Systems

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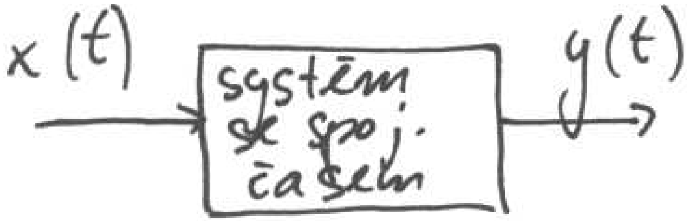
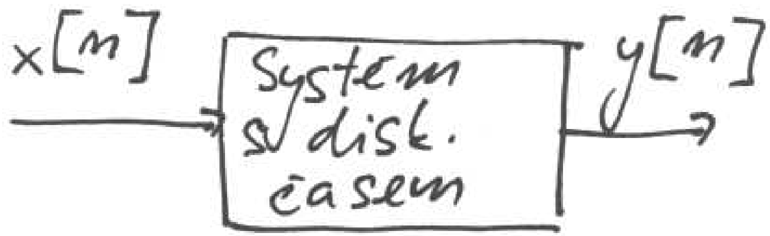
- Properties of linear systems.
- Convolution – diskrete and continuous time.
- Properties of convolution

Systems

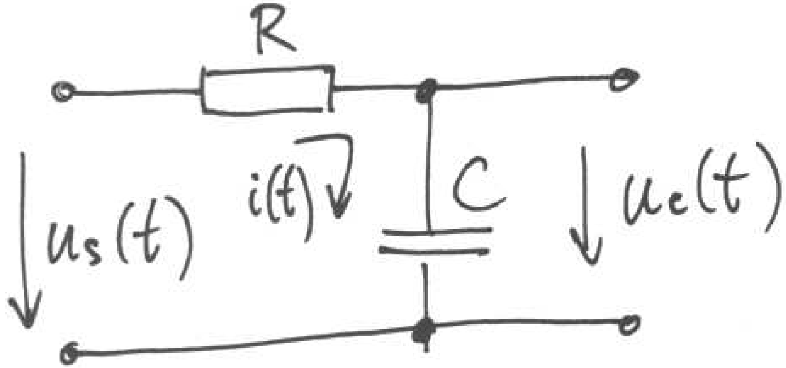
- What can be system?





Continuous time Systems: $x(t) \rightarrow y(t)$. **Discrete time systems**: $x[n] \rightarrow y[n]$.

Example 1: electric circuit $u_c(t)$, $u_s(t)$:



- $i(t) = \frac{u_s(t) - u_c(t)}{R}$
- $i(t) = C\frac{du_c(t)}{dt}$.
- $\frac{du_c(t)}{dt} + \frac{1}{RC}u_c(t) = \frac{1}{RC}u_s(t)$.

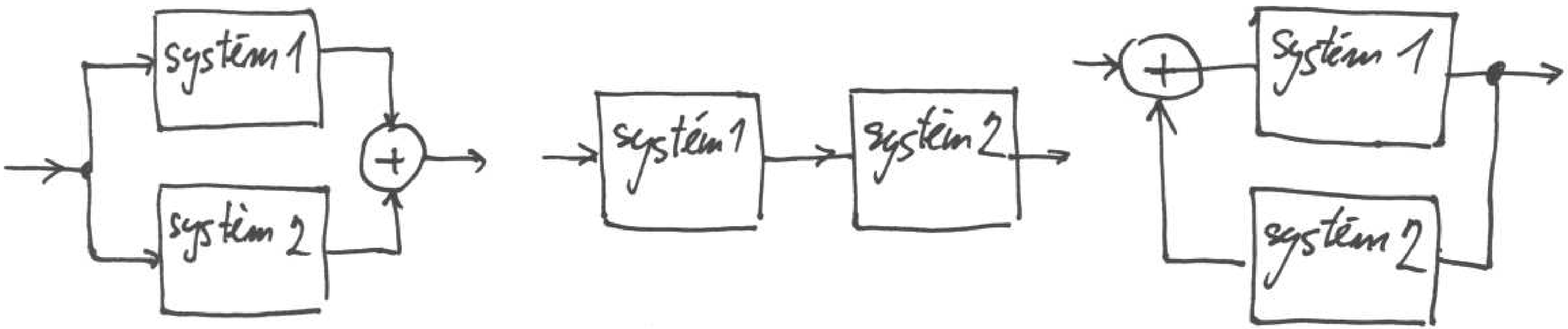
Example 2 discrete: amount of neurons in one's brain decreases by 0.1% plus the number of beers drunk:

$$y[n] = 0.999y[n-1] - x[n]$$

This difference equation can be solved for a given $x[n]$ at a certain time n and the initial condition $y[0]$ (the number of neurons at birth). We can also find out when $y[n]$ will equal to zero.

Combination of systems

paralel, in series, loop-back:

