IELe - complex numbers

Václav Šátek

Brno University of Technology, Faculty of Information Technology
Božetěchova 1/2. 612 66 Brno - Královo Pole
satek@fit.vutbr.cz

BRNO UNIVERSITY OF TECHNOLOGY FACULTY OF INFORMATION TECHNOLOGY

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Complex numbers

- superset (algebraic extension) of the real numbers, allow rooting of negative numbers
- solution of quadratic equation $x^2 + 1 = 0$
  - doesn't have solution in real numbers
    $D = b^2 - 4ac = -4 \dots$ Discriminant
    $x_{1,2} = \frac{-b \pm \sqrt{D}}{2a} \dots \sqrt{-4} = ?$
  - has a solution in complex numbers ;)
- **representation** of complex numbers: **ordered pair** $[x, y]$ in the Cartesian plane
  - $x \dots$ real part
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Complex numbers

- **algebraic form**: $z = x + jy \in \mathbb{C}$
  - $j \ldots$ imaginary unit (math 'i', electronics 'j')
- complex conjugate for z: $\bar{z} = x$-jy
- holds:
  - $\mathbf{j^2 = -1}$
  - $j^3 = j^2 \cdot j = -j$
  - $j^4 = j^2 \cdot j^2 = 1$
  - $\vdots$
- if:
  - $y = 0 \ldots$ real number
  - $x = 0 \ldots$ pure imaginary number

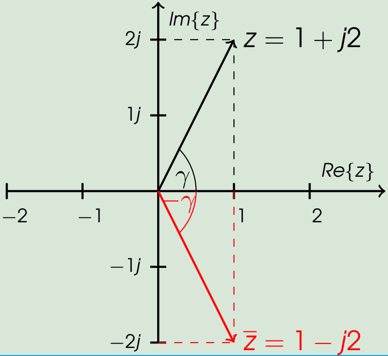
Complex numbers

- What is the solution of the equation $x^2 + 1 = 0$ in $\mathbb{C}$?
  $\sqrt{D} = \sqrt{-4} = \sqrt{j^2 4} = j2$
  $x_{1,2} = \frac{\pm j2}{2} = \pm j$
- roots of quadratic equation with negative discriminant are **complex conjugate**

- complex numbers are represented graphically using **Gauss** (complex) **plane**:
  - Cartesian orthogonal coordinate system
  - x axis – real part (Re) of a complex number
  - y axis – imaginary part (Im) of a complex number

Graphical representation of complex numbers $z = 1 + j2$, $\overline{z} = 1 - j2$

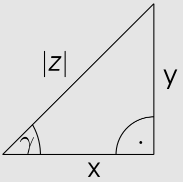
Complex number $z = 1 + j2$ and a conjugate $\overline{z} = 1 - j2$



- complex number is represented as a **vector with orientation** $\vec{0z}$
- vector is used in **polar form**: $z = |z|\,(\cos\gamma + j\sin\gamma)$
  - $|z|\ldots$ absolute value or modulus (size of vector $\vec{0z}$)
  - $\gamma\ldots$ angle with orientation (argument)

Goniometric functions

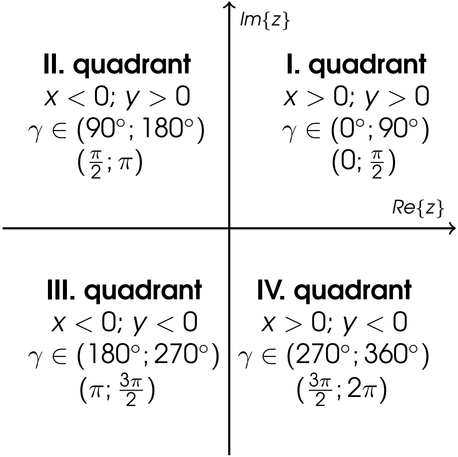
- right angle triangle



- $\sin\gamma = \frac{y}{|z|}$
- $\cos\gamma = \frac{x}{|z|}$
- $|z| = \sqrt{x^2 + y^2}$

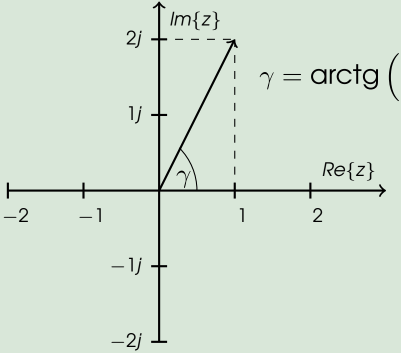
- $\mathrm{tg}\,\gamma = \frac{y}{x}$
  $\gamma = \mathrm{arctg}\,\left(\frac{y}{z}\right)$
- $D(\mathrm{tg}) = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

To obtain the correct result, it is necessary to determine the correct quadrant.



