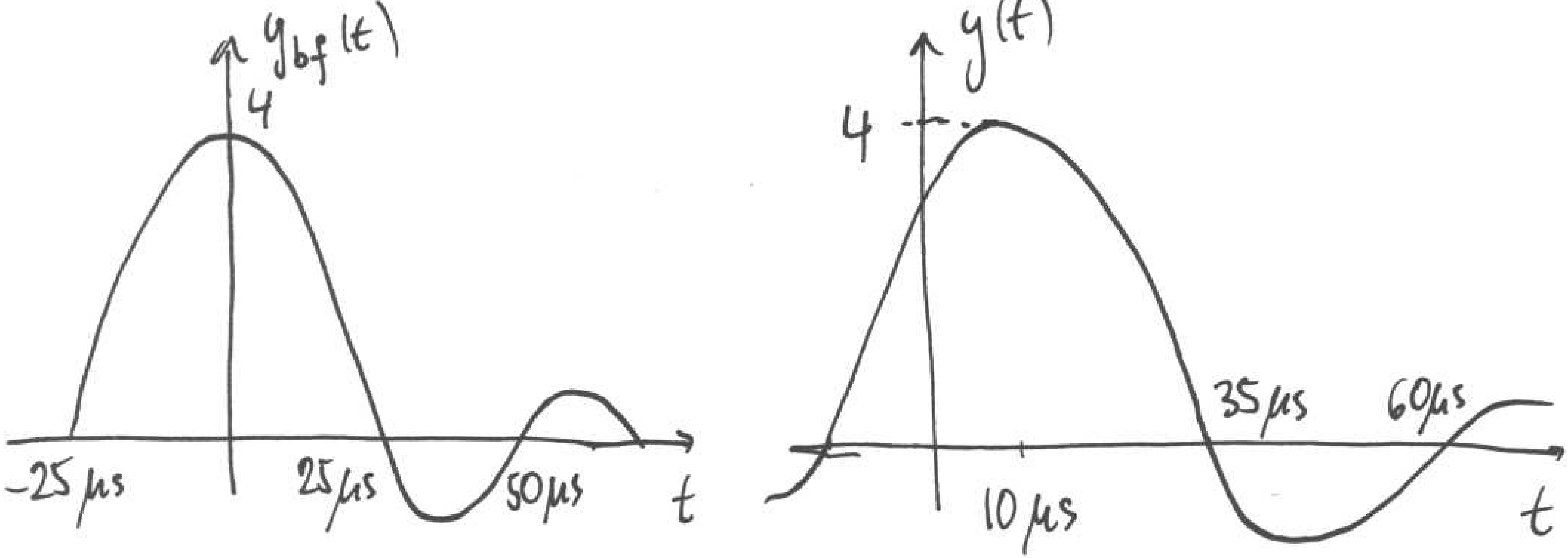
$$= 4\text{sinc}(40000\pi t).$$

The function will cross time axis for the first time for: $40000\pi t = \pi$, $t = \frac{1}{40000} = 25\ \mu$s.

The phase though is linear (not zero-values): $\phi = -\frac{\omega}{100000}\ldots$ What does it remind us? Shift of the signal in time !

$y(t) = y_{zp}(t-\tau) \quad \longrightarrow \quad Y(j\omega) = Y_{zp}(j\omega)e^{-j\omega\tau}$,

thus $\tau = \frac{1}{100000} = 10\ \mu$s.