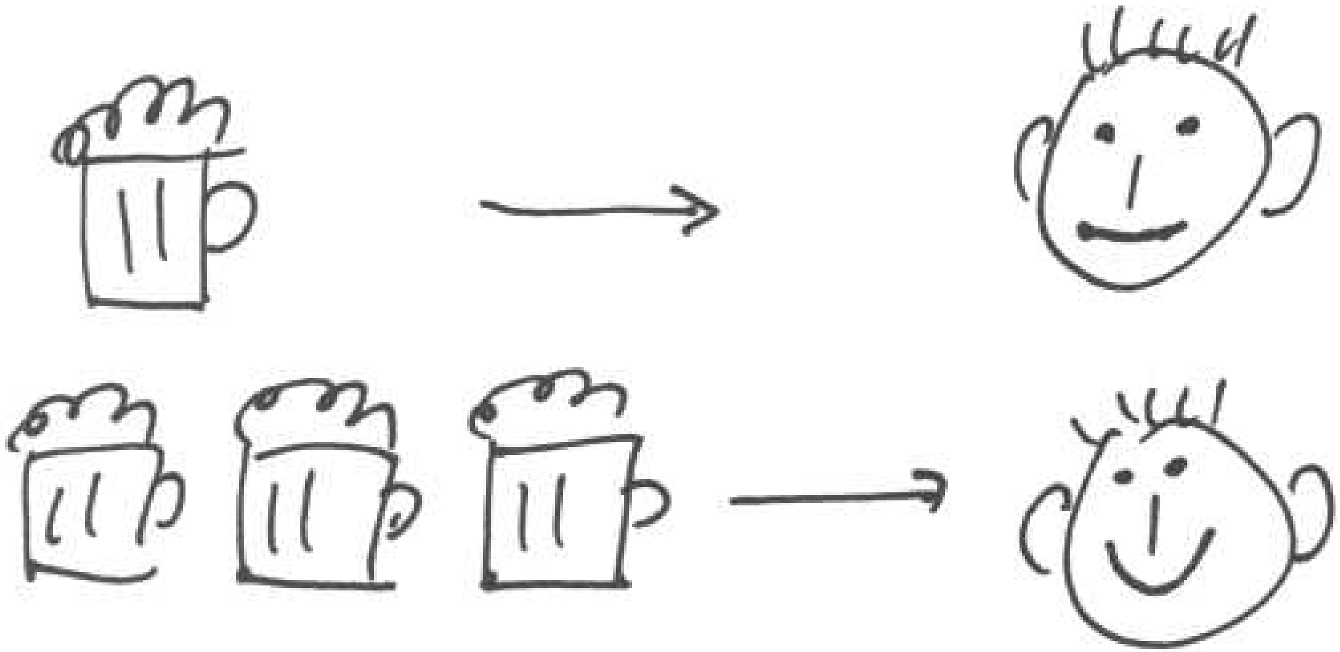


Fundamentals about systems

With memory / without memory

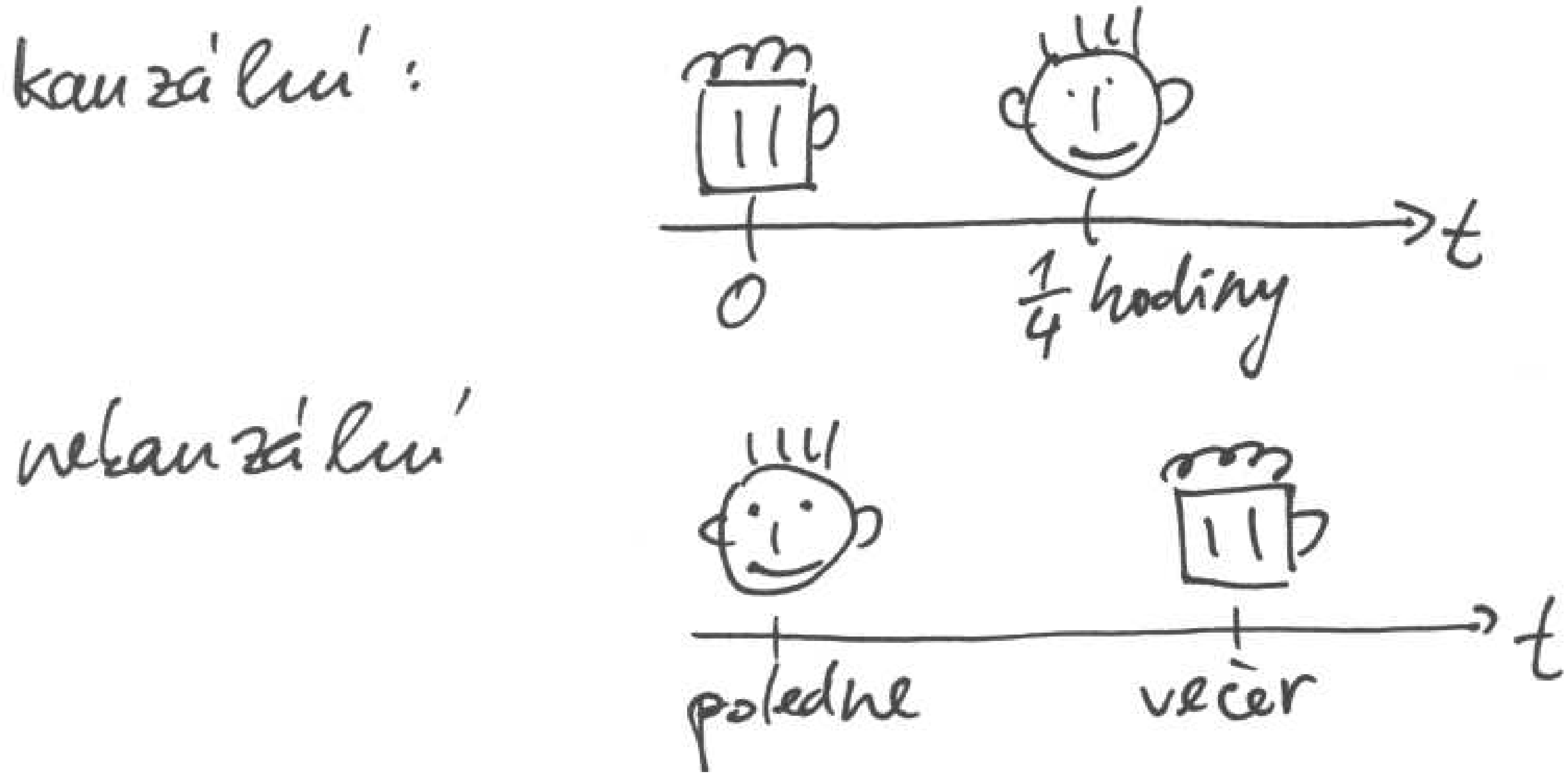
Example with memory: neurons in brain **Example without memory**: $y(t) = Kx(t)$.

Examples: **drinking of beer**: input - output:



Causal systems

Example:



Causal system: $y[n] = x[n] - x[n-1]$ Non-causal system: $y[n] = x[-n]$

Stability

Bounded input implies baunded output.

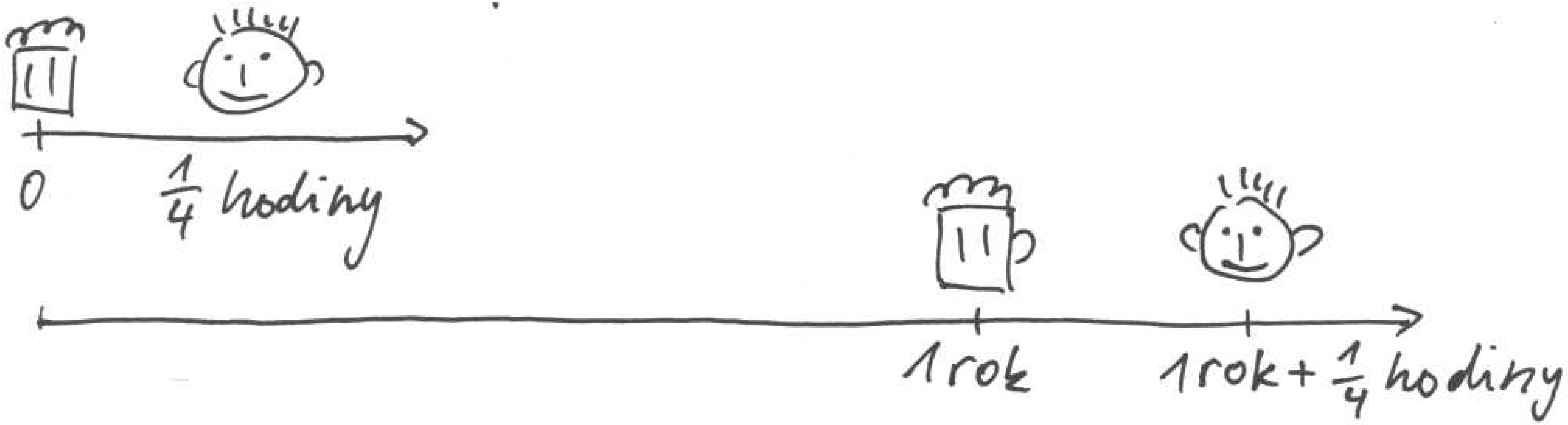
We can find such $B, C < \infty$ that for every t, n the following relations hold:
$|x(t)| < B \rightarrow |y(t)| < C \qquad |x[n]| < B \rightarrow |y[n]| < C$.