Computation of argument γ

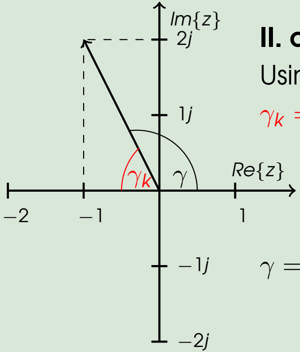
What is the argument γ for complex number $z = 1 + j2$?



$$\gamma = \text{arctg}\left(\frac{2}{1}\right) = \text{arctg}\,(2) \doteq 1.10715\,\text{rad}$$
$$\doteq \frac{180}{\pi} 1.10715 \doteq 63,44^\circ$$

I. quadrant ($x = 1 > 0$, $y = 2 > 0$), result is correct.

Graphical representation of $z_2 = -1 + j2$



II. quadrant, outside of the Domain of tg

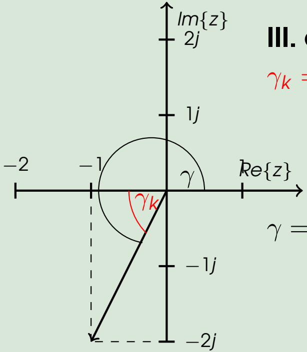
Using a 'calculator':

$$\gamma_k = \text{arctg}\left(\frac{2}{-1}\right) = \text{arctg}\,(-2) \doteq -1.1071\,\text{rad}$$
$$\doteq -\frac{180}{\pi}1.1071 \doteq -63,43^\circ$$

Need to add π (or 180°):

$$\gamma = \pi + \gamma_k = 3,1416 - 1,1071 = 2,0345\,\text{rad}$$
$$\doteq \frac{180}{\pi}2,0345 \doteq 116,57^\circ$$

Graphical representation of $z_3 = -1 - j2$



III. quadrant, outside of the Domain of tg

$$\gamma_k = \mathrm{arctg}\left(\frac{-2}{-1}\right) = \mathrm{arctg}\,(2) \doteq 1,1071\,\mathrm{rad} \doteq \frac{180}{\pi}1,1071 \doteq 63,43^\circ$$

$$\gamma = \pi + \gamma_k = 3,1416 + 1,1071 = 4,2487\,\mathrm{rad} \doteq \frac{180}{\pi}4,2487 \doteq 243,43^\circ$$

Is **algebraic form** $z = x + jy$ equivalent to the **polar form** $z = |z|\,(\cos\gamma + j\sin\gamma)$?

For $z = 1 + j2$

- $|z| = \sqrt{1^2 + 2^2} = \sqrt{5}$
- $\cos\gamma = \frac{x}{|z|} = \frac{1}{\sqrt{5}}$
- $\sin\gamma = \frac{y}{|z|} = \frac{2}{\sqrt{5}}$

After substitution

$$1 + j2 = \sqrt{5}\left(\frac{1}{\sqrt{5}} + j\frac{2}{\sqrt{5}}\right) = 1 + j2$$

Note. **When multiplying by the real number, real and imaginary parts are multiplied separately!**

COMPLEX NUMBERS OPERATIONS

Complex numbers operations

Addition: $z_1 + z_2$

$$(x_1 + jy_1) + (x_2 + jy_2) = (x_1 + x_2) + j(y_1 + y_2)$$

Subtraction: $z_1 - z_2$

$$(x_1 + jy_1) - (x_2 + jy_2) = (x_1 - x_2) + j(y_1 - y_2)$$

Multiplication: $z_1 \cdot z_2$

$$\begin{aligned}(x_1 + jy_1) \cdot (x_2 + jy_2) &= (x_1x_2 + jx_1y_2 + jy_1x_2 + j^2y_1y_2) \\ &= (x_1x_2 - y_1y_2) + j(x_1y_2 + y_1x_2)\end{aligned}$$

