Sampling

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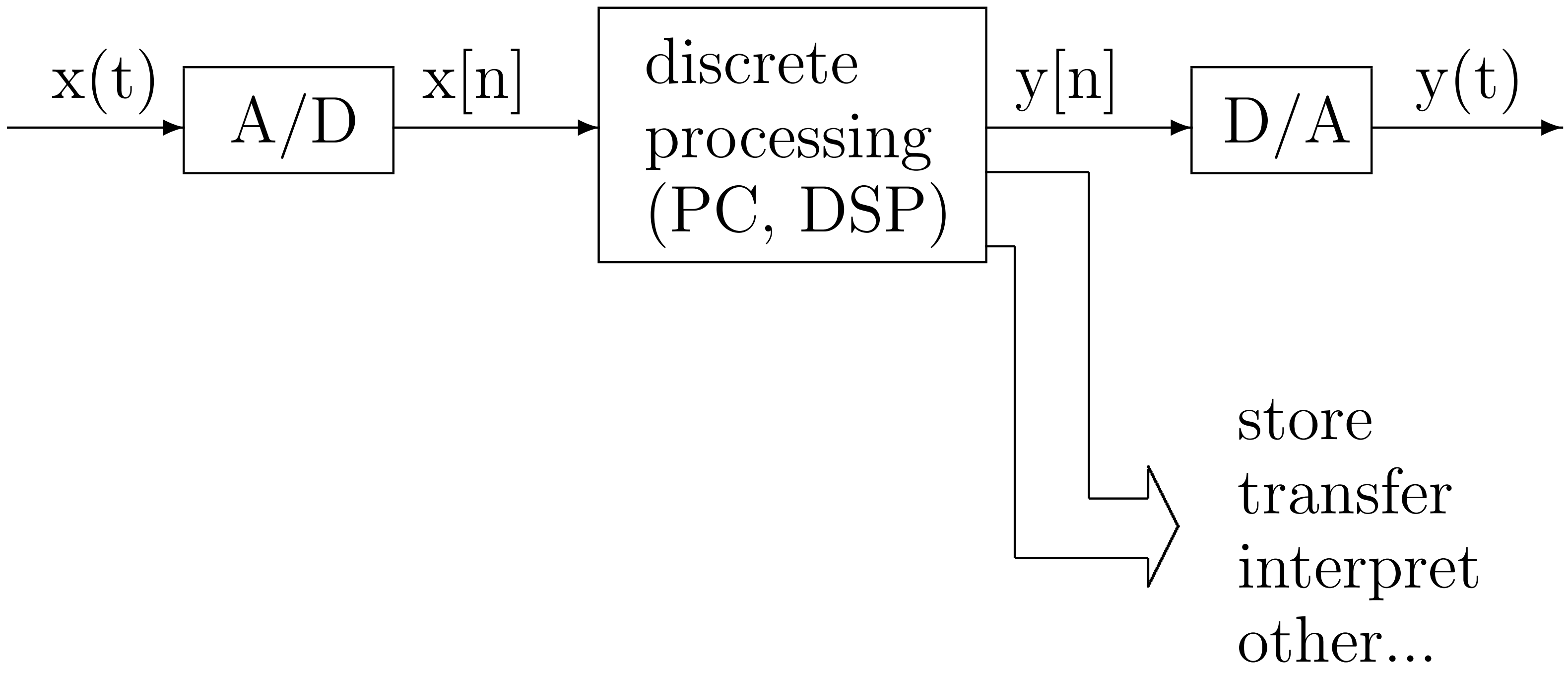
- Ideal sampling – spectrum of a sampled signal.
- Aliasing, Shannon theorem.
- Ideal reconstruction.
- Normalized time and frequency.

Recapitulation – Why we need discrete signal processing ?

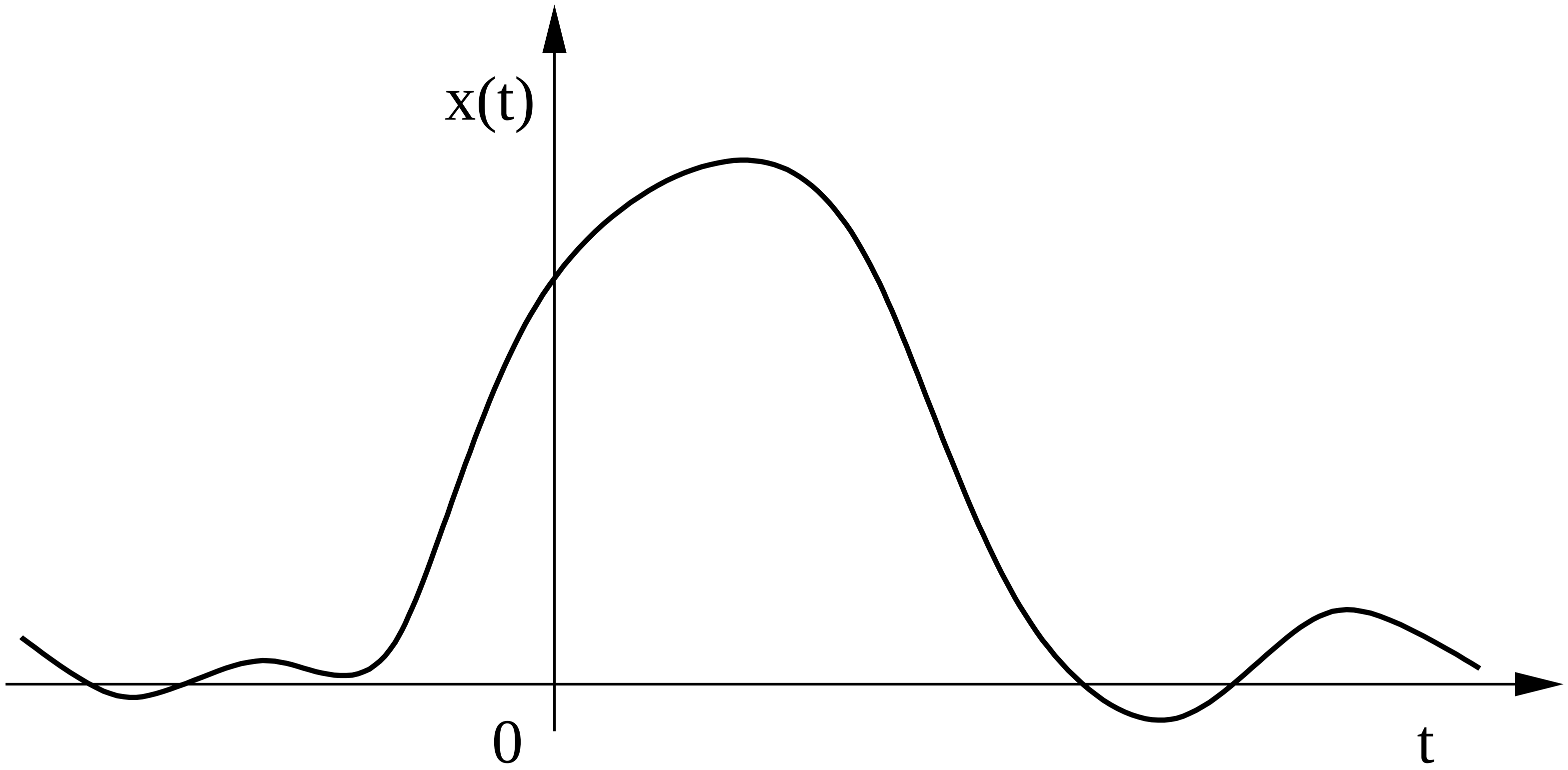
DSP has undisputed advantages compared to classical (although, DSP nowadays is becoming a standard) signal processing.

- reproductibility.
- no changes caused by material aging or temperature.
- no setting and calibration.
- possibile adaptive processing (functionality changes according to the input signal property).
- simulation = application.
- compatible with the boom of computer technology, Internet, mobile communication.