Example 1: $y(t) = tx(t), \forall t x(t) < \infty$. The system is unstable as at time $t = \infty$ the output is ∞.

Example 2: $y(t) = \exp^{x(t)}$

Time-invariant systems

System does not change its behaviour over time. That is, if for the input $x(t)$ the system outputs $y(t)$, given the input $x(t - t_0)$ the system will output $y(t - t_0)$.



Example 1: $y[n] = \sin(x[n])$. For time $[n - n_0]$ we get exactly the same sine but shifted by n_0 : $y[n - n_0] = \sin(x[n - n_0])$

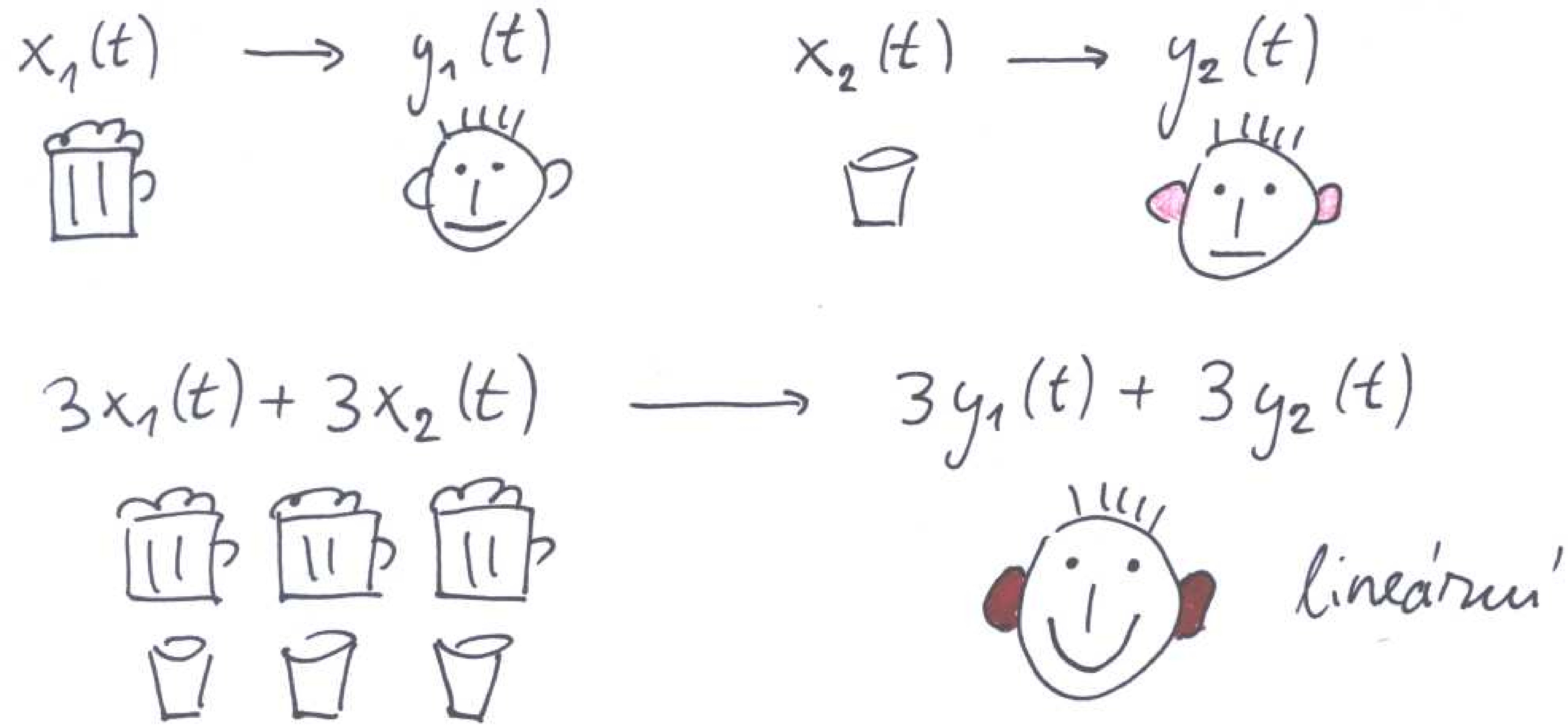
Example 2: $y[n] = nx[n]$

Linearity

$x_1(t) \longrightarrow y_1(t)$ a $x_2(t) \longrightarrow y_2(t)$.

- addition: $x_1(t) + x_2(t) \longrightarrow y_1(t) + y_2(t)$.
- scaling: $ax_1(t) \longrightarrow ay_1(t)$.

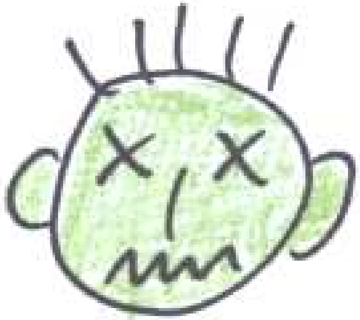
$$
\boxed{\begin{array}{l} ax_1(t) + bx_2(t) \longrightarrow ay_1(t) + by_2(t) \\ ax_1[n] + bx_2[n] \longrightarrow ay_1[n] + by_2[n] \end{array}}
$$



Example: $y(t) = tx(t)$ is not stable but is linear!

Linearity plays very important role in system analysis: each input can be represented as a sequence of scaled and shifted impulses. Given the system impulse response, we therefore can calculate the output to an arbitrary (that is different from impulse) input by means of convolution.

$$\not\longrightarrow 3y_1(t) + 3y_2(t)$$



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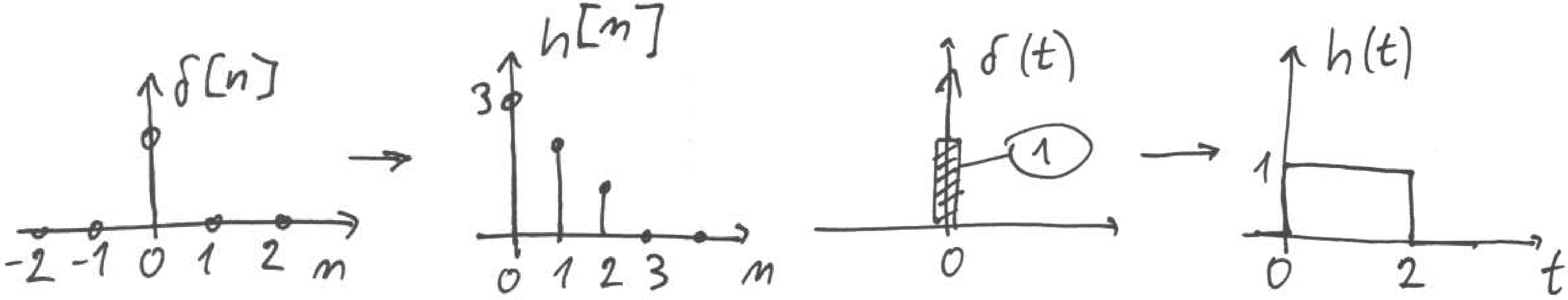
LTI SYSTEMS