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Division: $\frac{z_1}{z_2}$

Using the **complex conjugate** to the denominator and multiplying the nominator and denominator (real number in denominator).

$$\frac{x_1 + jy_1}{x_2 + jy_2} = \frac{(x_1 + jy_1) \cdot (x_2 - jy_2)}{(x_2 + jy_2) \cdot (x_2 - jy_2)} = \frac{(x_1 x_2 + y_1 y_2) + j(y_1 x_2 - x_1 y_2)}{x_2^2 + y_2^2}$$
$$= \frac{x_1 x_2 + y_1 y_2}{x^2 + y^2} + j\frac{y_1 x_2 - x_1 y_2}{x^2 + y^2}$$

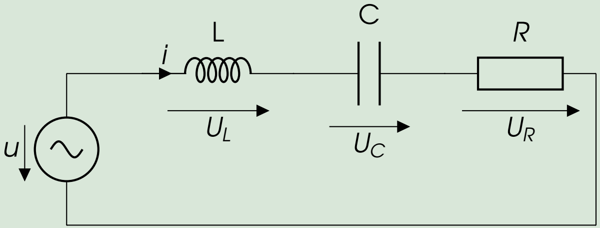
Example: $z_1 = 2 + j3$, $z_2 = 2 - j3$

$$\frac{2 + j3}{2 - j3} = \frac{4 - 9}{4 + 9} + j\frac{6 - 2 \cdot (-3)}{4 + 9} = -\frac{5}{13} + j\frac{12}{13}$$

EXAMPLE: RLC CIRCUITS

- Calculate overall voltage U in the serial RLC circuit.

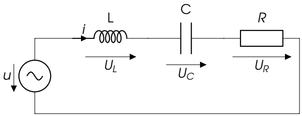
$U_R = 4\,\text{V}$, $U_C = -j1\,\text{V}$, $U_L = j3\,\text{V}$, $I = 2\,\text{A}$



Using II. Kirchhoff's law:

$$U_R + U_C + U_L - U = 0$$
$$U = U_R + U_C + U_L$$
$$U = 4 + j2$$

Serial RLC circuit: $U_R = 4\,\text{V}$, $U_C = -j1\,\text{V}$, $U_L = j3\,\text{V}$, $I = 2\,\text{A}$



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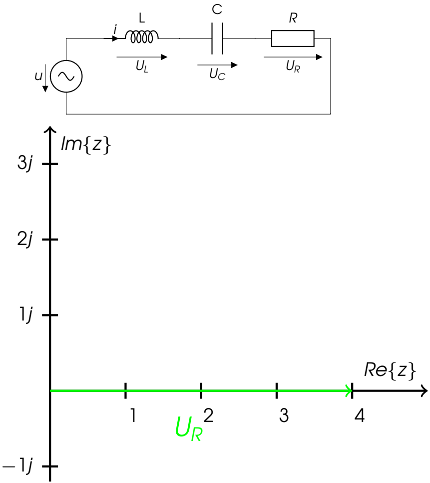

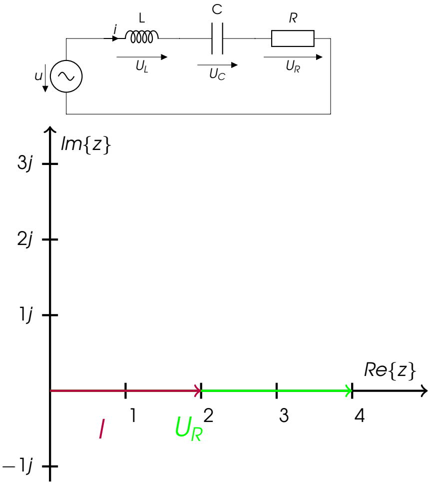

L
C
R
i
u
U_L
U_C
U_R
$Im\{z\}$
$Re\{z\}$
3j
2j
1j
−1j
1
2
3
4
I
U_R

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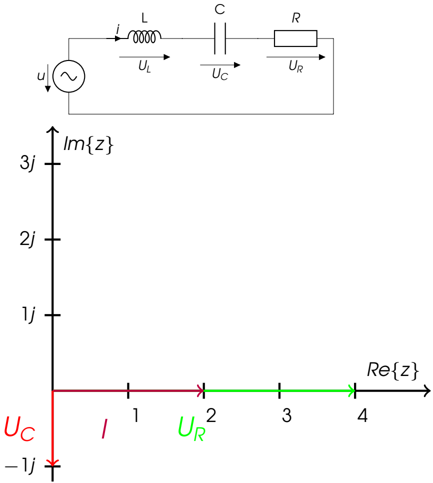