Signal Provcessing Stages



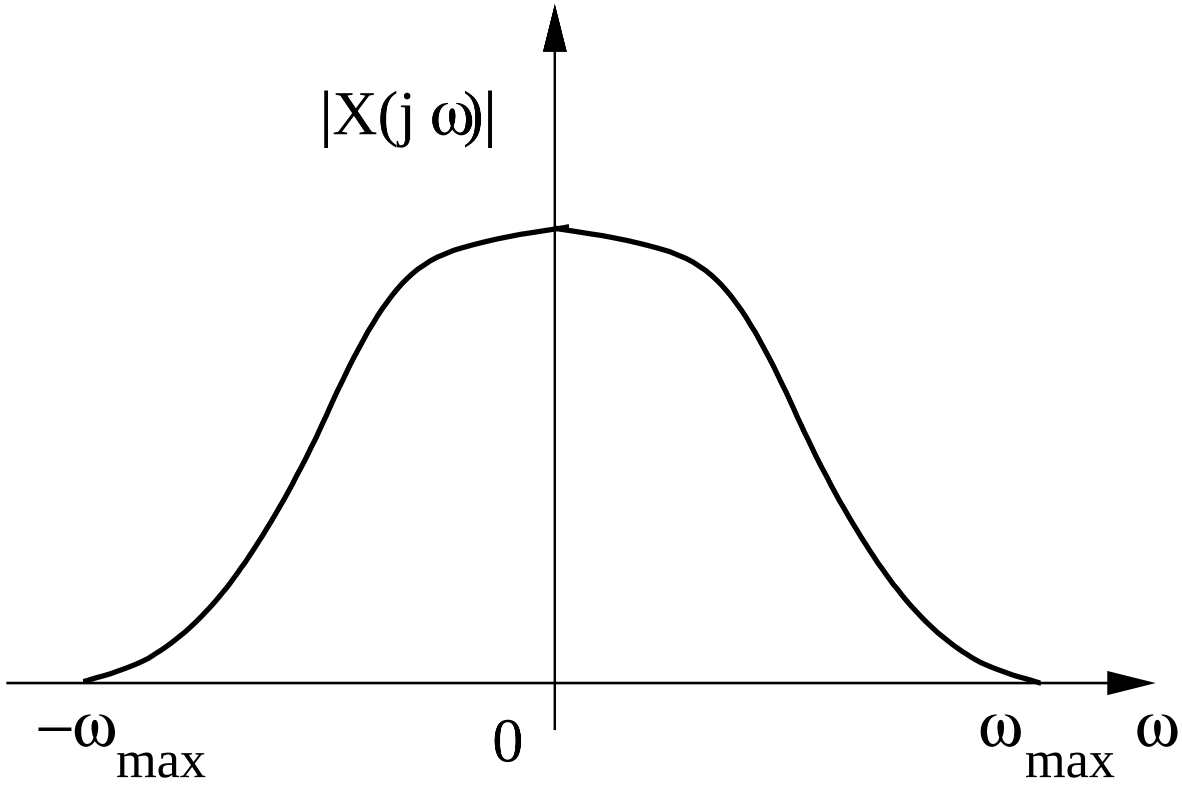
A continuous time signal is defined at every time point t, from $-\infty$ to ∞ (infinite number of values)



Fourier transform is used to represent a signal in **frequency domain**:

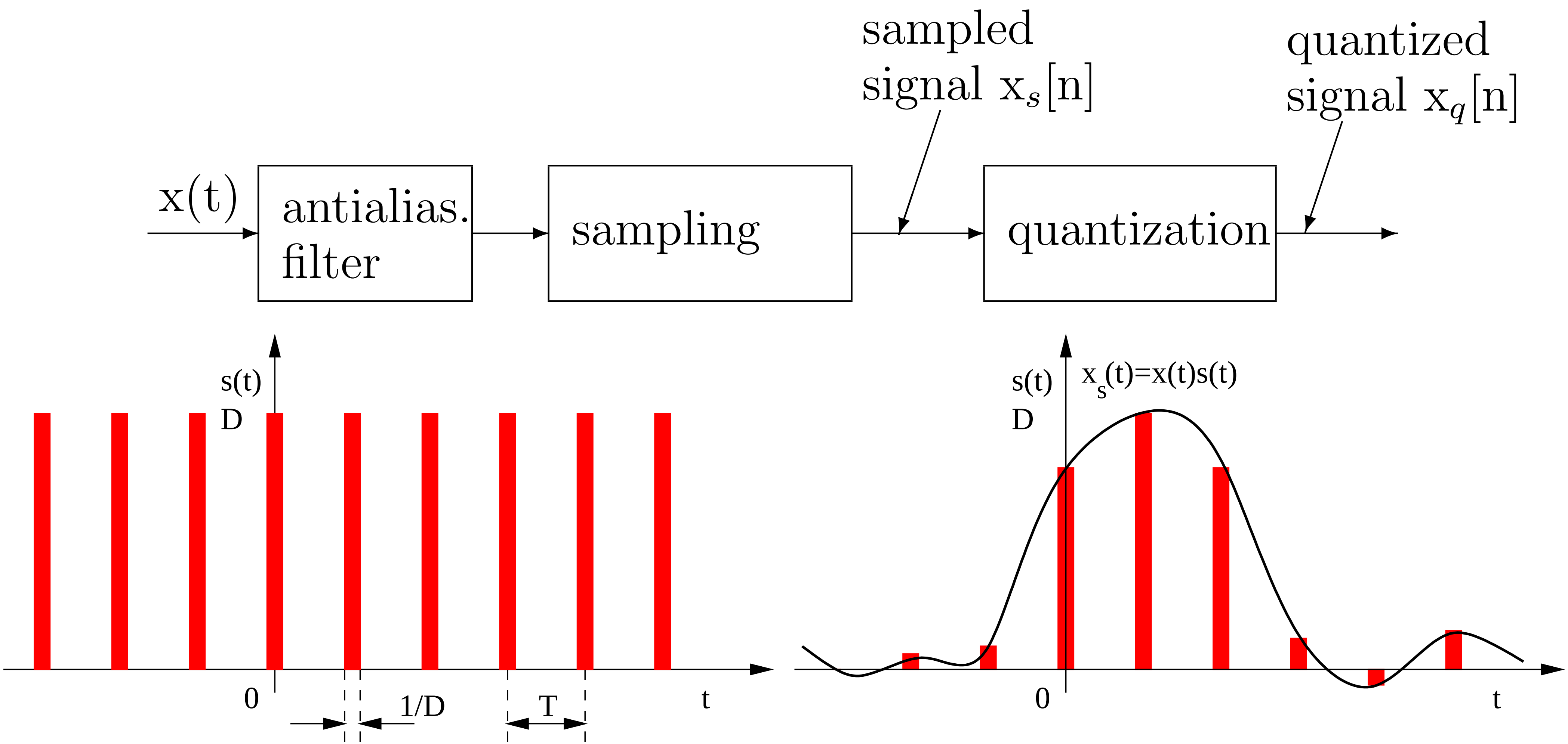
$$X(j\omega) = \int_{-\infty}^{-\infty} x(t)e^{-j\omega t}dt, \tag{1}$$

where $X(j\omega)$ is called **spectral function** or shortly **spectrum**. Real signals are normally bounded in frequency (the energy is concentrated in the band $(0, \omega_{max})$).