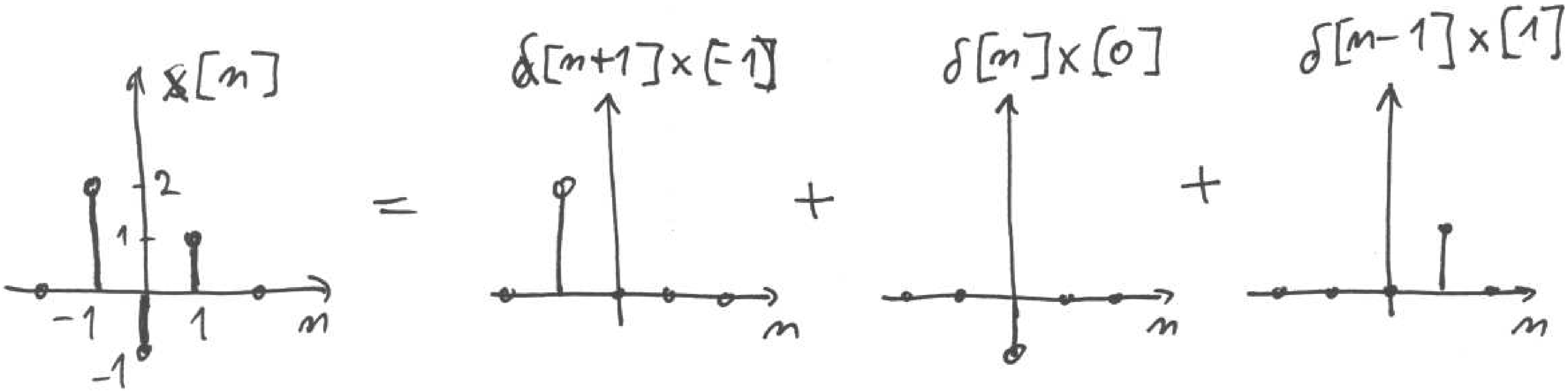
- LTI – linear, time-invariant
- **impuls response** - reaction to the impulse signal



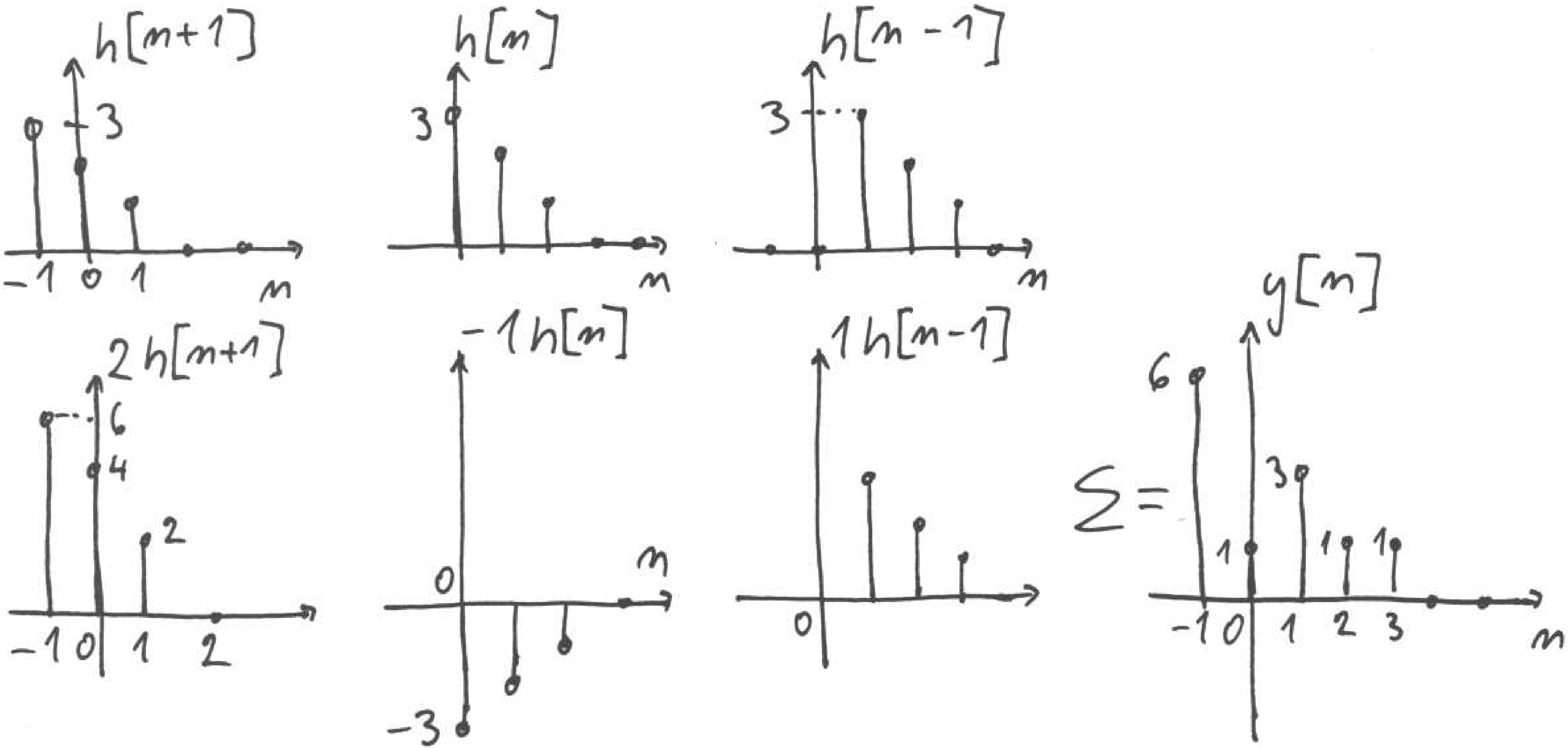
We are interested in the system response to the generic signals $x(t)$, $x[n]$.