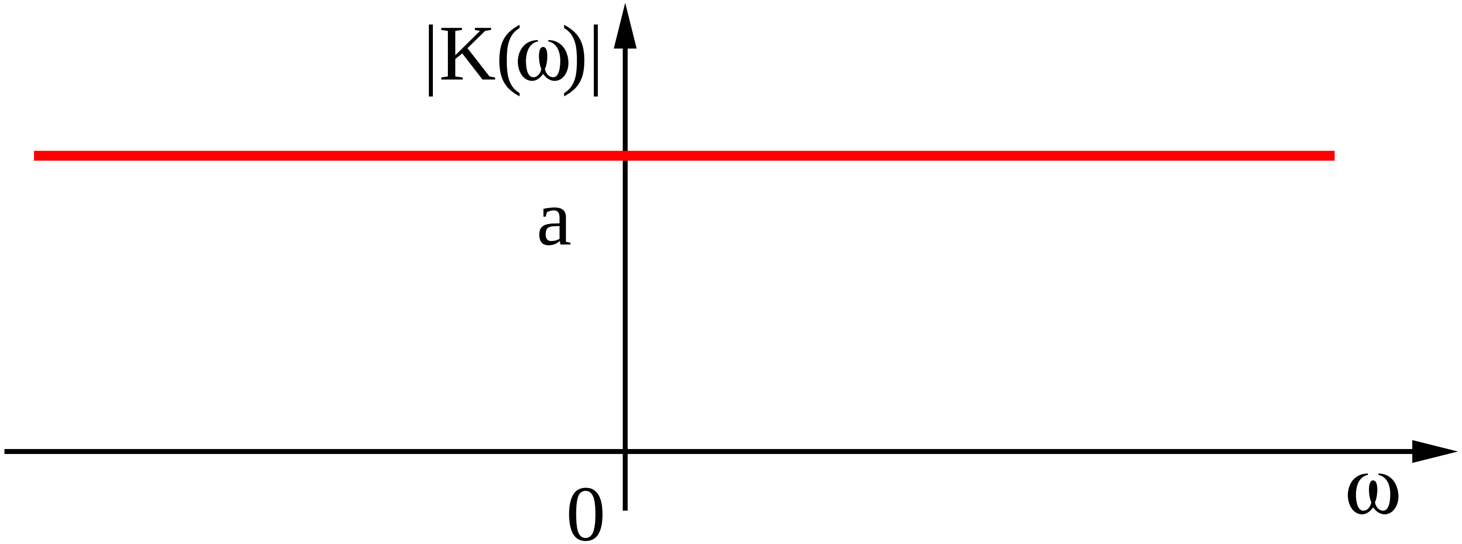


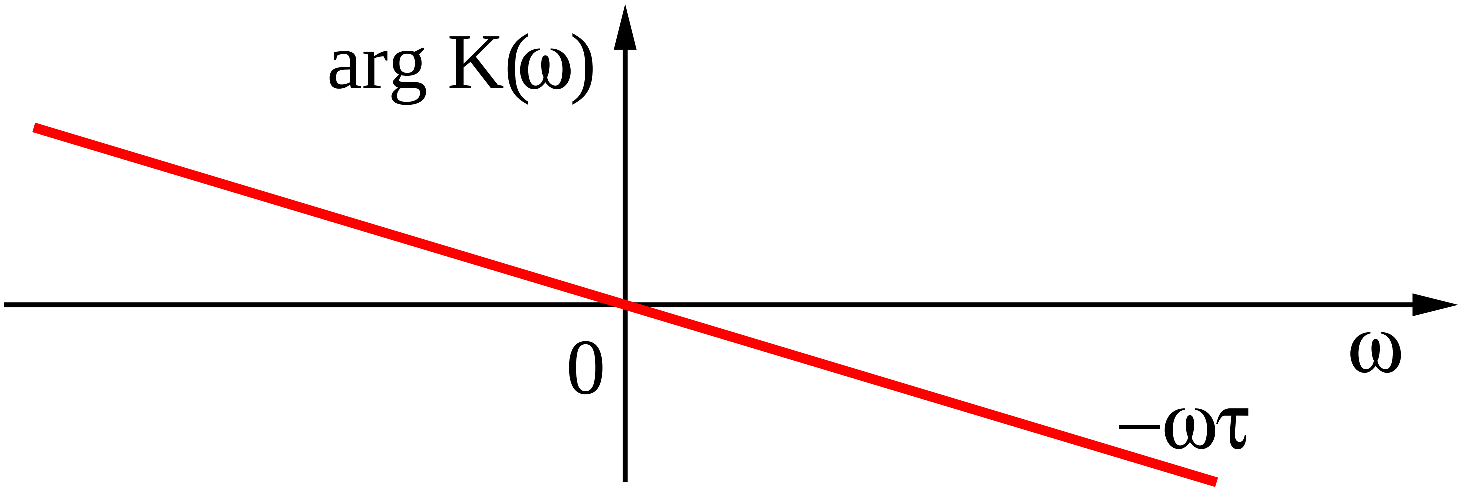
We see that:

- the signal is stretched 50 times.
- low-band filters 'do not like' sharp edged signals in the input.
- impulse response of the ideal low-band filter apparently is function sinc.
- is it really possible for the signal $y(t)$ begin at time -0.5 μs (before the input $x(t)$ is actually given to the filter)?