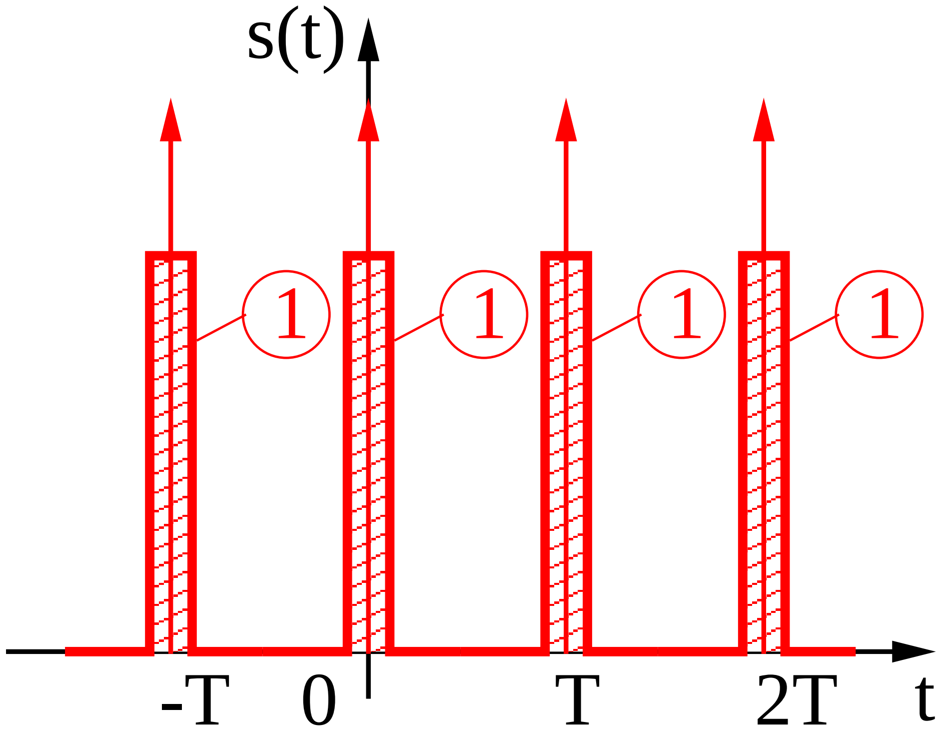
Analog to discrete (AD) transform

Sampled signal is obtained by multiplying the given analog signal with some signal periodic in time. Signal $s(t)$ is called sampling signal.



Theoretically, sampling is understood as multiplying a signal by a sequence of Dirac impulses. We will have to derive spectral function formula for such signal :-(

Spectral function of a periodic sequence of square impulses



Prior to spectral function, we compute coefficients of Fourier series for a periodic sequence of Dirac impulses. We know that for a periodic sequence of square impulses with width ϑ, height D and period T, we compute FS coefficients using:

$$c_k = \frac{D\vartheta}{T} \mathsf{sinc}\left(\frac{\vartheta}{2} k\Omega\right)$$

Consider, we express a Dirac impulse by means of a square signal: D has to equal to $D = \frac{1}{\vartheta}$ for the square's arrea to make 1. If $\vartheta \to 0$, $\frac{1}{\vartheta} \to \infty$, the coefficients become:

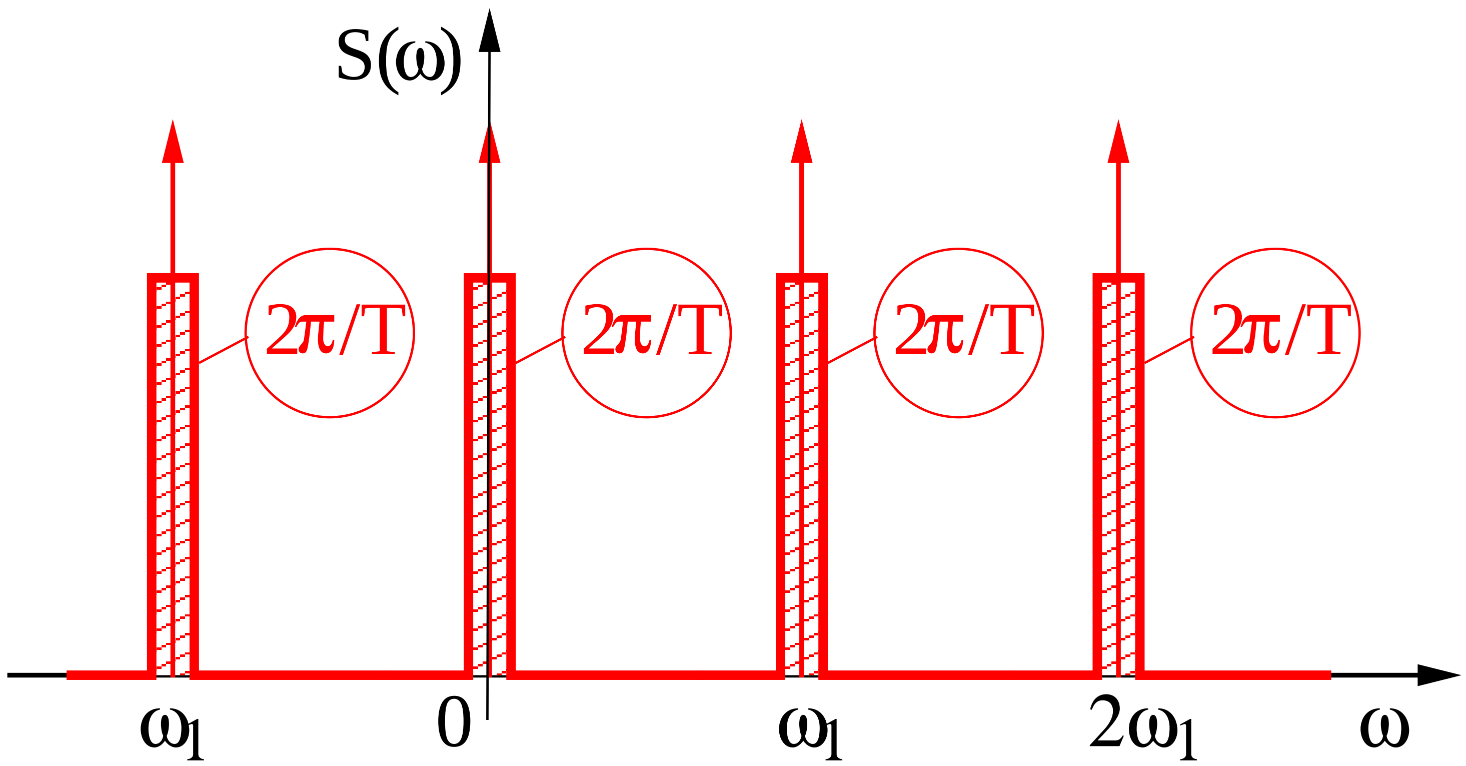
$$c_k = \lim_{\vartheta \to 0} \frac{\frac{1}{\vartheta}\vartheta}{T} \text{sinc}\left(\frac{\vartheta}{2} k\Omega\right) = \frac{1}{T}\text{sinc}(0) = \frac{1}{T}$$

Thus, all FS coefficients of a periodic series of Dirac impulses are equal to $1/T$! To convert FS coefficients to spectral function we have to periodize Dirac impulses (place them to multiples of the basic angular frequency) and their potency set to the value of FS coefficients. For our signal we obtain:

$$S(j\omega) = \sum_{k=-\infty}^{+\infty} 2\pi c_k \delta(\omega - k\omega_1) = \sum_{k=-\infty}^{+\infty} \frac{2\pi}{T} \delta(\omega - k\omega_1)$$

The result is interesting: spectrum of a periodic sequence of Dirac impulses (with period T) is again a periodic sequence of Dirac impulses (now with period $\omega_1 = \frac{2\pi}{T}$)

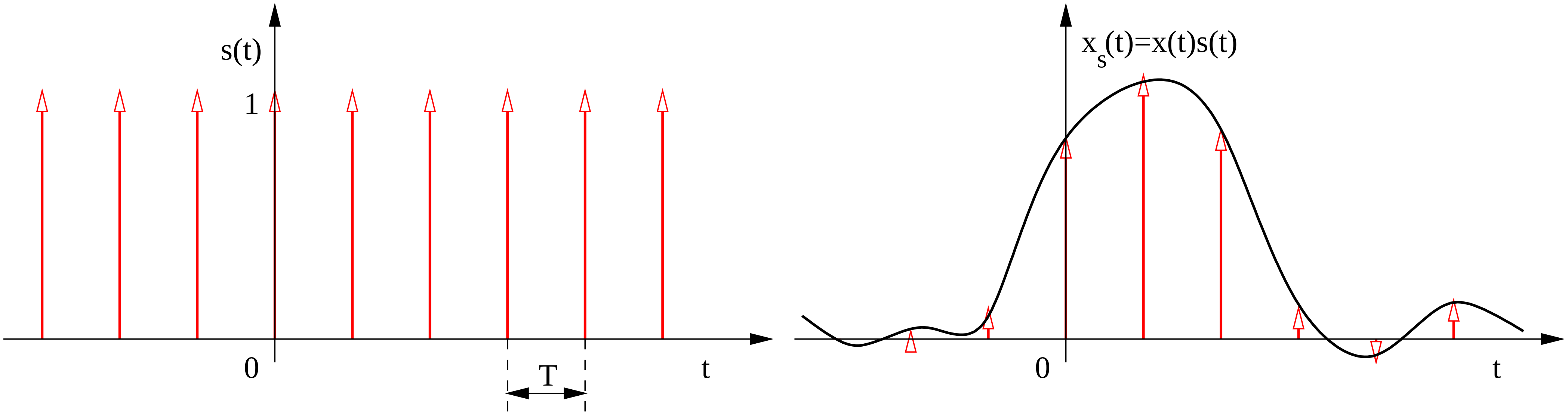
Spectrum of a sampling signal.