Decomposition of signal into discrete unit impulses

$$x[n] = \sum_{k=-\infty}^{+\infty} x[k]\delta[n-k].$$



Example for $h[n]$ a $x[n]$:



Solution:

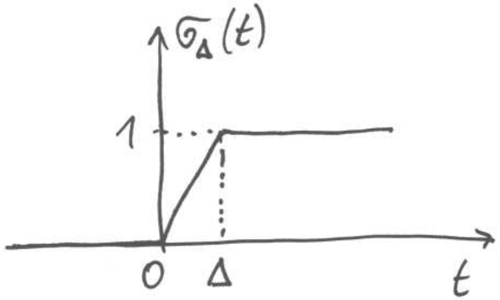
$$\boxed{y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]}$$

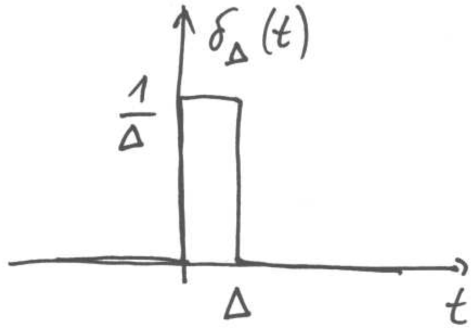
$$\boxed{y[n] = x[n] \star h[n]}$$

k	-5	-4	-3	-2	-1	0	1	2	3	4	5	y[n]
x[k]	0	0	0	0	2	-1	1	0	0	0	0	
n=-2	0	1	2	3	0	0	0	0	0	0	0	0
n=-1	0	0	1	2	3	0	0	0	0	0	0	6
n= 0	0	0	0	1	2	3	0	0	0	0	0	1
n= 1	0	0	0	0	1	2	3	0	0	0	0	3
n= 2	0	0	0	0	0	1	2	3	0	0	0	1
n= 3	0	0	0	0	0	0	1	2	3	0	0	1
n= 4	0	0	0	0	0	0	0	1	2	3	0	0

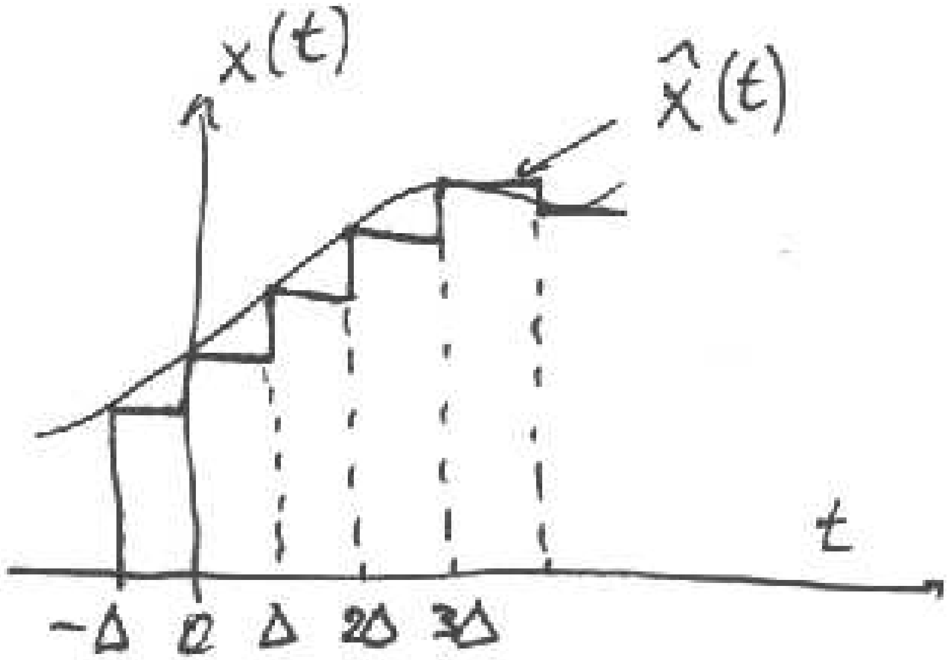
LTI systems for continuous time

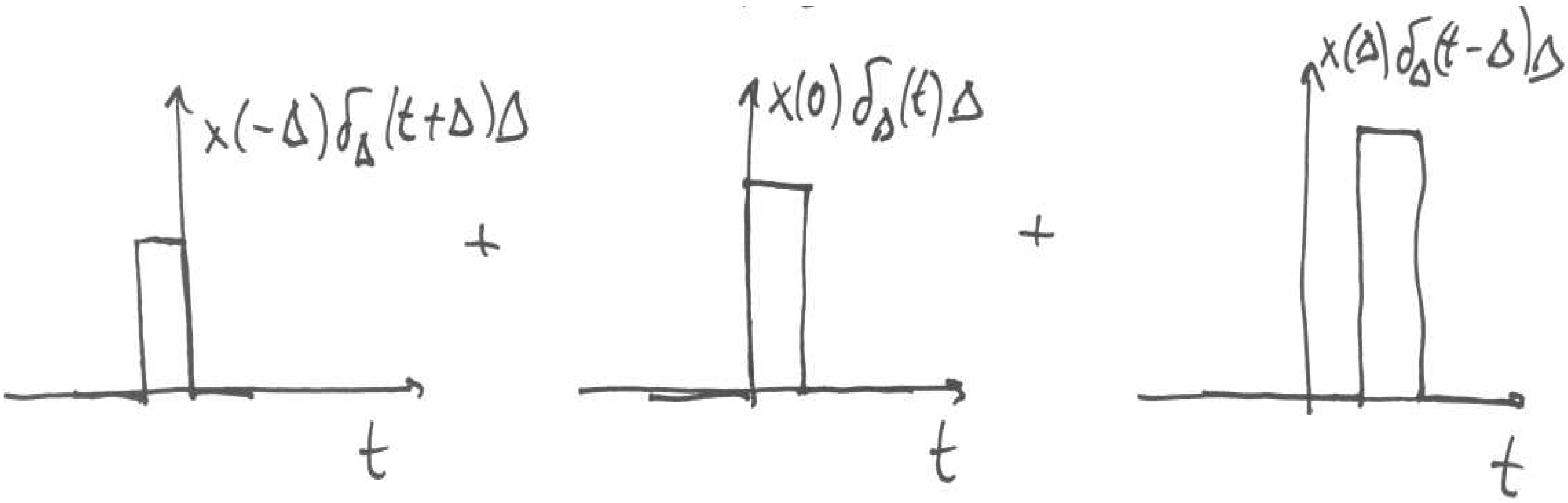
$$\delta_\Delta(t) = \begin{cases} \frac{1}{\Delta} & \text{for } 0 \le t \le \Delta \\ 0 & \text{elsewhere} \end{cases}$$



