Brief evaluation of problems in a subject "Theory of curcuits"

December 9, 2013

1 Intro

Theory of curcuit deals with analysis, synthesis, description, design and mutual connections of electronic elements.
There are very complex and vast connections of electronic elements with very complicated function. However, this connections can be dissasembled into individual functioning curcuits, whose description does not have to be so much complicated. Individual curcuits can be divided into:

1. Linear curcuits
   (a) mutual connections of resistors and DC (direct current) voltage sources
   (b) mutual connections of resistors, capacitors, coils and AC (alternate current) voltage sources
2. Non-linear curcuits
   (a) mutual connections of passive non-linear semicondacting elements, resistors, capacitors, coils and voltage sources (DC/AC)
   (b) mutual connections of active semicondacting elements (transistors), passive elements (diods), RLC curcuits and voltage sources

2 Mutual connections of resistors and DC (direct current) voltage sources

The most important axiom of curcuits is Ohm's law, which defines a relation between voltage U, resistance R and current I flowing through an electric curcuit (figure 1).

$$I = \frac{U}{R} \tag{1}$$

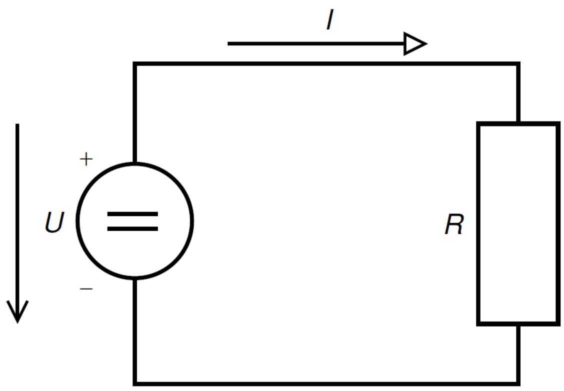

Figure 1: A simple curcuit for Ohm's law

In connection with Ohm's law and (figure 1), there is another elementary curcuit with two resistors in serial and one voltage source (figure 2).

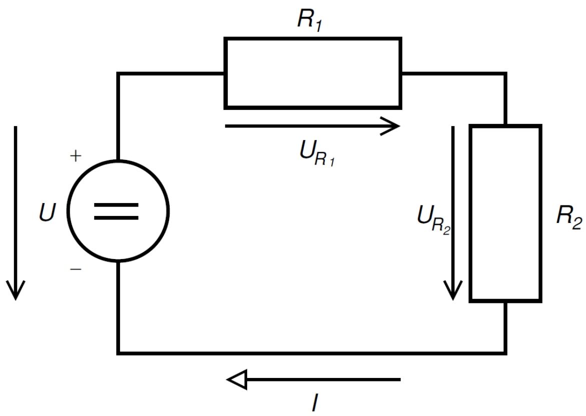

Figure 2: Voltage divider

Even if this curcuit is very simple (called voltage divider), it is a basis for methods for solving electrical curcuits. Often, some complex curcuits are transformed into this one.
Two very important laws hold for this curcuit:

- II. Kirchhoff's law ($U = U_{R_1} + U_{R_2}$)
- Ohm's law ($I = \frac{U}{R_1+R_2}$)

More, the following axiom holds for the same curcuit: voltages U_{R_1} and U_{R_2} are divided directly between R_1 and R_2:

$$\frac{U_{R_1}}{U_{R_2}} = \frac{R_1}{R_2} \tag{2}$$

From the curcuit (figure 2) and II. Kirchhoff's law, there is only a small step to a curcuit (figure 3), where R_1 and R_2 are connected in parallel.