Ideal amplifier

$$y(t) = a\,x(t-\tau), \quad Y(\omega) = a\,X(\omega)e^{-j\omega\tau} \quad H(j\omega) = ae^{-j\omega\tau}$$

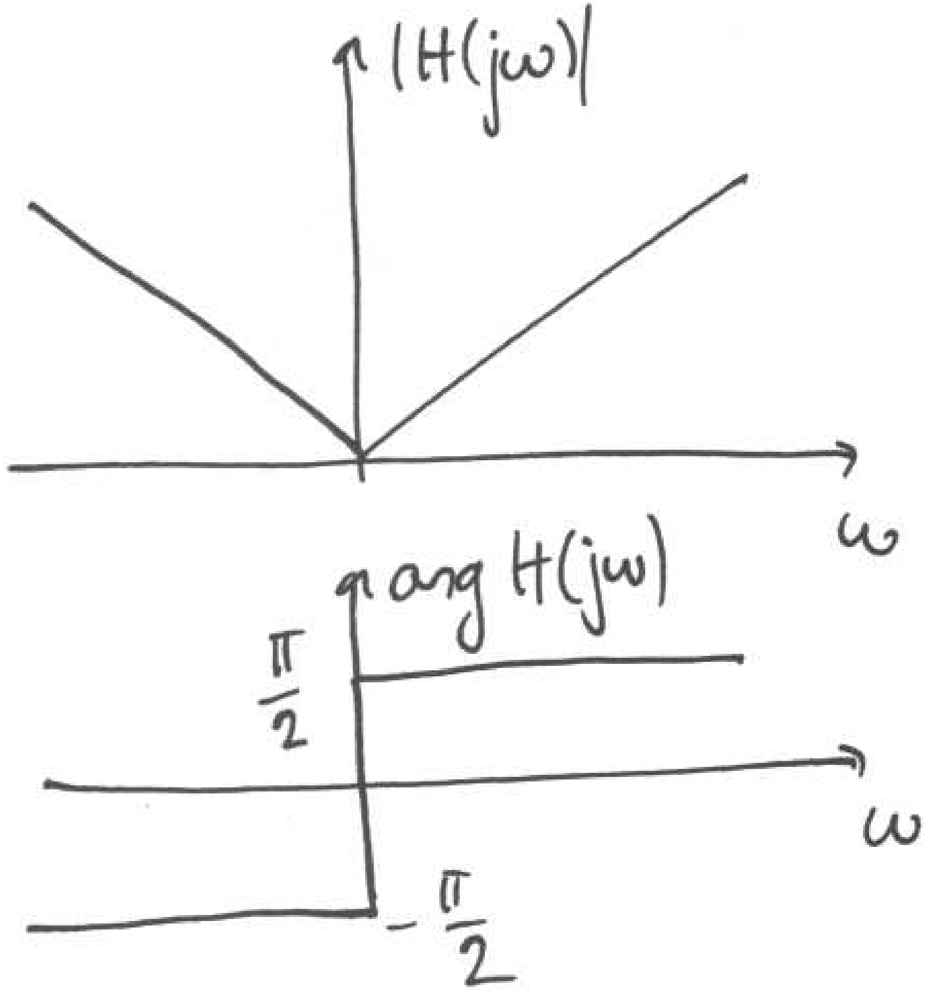




Derivative unit

$y(t) = \dfrac{dx(t)}{dt}$. If we set the input $x(t) = e^{j\omega t}$, the output is $y(t) = \dfrac{de^{j\omega t}}{dt} = e^{j\omega t} j\omega$

$H(j\omega) = j\omega.$



LAPLACE TRANSFORM

We know that given a complex exponential e^{st} to the input, an LTI system produces the output:

$$y(t) = e^{st}H(s), \quad \text{where} \quad H(s) = \int_{-\infty}^{+\infty} h(\tau)e^{-s\tau}d\tau.$$

So far we were interested in the case when $s = j\omega$, now we will be interested in the entire complex plane "s". **Laplace transform:**

$$X(s) = \int_{-\infty}^{+\infty} x(t)e^{-st}dt,$$

where $s = \sigma + j\omega$ is a complex variable. $X(s)$ is a complex function over complex plane called **projection** and is denoted as $\mathcal{L}$, $x(t) \xrightarrow{\mathcal{L}} X(s)$. Note, that:

$$X(s)|_{s=j\omega} = \mathcal{F}\{x(t)\}.$$

What do we need LT for? We are not interested in the transformed signals but rather the behaviour and stability of the systems defined by differential equations (LTI systems).

Fundamental properties of LT

- Convolution in time domain $x_1(t) \star x_2(t) \longrightarrow X_1(s)X_2(s)$, corresponds to multiplication in frequency domain

$$x(t) \longrightarrow X(s), \quad h(t) \longrightarrow H(s)$$

$$y(t) = x(t) \star h(t) \longrightarrow Y(s) = X(s)H(s).$$

  $H(s)$ is called **transfer** function or **system** function.