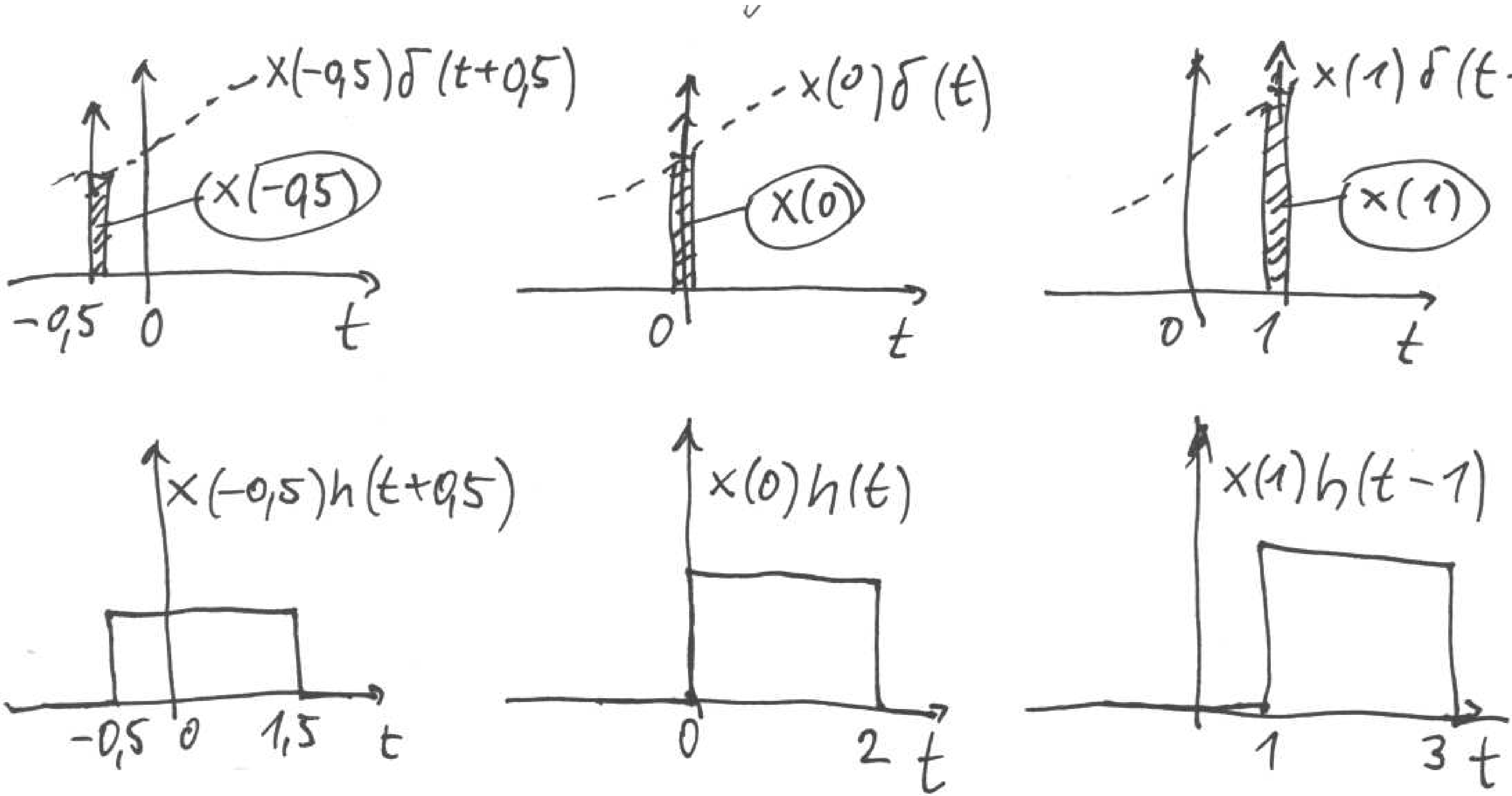


$$\hat{x}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)\delta_\Delta(t - k\Delta)\Delta.$$



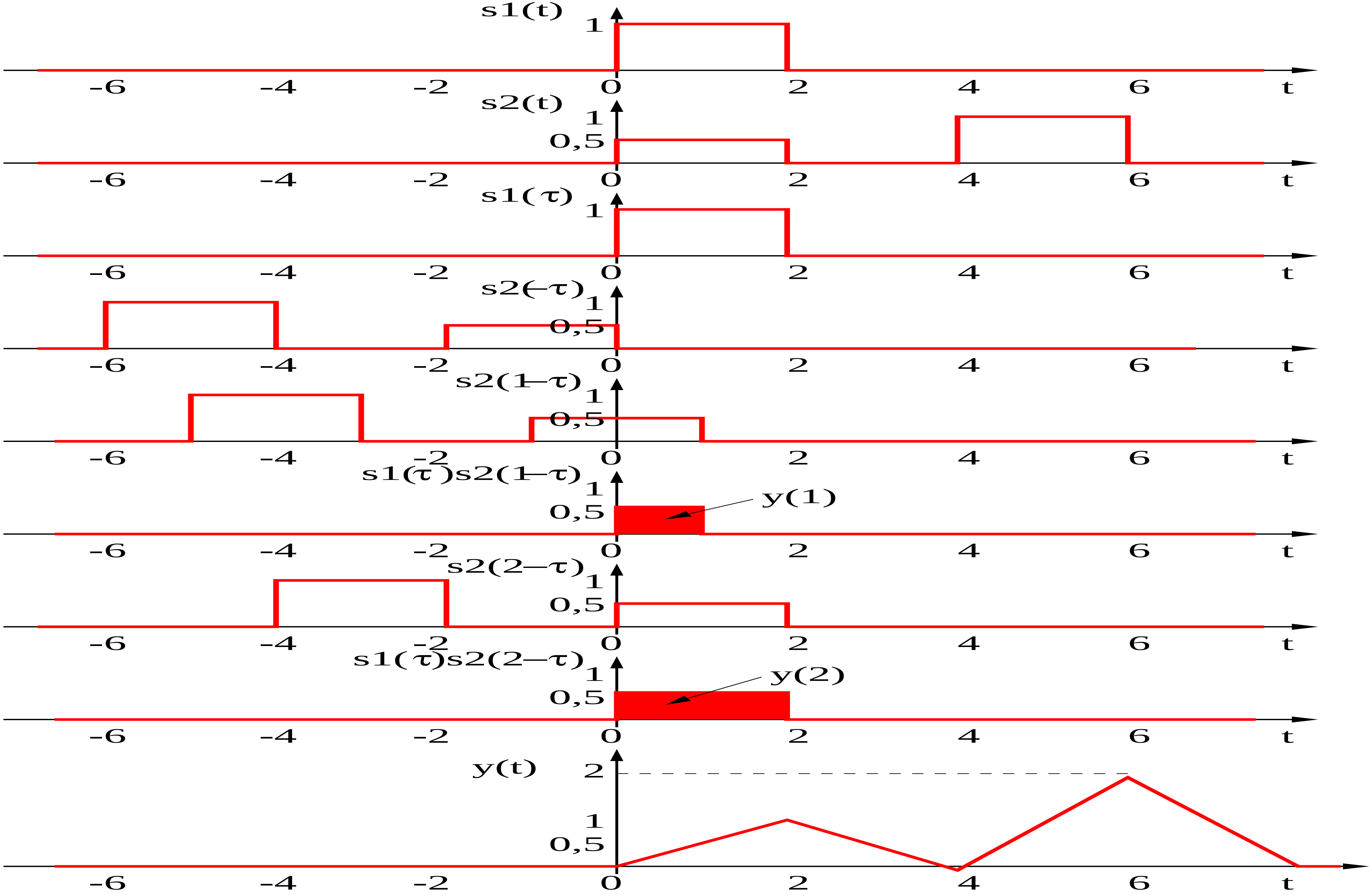


$$\delta(t) \to h(t), \quad \delta(t-\tau) \to h(t-\tau), \quad x(\tau)\delta(t-\tau) \to x(\tau)h(t-\tau)$$