S(ω)
2π/T
2π/T
2π/T
2π/T
ω1
0
ω1
2ω1
ω

Multiplication by a sequence of Dirac impulses

When we multiply a signal by a sequence of Dirac impulses $x(t)$, we obtain a sequence of Dirac impulses but their potency (area of a Dirac signal) becomes the value of the original signal at time nT:

$$x_s(t) = x(t)s(t)$$



T is **sampling period** and $F_s = \dfrac{1}{T}$ is **sampling frequency**

Spectrum of a sampled signal

In time domain, we multiplied the original signal and the sampled signal which means we have to apply **convolution** in frequency domain:

$$X_s(j\omega) = \mathcal{F}\{x(t)s(t)\} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} S(\nu)X(\omega-\nu)d\nu =$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left[\frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta(\nu-k\omega_1)\right]X(\omega-\nu)d\nu = \frac{1}{T}\sum_{k=-\infty}^{\infty}\int_{-\infty}^{+\infty}X(\omega-\nu)\delta(\nu-k\omega_1)d\nu =$$

$$= \frac{1}{T}\sum_{k=-\infty}^{\infty}X(\omega-k\omega_1).$$

We used a previously derived formula:

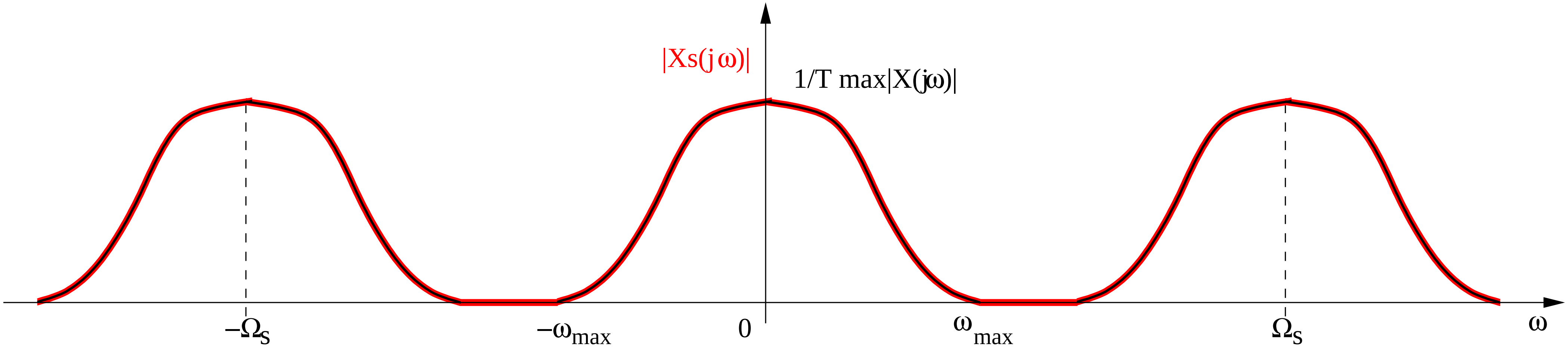
$$\int f(x)\delta(x-x_0)dx = f(x_0),$$

and use substitution : $x=\nu$, $f(x)=X(\omega-\nu)$, $x_0=k\omega_1$. **Spectrum of the original signal periodizes and all coppies add up!**

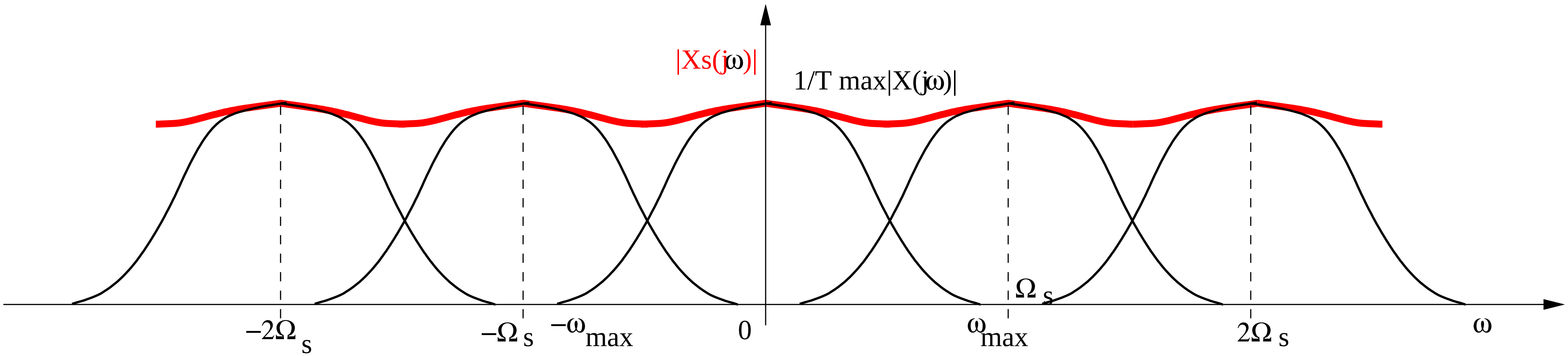
Sampling theorem and aliasing

Depending on the relation of the maximum frequency in the signals spectrum ω_{max} and the sampling frequency $\Omega_s = 2\pi F_s$, we have two cases

1) $\Omega_s > 2\omega_{max}$: Single copies of a sampled signal do not overlap and the signal can be perfectly reconstructed from the spectrum



2) $\Omega_s \leq 2\omega_{max}$: Single coppies of a sampled signal operlap and the original spectrum is modified. We cannot reconstruct the original signal from the spectrum of the sampled signal. This phenomenon is called **aliasing**.



The condition for a correct sampling is called **Shannon Kotelniko Nyquist sampling theorem**

$$\Omega_s > 2\omega_{max}$$
or
$$F_s > 2f_{max}$$

Notes

- the condition should be satisfied even when we do not need reconstruction.
- it is not possible practicly reconstruct ideal “rectangular” low pass filter with passing in the interval $-\omega_{max}$ to $+\omega_{max}$ and supressing the signal elsewhere. Most of the reconstruction filters have cut off band of 30–40 dB.

Reconstruction

we apply a low-pass filter with cutting frequency $\Omega_s/2$ on the sampled signal:

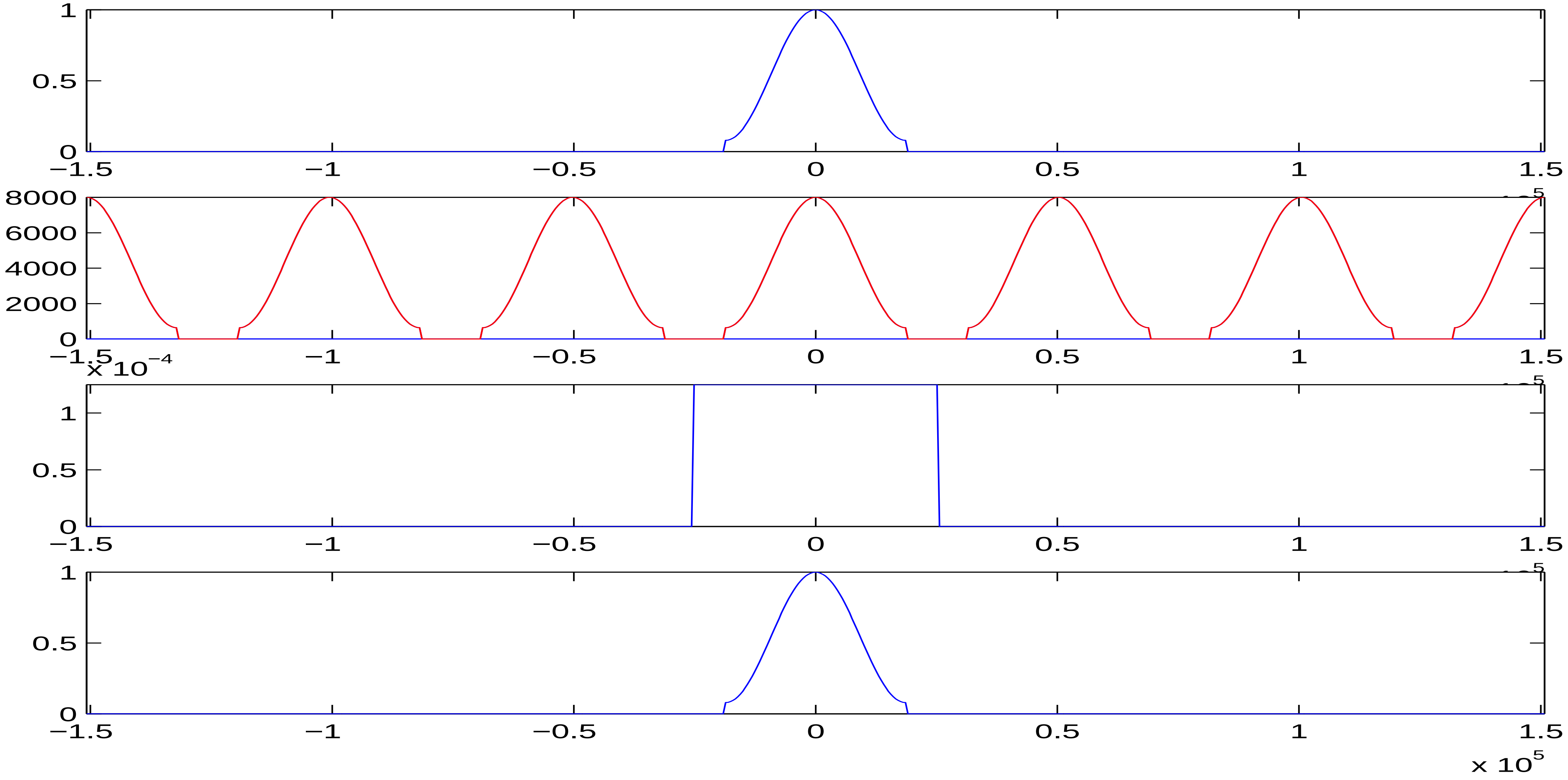
$$H_r(j\omega) = \begin{cases} T \text{ for } & -\Omega_s/2 < \omega < \Omega_s/2 \\ 0 \text{ elsewhere} \end{cases}$$

The value of a frequency characteristic in the pass band is T so we get the same magnitude as in the original spectrum ($\frac{1}{T}T = 1$).

1. Example of sampling and reconstruction – OK

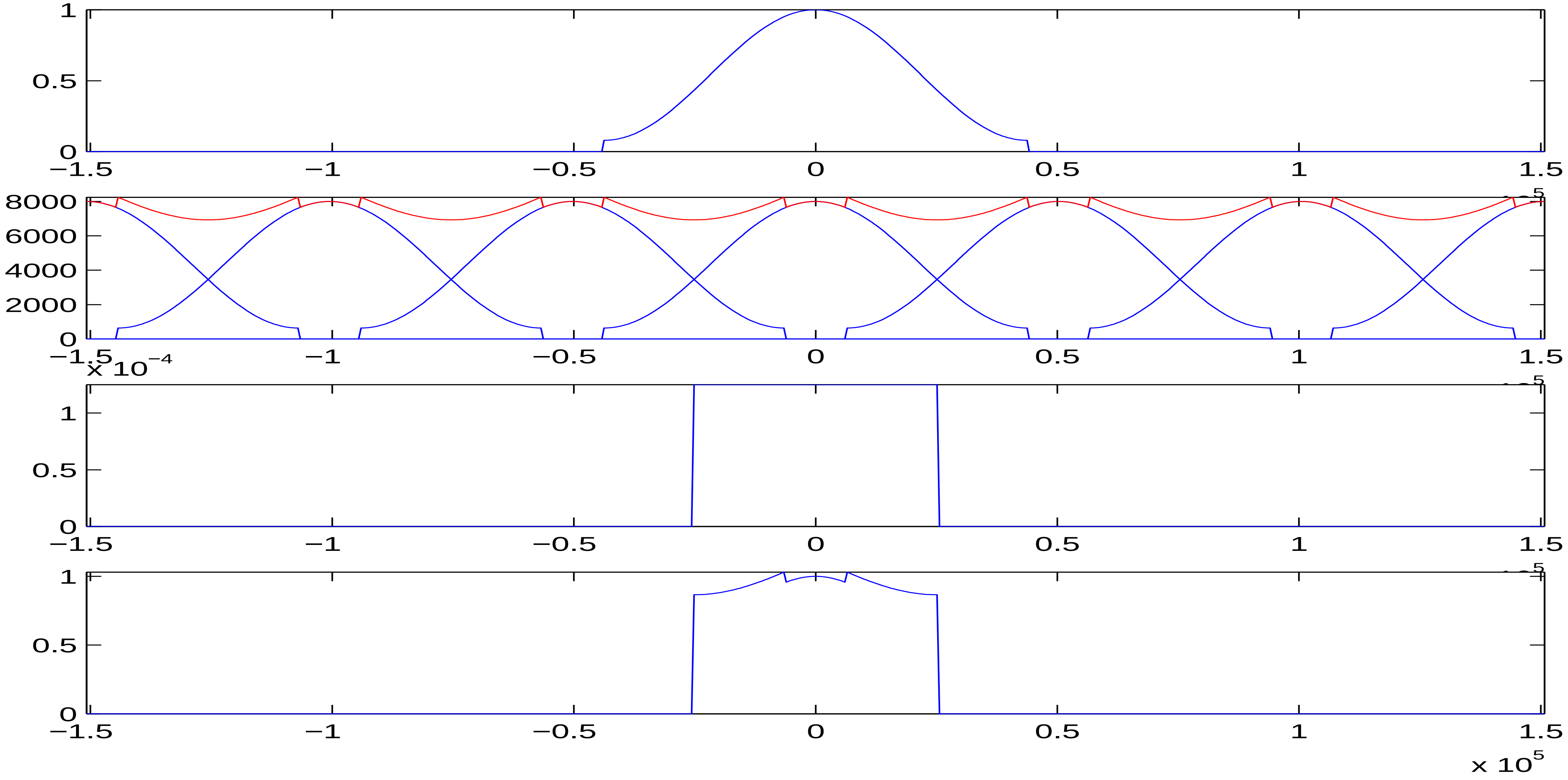
$F_s = 8000\ Hz$, $f_{max} = 3000\ Hz$, a tedy $\Omega_s = 16000\pi$ rad/s, $\omega_{max} = 6000\pi$ rad/s.