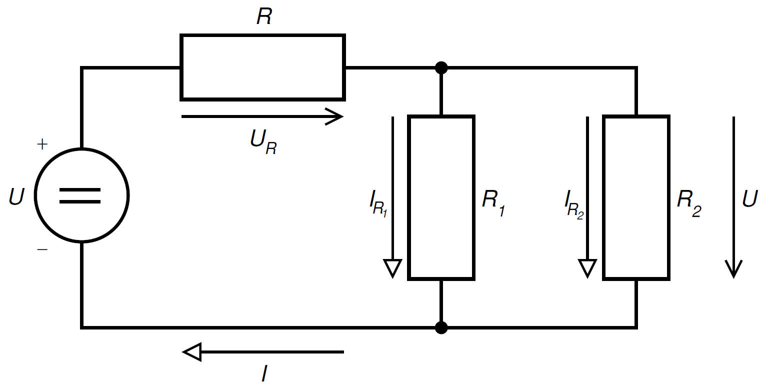

Figure 3: A curcuit with two loops

In this curcuit, I. Kirchhoff's law holds: $I = I_{R_1} + I_{R_2}$. Together with II. Kirchhoff's law $U = U_R + U_{R_{1,2}}$ and Ohm's law we get:

$$I = \frac{U}{R + \frac{R_1 R_2}{R_1 + R_2}} \tag{3}$$

Next, another important axiom holds for this curcuit: currents I_{R_1} and I_{R_2} are divided in an indirect proporsion to R_1 and R_2:

$$\frac{I_{R_1}}{I_{R_2}} = \frac{R_2}{R_1} \tag{4}$$

From a point of view of possible connections of resistors and one DC voltage source a curcuit on figure 4 is interesting.

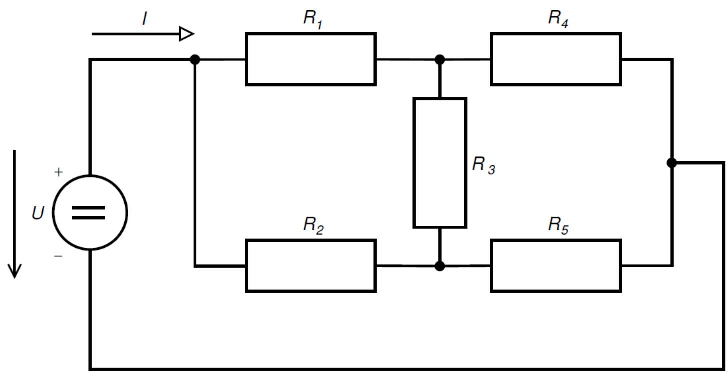

Figure 4: More complex connection of resistors

In this curcuit it is not unambiguous to decide which resistors are connected in serial and which ones in parallel. Mathematically, it can be proved that "triangual" connection of resistors R_1, R_2, R_3 can be transformed into an equivalent curcuit (figure 5). In this simplified curcuit, it is now easy to simplify it except for Ohm's law $I = \frac{U}{R_{ekv}}$.

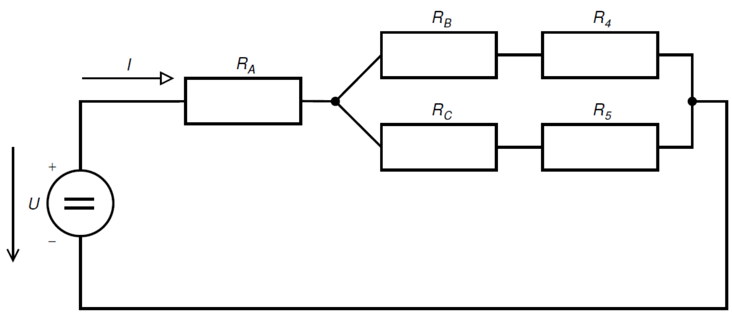

Figure 5: A simplified curcuit from the figure 4

From a point of view of resistor's nets and one voltage source, a curcuit on figure 6 has important meaning (called $R - 2R$ distributed element model). It is used in analog-to-digital (A/D) and digital-to-analog (D/A) converters (e.g. scanners, digital cameras), where a continuous analog signal is converted into a digital one (for postprocessing in digital computers). Eventually in conversion of a digital signal to an analog one.