- Note the derivation $\dfrac{dx(t)}{dt} \longrightarrow sX(s)$.

Systems defined by differential equations

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

We are interested in the transfer function that is telling us what's the system's reaction on the input:

$$\left(\sum_{k=0}^{N} a_k s^k\right) Y(s) = \left(\sum_{k=0}^{M} b_k s^k\right) X(s),$$

$$H(s) = \frac{\sum_{k=0}^{M} b_k s^k}{\sum_{k=0}^{N} a_k s^k},$$

From the above notation, we can easily get to the frequency characteristics:

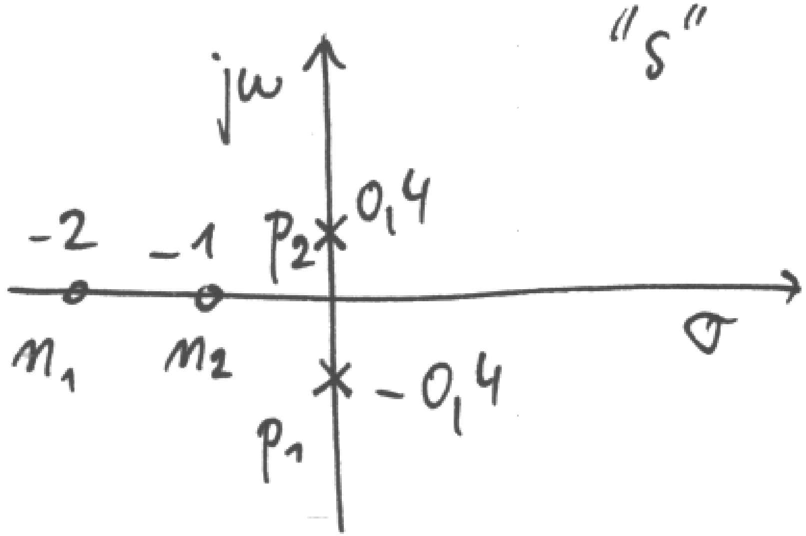
$$\boxed{H(j\omega) = H(s)\big|_{s=j\omega}}$$

the numinator and the denominator are composed of **polynomials**. Polynomials are defined either by coefficients or **roots**.

$$H(s) = \frac{b_M}{a_N} \frac{\prod_{k=1}^{M}(s - n_k)}{\prod_{k=1}^{N}(s - p_k)}.$$

The roots are the values of s for which the polynomial equals to zero. The roots of the numinator n_k are called **zero points** or **zeros**. The roots of the denominator p_k are called **poles**.

Example: $H(s) = \frac{s^2+3s+1}{s^2+0.16} = \frac{(s+2)(s+1)}{(s+j0.4)(s-j0.4)}$
$n_1 = -2, n_2 = -1, p_1 = -j0.4, p_2 = +j0.4$



From zeros and poles, we can also estimate graphical represenation of $H(j\omega)$.

Stability of causal systems

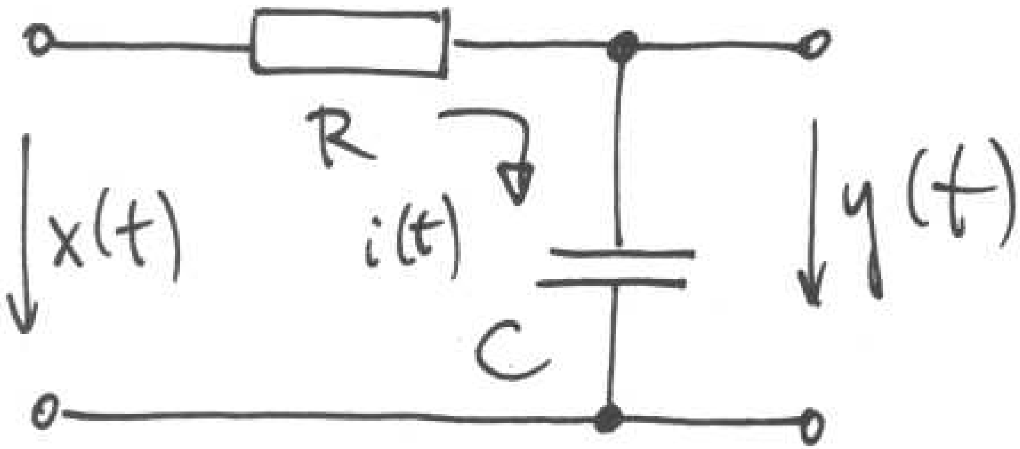
Causal system is stable when all poles lie in the left half-plane of the complex plane: $\Re\{p_k\} < 0$.