$T = \frac{1}{8000}$ s



2. Example of sampling and reconstruction – BAD

$F_s = 8000\ Hz$, $f_{max} = 7000\ Hz$, a tedy $\Omega_s = 16000\pi$ rad/s, $\omega_{max} = 14000\pi$ rad/s.
$T = \frac{1}{8000}$ s

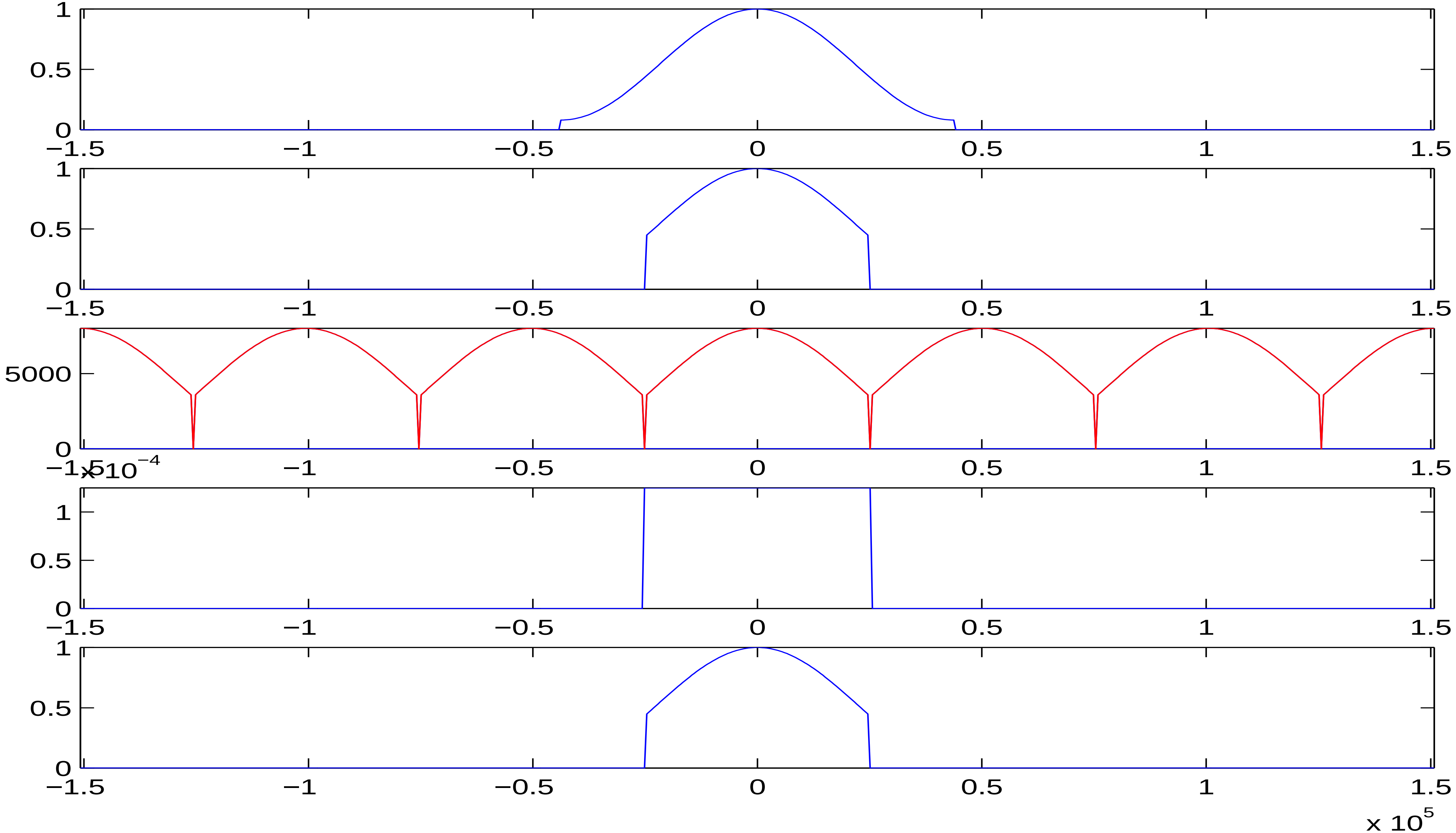


Antialiasing filter

What can we do when we have a sampled signal where the sampling theorem is not satisfied and we cannot change the sampling frequency? Prior to sampling we have to apply an anti-aliasing filter on the sampled signal, that filters off frequencies higher than the half sampling period. The original spectrum is modified before sampling and high freuencies are lost. We cannot restore the original signal but the preprocessed signal that does not contain the high frequencies from the original signal is no longer affected by aliasing.

$$H_{aa}(j\omega) = \begin{cases} 1 & \text{for} \quad -\Omega_s/2 < \omega < \Omega_s/2 \\ 0 & \text{elsewhere} \end{cases}$$

Example 2. with applying an anti-aliasing filter:



Reconstruction in time domain

... or what happens during low pass filtering. Filtering with reconstruction filter with frequency characteristic $H_r(j\omega)$:

$$H_r(j\omega) = \begin{cases} \dfrac{1}{T} & \text{for} \quad -\Omega_s/2 < \omega < \Omega_s/2 \\ 0 & \text{elsewhere} \end{cases}$$

corresponds to convolution with its impulse response $h_r(t)$. Impulse response is an inverse Fourier transform of a square signal:

$$h_r(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H_r(j\Omega_s)e^{+j\omega t}d\omega = \frac{1}{2\pi}\int_{-\Omega_s/2}^{\Omega_s/2} Te^{+j\omega t}d\omega = \frac{T}{2\pi}\int_{-\Omega_s/2}^{\Omega_s/2} e^{+j\omega t}d\omega =$$

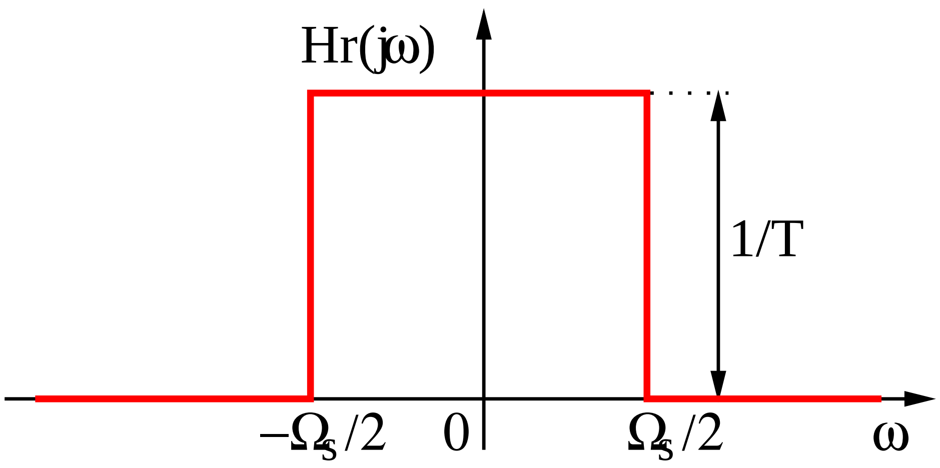
We know that: $\int_{-b}^{b} e^{\pm jxy}dy = 2b\,\text{sinc}(bx)$, where $\vartheta = \Omega_s/2$, $y = \omega$ a $x = t$. We get:

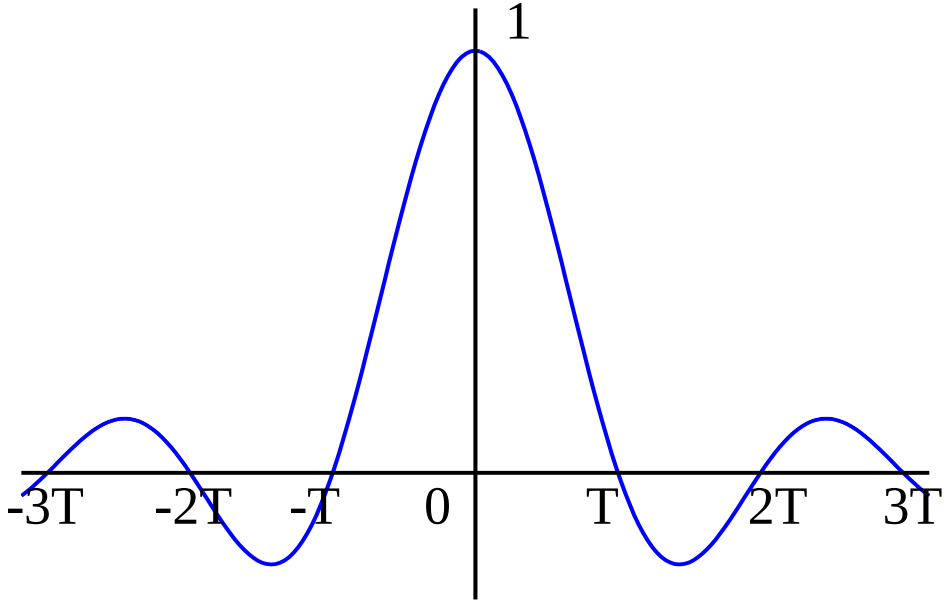
$$h_r(t) = \frac{T}{2\pi}\Omega_s \text{sinc}\left(\frac{\Omega_s}{2}t\right) = \text{sinc}\left(\frac{\Omega_s}{2}t\right)$$

SINC reachs maximum value 1 and is equal to zero at :

$$\frac{\Omega_s}{2} t = \pi, \quad \text{thus} \quad t = \frac{2\pi}{\Omega_s} = \frac{2\pi T}{T} = T.$$

This is interesting:





The reconstructed signal in time domain is:

$$x_r(t) = h_r(t) \star x_s(t)$$

We skip the proof. We have to note though that when we do convolution with a sequence of Dirac impulses ($x_s(t)$ is such a signal), for each Dirac impulse, we obtain an impulse responce at each point nT multiplied with the signal's value at this time point.