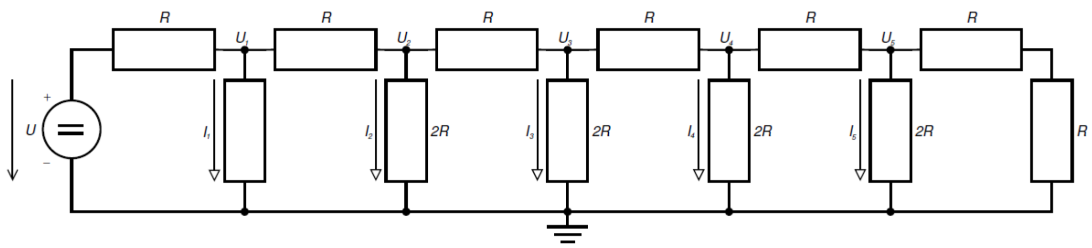

Figure 6: A distributed element model

The following holds:

$$
\begin{array}{ll}
U_1 = \frac{U}{2} = 2^{-1}U & I_1 = \frac{I}{2} = 2^{-1}I \\
U_2 = \frac{U_1}{2} = 2^{-2}U & I_2 = \frac{I_1}{2} = 2^{-2}I \\
U_3 = \frac{U_2}{2} = 2^{-3}U & I_3 = \frac{I_2}{2} = 2^{-3}I \\
\ldots & \ldots
\end{array}
\tag{5}
$$

Flowing currents are decreased exponentially (with a base 2).

In connection with the previous curcuits, a simple curcuit with a resistor's net and two voltage sources (figure 7) follows.

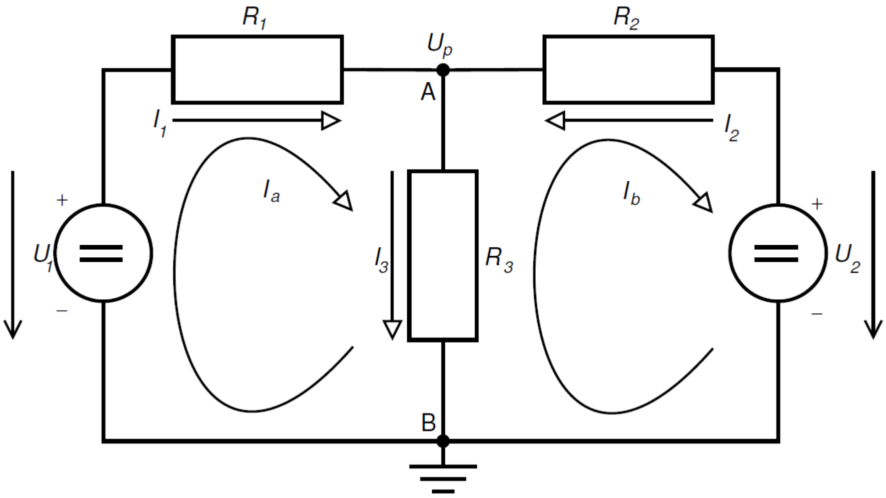

Figure 7: A curcuit with two voltage sources

To solve this curcuit, two methods can be used:

1. A method of loop currents:

$$\begin{gathered} R_1 I_A + R_3(I_A - I_B) - U_1 = 0 \\ R_2 I_B + U_2 + R_3(I_B - I_A) = 0 \\ \Rightarrow I_a = \ldots I_b = \ldots \end{gathered} \tag{6}$$

2. A method of loop's voltages:

$$\begin{gathered} I_1 + I_2 = I_3 \\ \frac{U_1 - U_p}{R_1} + \frac{U_2 - U_p}{R_2} = \frac{U_p}{R_3} \\ \Rightarrow U_p = \ldots \end{gathered} \tag{7}$$

The stated methods are used whenever we need to compute all currents and all voltages in a curcuit, i.e. $U_{R_1}, I_{R_1}, U_{R_2}, I_{R_2}, U_{R_3}, I_{R_3}$. If we want to compute only one current (e.g. I_3), we can use "one crocodile"'s method or Thevenin's theorem.
"One crocodile"'s method describes an original way how to catch one crocodile. It is simple - we catch two crocodiles and release one (i.e. we compute all currents I_1, I_2, I_3 and I_1, I_2 forget).
Based on Thevenin's theorem we mark on the previous curcuit (figure 7) two points A, B , among which I_3 flows and redraw it into an equivalent curcuit (figure 8).