$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau.$$

$y(t) = x(t) \star h(t).$


s1(t)
s2(t)
s1(τ)
s2(−τ)
s2(1−τ)
s1(τ)s2(1−τ)
y(1)
s2(2−τ)
s1(τ)s2(2−τ)
y(2)
y(t)
t
-6
-4
-2
0
2
4
6
0,5
1
2

Convolution – conclusions

$$y[n] = x[n] \star h[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

$$y(t) = x(t) \star h(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau.$$

Properties of convolution

Comutativity:

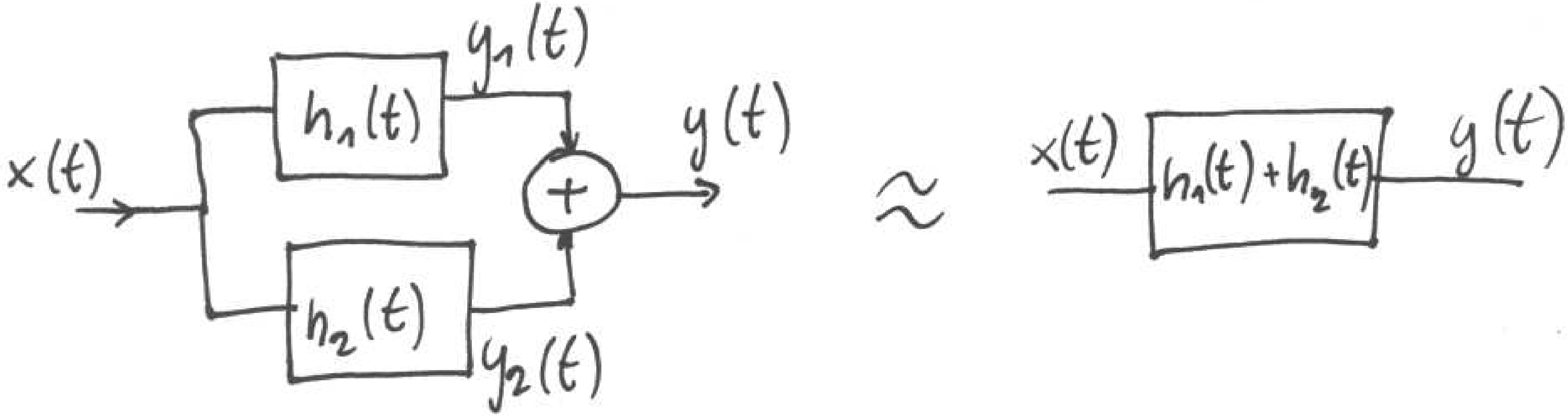
$$y[n] = x[n] \star h[n] = h[n] \star x[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k] = \sum_{k=-\infty}^{+\infty} h[k]x[n-k]$$

$$y(t) = x(t) \star h(t) = h(t) \star x(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{+\infty} h(\tau)x(t-\tau)d\tau.$$

Distributivity – paralel combination of systems:

$$y(t) = y_1(t) + y_2(t) = x(t) \star h_1(t) + x(t) \star h_2(t) = x(t) \star [h_1(t) + h_2(t)].$$

$h(t) = h_1(t) + h_2(t)$.



Check it:

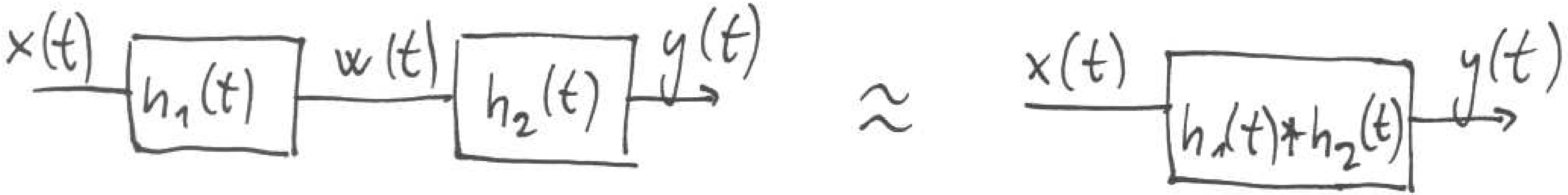
$$\int x(\tau)h_1(t-\tau)d\tau + \int x(\tau)h_2(t-\tau)d\tau = \int x(\tau)[h_1(t-\tau)+h_2(t-\tau)]d\tau = x(t)\star[h_1(t)+h_2(t)],$$

$$y[n] = x[n] \star [h_1[n] + h_2[n]].$$

Associativity – combination of systems in series:

$$y(t) = [x(t) \star h_1(t)] \star h_2(t) = x(t) \star [h_1(t) \star h_2(t)].$$

Impulse response $h(t) = h_1(t) \star h_2(t)$.



Check it:

$$y(t) = \int_v \left[\int_\tau x(\tau) h_1(v-\tau) d\tau \right] h_2(t-v) dv = \int_v \int_\tau x(\tau) h_1(v-\tau) h_2(t-v) d\tau dv =$$

$$= \text{swap order of integration} = \int_\tau \int_v x(\tau) h_1(v-\tau) h_2(t-v) dv d\tau =$$

$$\int x(\tau) \left[\int_v h_1(v-\tau) h_2(t-v) dv \right] d\tau = \ldots$$

$v = g + \tau$ and take into account: $\int_g h_1(g) h_2(t-\tau-g) dg = h(t-\tau)$ thus, solution is:

$$\ldots = x(t)[h_1(t) \star h_2(t)].$$

For discrete systems:

$$y[n] = x[n] \star [h_1[n] \star h_2[n]].$$

Systems with memory and without it :

Without memory: $h[0] = K\delta[n]$, thus: $y[n] = x[n] \star h[n] = \sum_{k=-\infty}^{+\infty} x[k]K\delta[n-k] = Kx[n]$.