For one Dirac in sampled signal:

$$x(nT)\delta(t - nT) \longrightarrow x(nT)\mathsf{sinc}\left(\frac{\Omega_s}{2}(t - nT)\right),$$

For the whole sequence of Diracs all impulse responses add up:

$$y_r = \sum_{n=-\infty}^{\infty} x(nT)\mathsf{sinc}\left(\frac{\Omega_s}{2}(t - nT)\right).$$

Functions sinc interpolate values between single samples. Each sinc function passes zero values for all neighboring samples, thus the value of the reconstructed signal at nT is given by $x(nT)$ and between samples interpolation is made using mainly two adjusted samples but also all other samples.

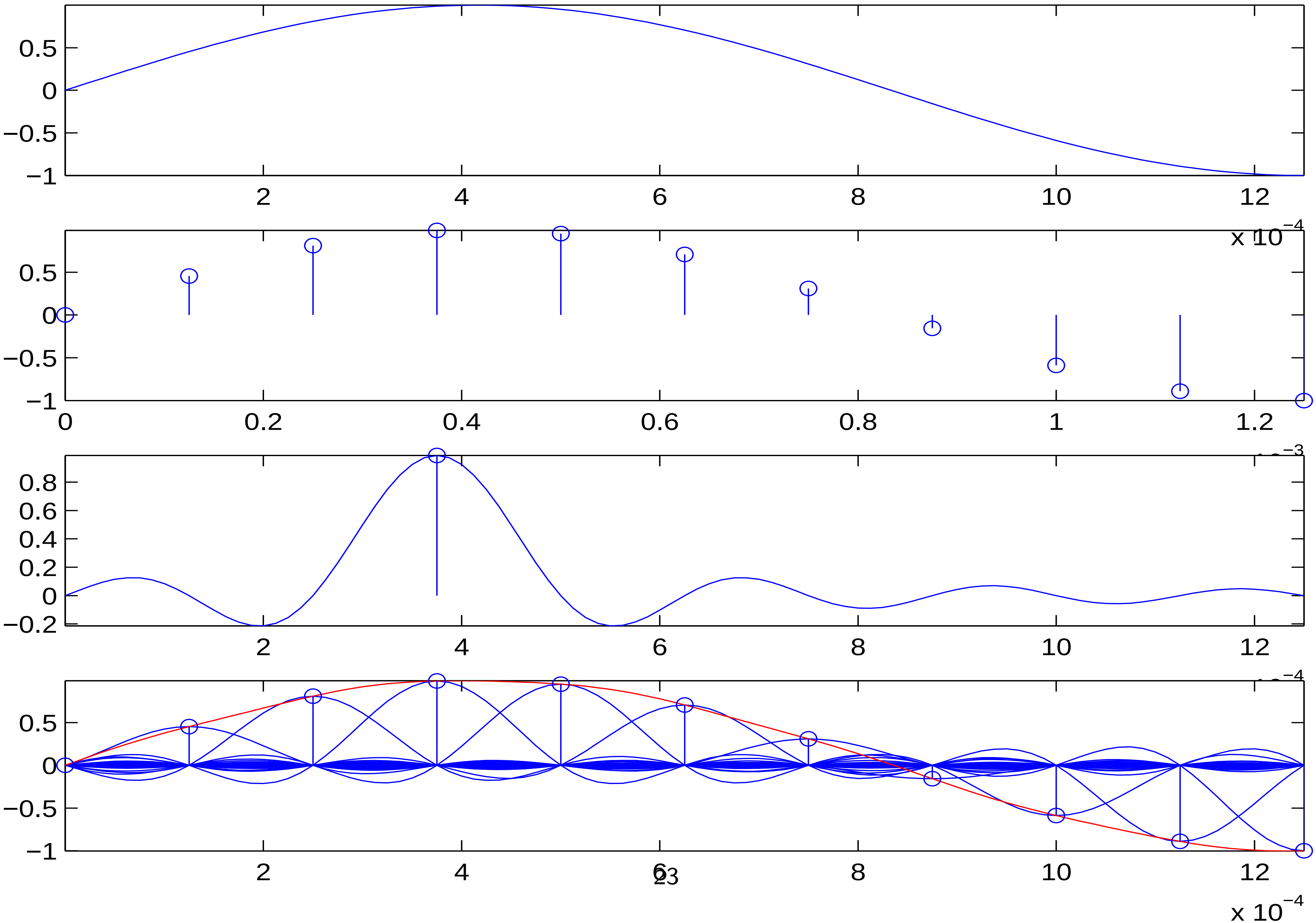
Reconstruction in time domain

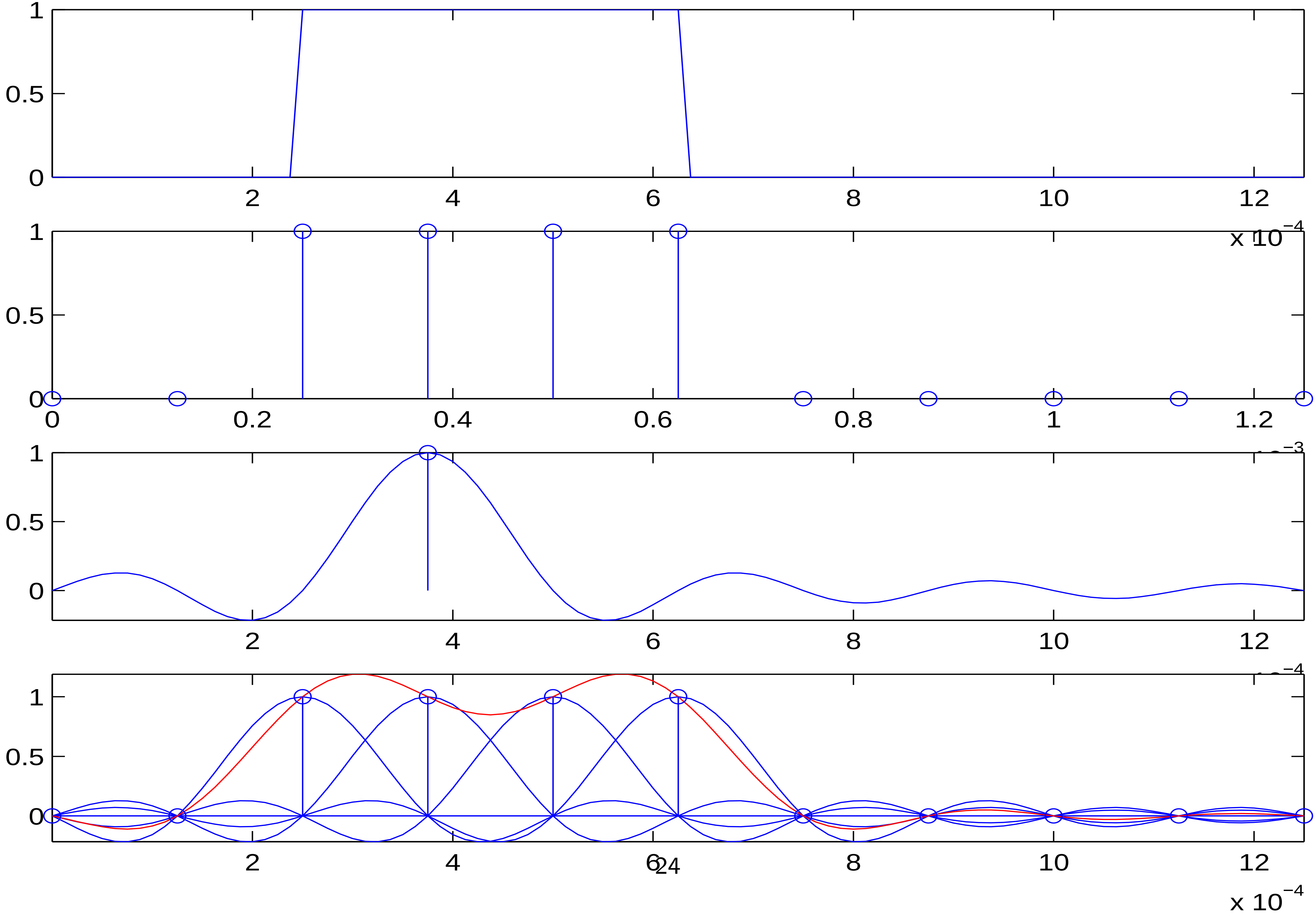
Illustration 1:

$F_s = 8000$ Hz, $\Omega_s = 16000\pi$ rad/s, $T = 1/8000$ s. We sample signal $x(t) = \sin(2\pi 600t)$ using sample frequency $Fs = 600\ Hz$.