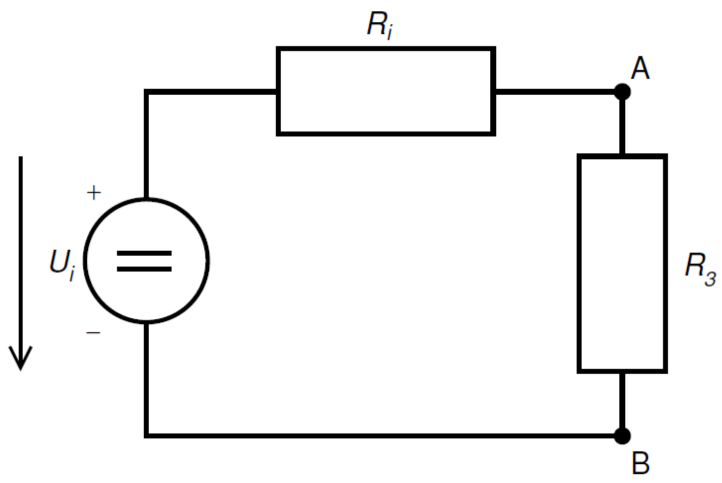

Figure 8: An equivalent curcuit for curcuit on figure 7

Then:

$$I_3 = \frac{U_i}{R_i + R_3} \tag{8}$$

R_i can be computed as a resistance (figure 7) between A, B (withnout resistor R_3), voltage sources are short-curcuited (figure 9).

$$R_i = \frac{R_1 R_2}{R_1 + R_2} \tag{9}$$

U_i can be computed as a voltage between A, B (without R_3) - figure 10.
E.g.:

$$\begin{array}{ll} IR_1 + U_i - U_1 = 0 & IR_1 + IR_2 + U_2 - U_1 = 0 \\ & I = \frac{U_1 - U_2}{R_1 + R_2} \\ \frac{U_1 - U_2}{R_1 + R_2} R_1 + U_i - U_1 & \\ & \Rightarrow U_i = \ldots \end{array} \tag{10}$$