Example: what is the capacitor tention $y(t)$ with respect to the source tention $x(t)$?

$i(t) = \frac{x(t)-y(t)}{R}$

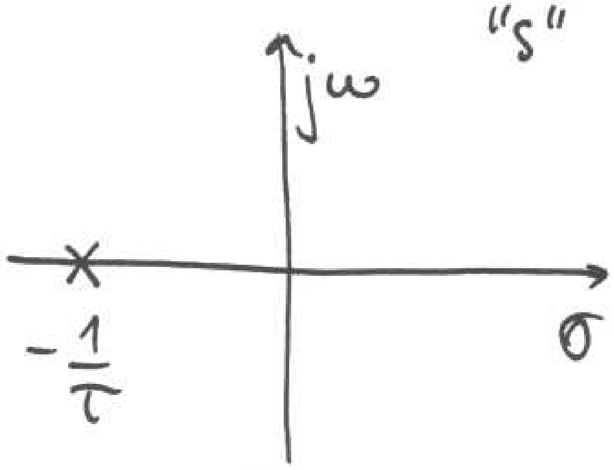
current on the capacitor : $i(t) = C\frac{dy(t)}{dt}$

it gives : $RC\frac{dy(t)}{dt} + y(t) = x(t)$



$\tau = RC$ is so called **time constant** of the system. The coefficients of the differential equation are : $a_1 = \tau, \quad a_0 = 1, \quad b_0 = 1$.

The transfer funtion have no zeros and one pole for $(s\tau + 1) = 0$, thus $p_1 = -\frac{1}{\tau}$.



$\Rightarrow$ the system is **stable**.

Zero-pole notation:

$$H(j\omega) = \frac{b_M}{a_N} \frac{\prod_{k=1}^{M} (j\omega - n_k)}{\prod_{k=1}^{N} (j\omega - p_k)}.$$

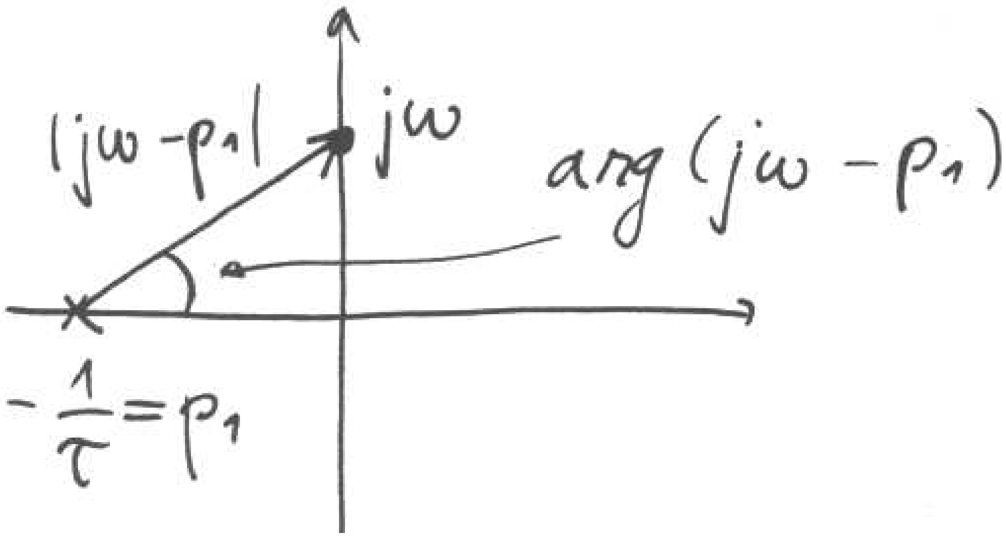
For a given $j\omega$ each braces is a complex number, that can be represented as a vector.

To obtain the value of the frequency characteristic for the frequency ω we have to

- multiply together moduls of all the numbers from the numinator and sum up argumets.
- divide moduls of all the numbers from the denuminator and subtract arguments.

Back to the example...

$H(s) = \frac{1}{\tau} \frac{1}{s-(-\frac{1}{\tau})}$.



- we start off with $j\omega = 0$: $|H(0)| = 1$, $\arg H(0) = 0$.
- increasing ω: then $|s - (-\frac{1}{\tau})|$ increases, but is in the denominator. Thus the fraction decreases. The ungle increases and is used with the inverted sign to get the final result.
- $j\omega = \infty$: $|H(j\infty)| = 0$, $\arg H(0) = -\frac{\pi}{2}$.

Example: $R = 1\ k\Omega$, $C = 1\ \mu$F, $\tau = RC = 1$ ms.