$h(t) = K\delta(t)$, therefore: $y(t) = \int_{-\infty}^{+\infty} x(\tau)K\delta(t-\tau)d\tau = Kx(t)$.

identity (wire):

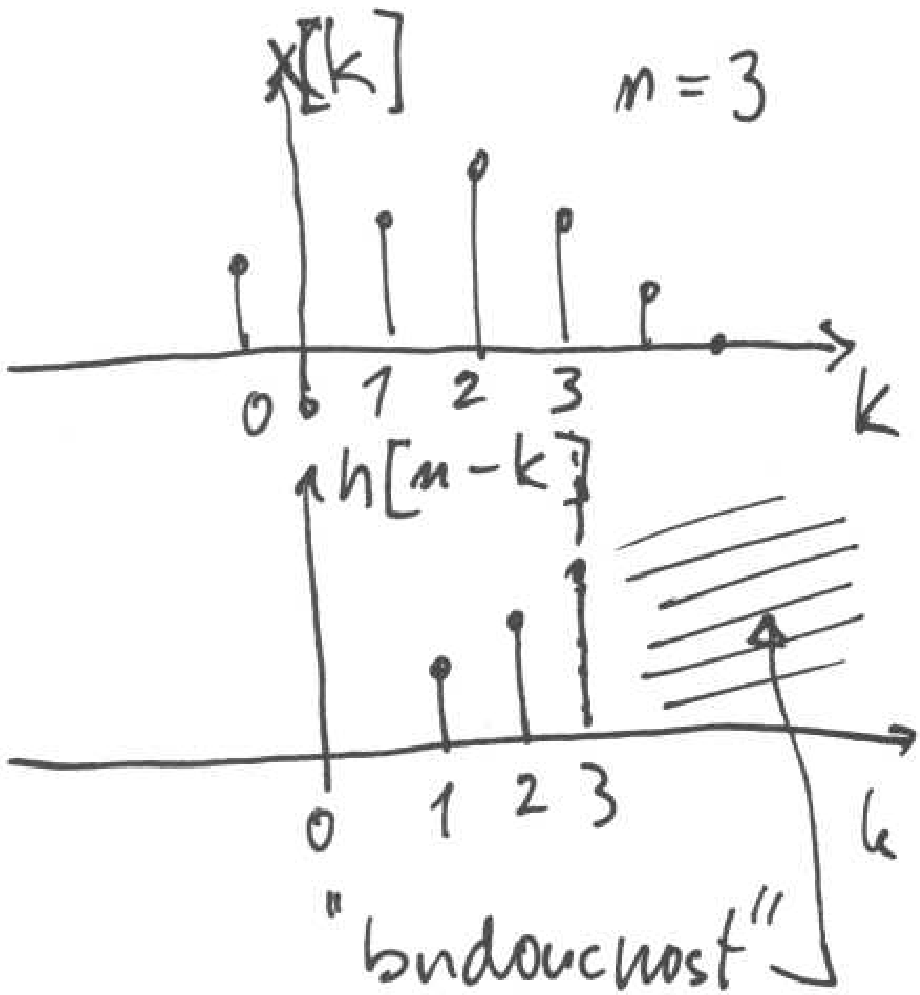
$$h[n] = \delta[n]$$

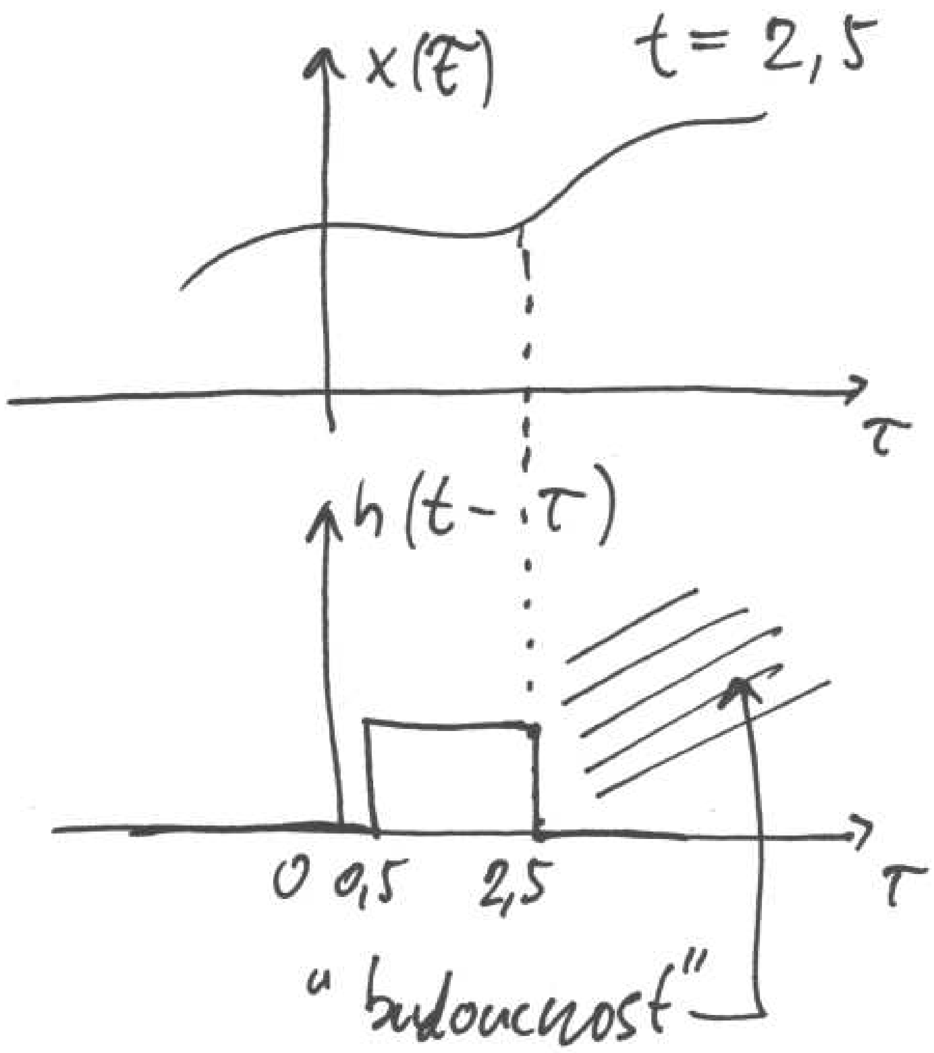
$$h(t) = \delta(t).$$

Causality:

$$h[n] = 0 \text{ pro } n < 0$$

$$h(t) = 0 \text{ pro } t < 0$$





$$\sum_{k=-\infty}^{+\infty} x[k]h[n-k] \rightarrow \sum_{k=-\infty}^{n} x[k]h[n-k], \qquad \sum_{k=-\infty}^{+\infty} h[k]x[n-k] \rightarrow \sum_{k=0}^{\infty} h[k]x[n-k]$$

$$\int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau \rightarrow \int_{-\infty}^{t} x(\tau)h(t-\tau)d\tau,$$

$$\int_{-\infty}^{+\infty} h(\tau)x(t-\tau)d\tau \rightarrow \int_{0}^{+\infty} h(\tau)x(t-\tau)d\tau.$$

Stability:

$$\sum_{k=-\infty}^{+\infty} |h[k]| < \infty, \qquad \int_{-\infty}^{+\infty} |h(t)|dt < \infty.$$