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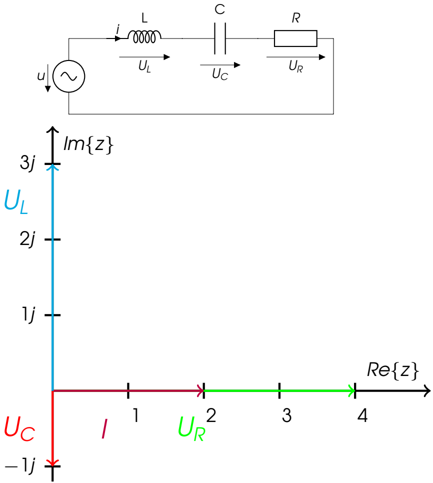

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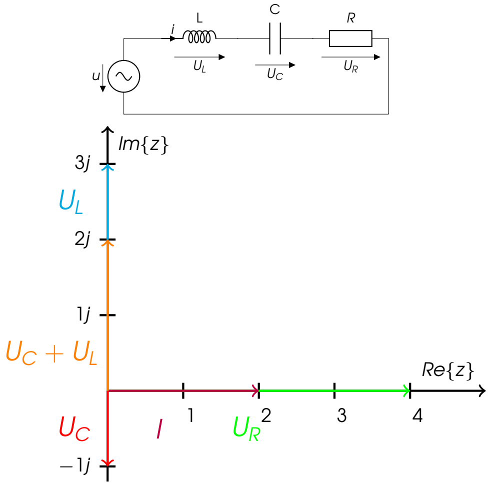
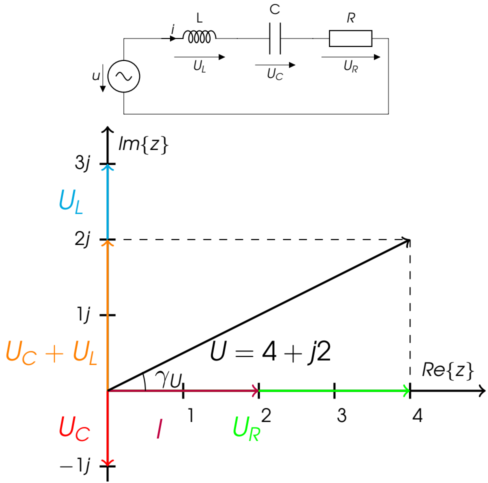
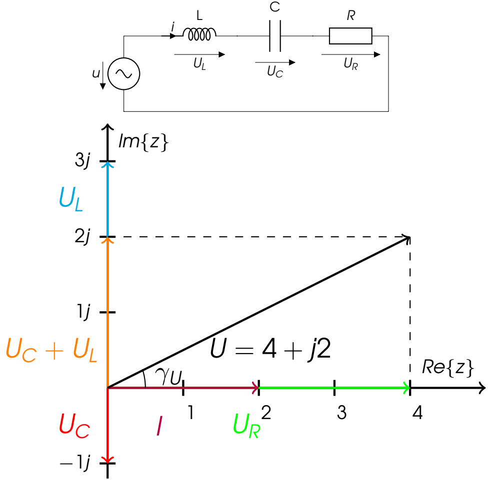




L
C
R
i
U_L
U_C
U_R
u
$Im\{z\}$
$3j$
U_L
$2j$
$1j$
$U_C + U_L$
$U = 4 + j2$
γ_U
$Re\{z\}$
1
2
3
4
U_C
I
U_R
$-1j$



Argument γ_U of the voltage U depends on the size of the voltage on the capacitor U_C and coil U_L (on their addition). When $|U_C| = |U_L|$ $\gamma_U = 0$ ($U_C + U_L = 0$), **resonance** occurs.

Thank you for your attention!