Illustration 2: $F_s = 8000$ Hz, $\Omega_s = 16000\pi$ rad/s, $T = 1/8000$ s. We sample a square impuls signal with values 1 at $2T$ to $5T$, and 0 elsewhere.

What happened??? Why the reconstructed signal is not equivalent to the original one?


0.5
0
−0.5
−1
2
4
6
8
10
12
x 10−4
0.5
0
−0.5
−1
0
0.2
0.4
0.6
0.8
1
1.2
0.8
0.6
0.4
0.2
0
−0.2
2
4
6
8
10
12
0.5
0
−0.5
−1
2
4
6
8
10
12
x 10−4


1
0.5
0
2
4
6
8
10
12
x 10^-4
0
0.2
0.4
0.6
0.8
1
1.2

Denotation of a sampled signal

signal $x_s(t)$ can be expressed as a Dirac impulse with potency equal to $x_s(nT)$. As a sampled signal is a **sequence of numbers - discrete signal**, we use standard indeces n: $x_s[n]$.

When working with a sampled signal, we have to know its sampling frequency (implicitly, T, or explicitly, given in the WAV file header).

When working with a sampled signal, we like to avoid real time nT, and instead work with n. This can be viewed as normalized time that is obtained by dividing with the sampling period T:

$$n = \frac{nT}{T}$$

Thus the sampling period of signal $x_s[n]$ becomes 1. Along with time normalization we need to define **normalization of frequency**:

- normal fequency: $f' = \frac{f}{F_s}$, thus F_s becomes 1.
- circular frequvency: $\omega' = \frac{\omega}{F_s}$, thus F_s becomes 2π.

Note that

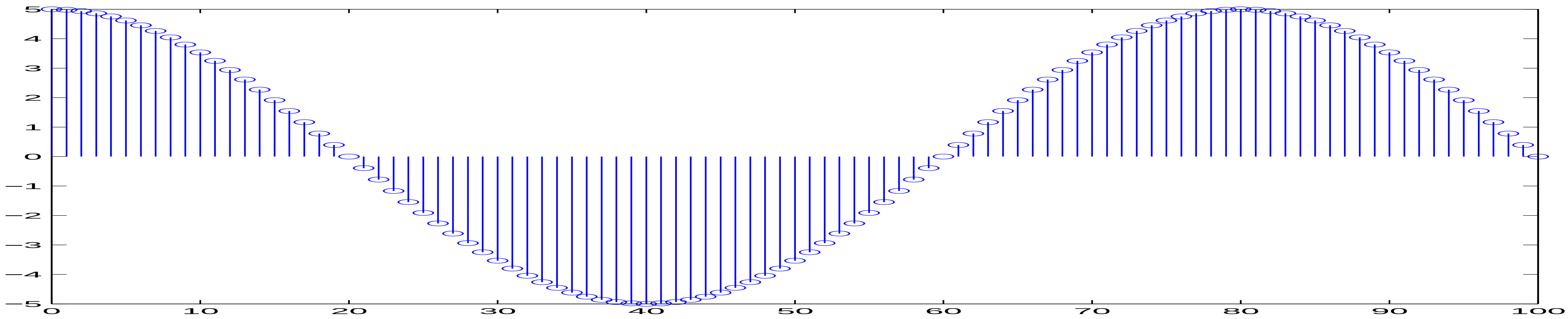
- For convinience and simplification, simbol ω can denote also normalized frequency
- When we work with normalized frequency, we usually dont consider any information on sampling period T.
- When we transform original frequency to normalized, the values in spectrum do not change. Only units and values on x axis change.

Example 1. We are given a continuous time cosine signal with frequency 100 Hz, amplitude 5 and zero initial phase. What is its discrete representation with sampling frequency $F_s = 8000$ Hz. Plot the signal for $n = 0 \ldots 200$.

Solution: $f_1 = 100$ Hz, $\omega_1 = 200\pi$ rad/s, $C_1 = 5$.
$x(t) = C_1 \cos(\omega_1 t + \phi_1) = 5\cos(200\pi t)$.

Normalized frequency is given by : $f_1' = \dfrac{f_1}{F_s} = \dfrac{100}{8000} = 0.0125$. $\omega_1' = 0.025\pi$.
$x[n] = C_1 \cos(\omega_1' n + \phi_1) = 5\cos(0.025\pi n)$.

Matlab: `n = 0:100; xn = 5 * cos(0.025 * pi * n); stem(n,xn)`

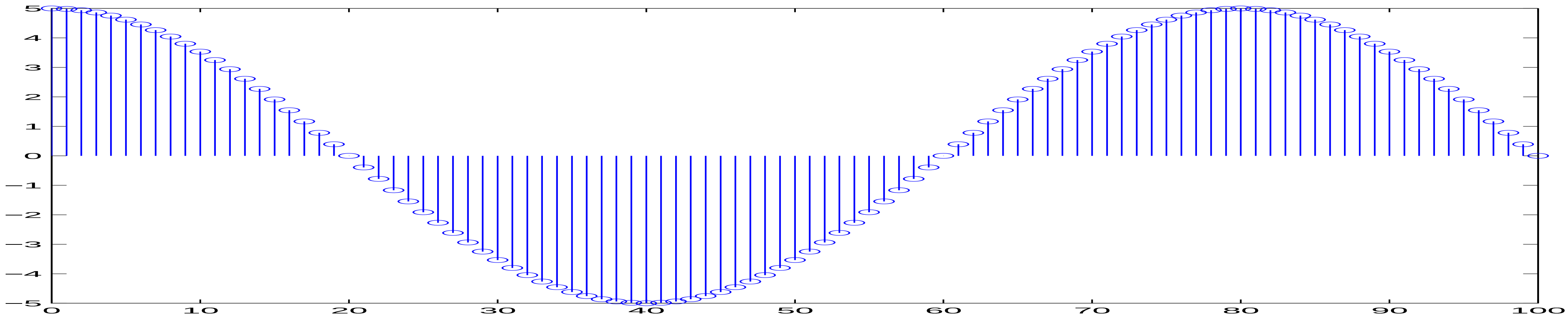


Example 2. We are given a continuous time cosine signal with frequency 8100 Hz, amplitude 5 and zero initial phase. What is its discrete representation with sampling frequency $F_s = 8000$ Hz. Plot the signal for $n = 0 \ldots 200$.

Solution: $f_1 = 8100$ Hz, $\omega_1 = 16200\pi$ rad/s, $C_1 = 5$.
$x(t) = C_1 \cos(\omega_1 t + \phi_1) = 5\cos(16200\pi t)$.

Normalized frequency is given by : $f_1' = \dfrac{f_1}{F_s} = \dfrac{8100}{8000} = 1.0125$. $\omega_1' = 2.025\pi$.
$x[n] = C_1 \cos(\omega_1' n + \phi_1) = 5\cos(2.025\pi n)$.

Matlab: `n = 0:100; xn = 5 * cos(2.025 * pi * n); stem(n,xn)`



What happened? Why did we get the same signal?