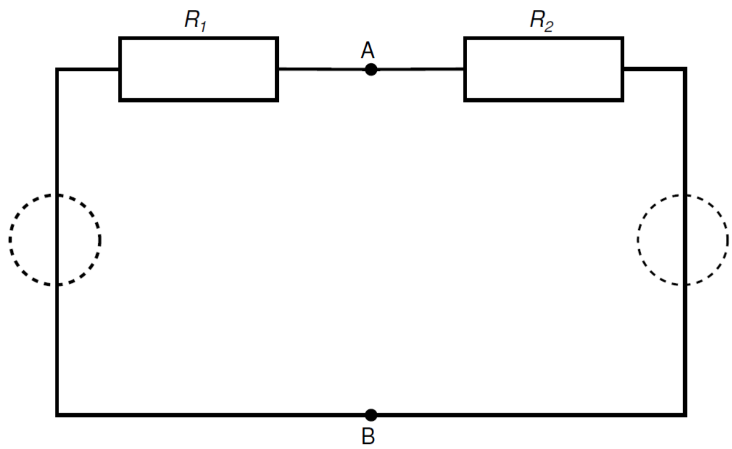

Figure 9: Determination of R_i

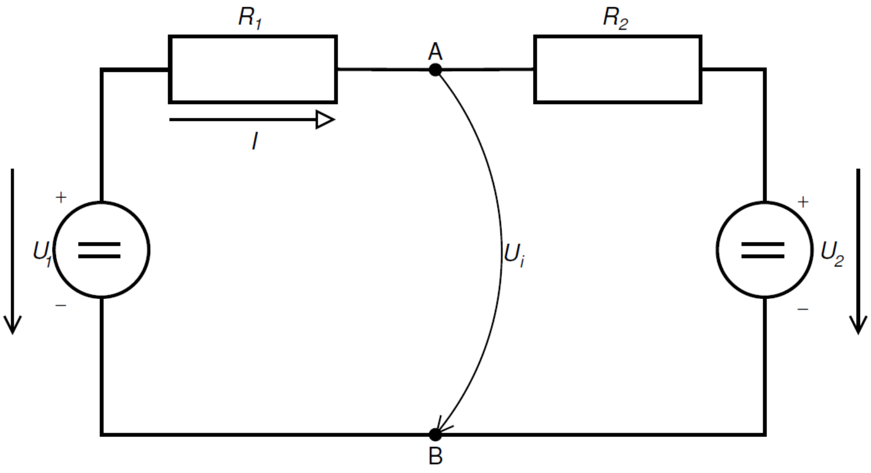

Figure 10: Determination of U_i

Higher amount of voltage sources and resistor's nets is dependend on a maliciousness of a lecturer or a guarantor of this subject (figure 11).

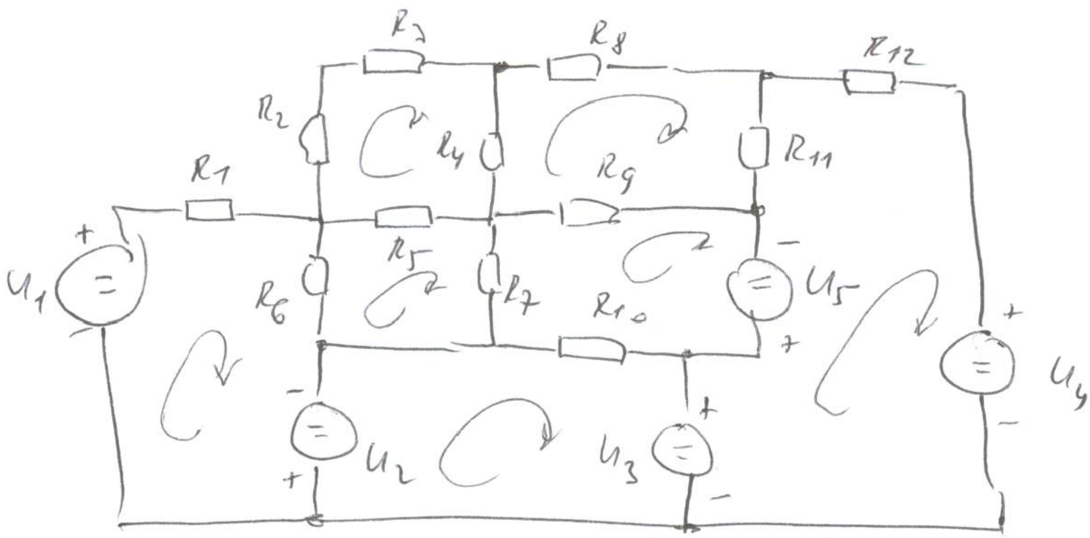

Figure 11: A complicated curcuit

For example, 7 loop's current equations or 2 voltage's loop equations.

3 Mutual connections of resistors, capacitors, coils and AC (alternate current) voltage sources

For alternating harmonic voltages the following equation is usually used:

$$u = U_m \sin(\omega t) \tag{11}$$

Ohm's law for alternating voltage and resistance R is in an expected form:

$$i = \frac{u}{R} = \frac{U_m}{R}\sin(\omega t) = I_m \sin(\omega t) \tag{12}$$

(resistor's voltage and resistor's current are in the same phase).
Ohm's law for an alternating voltage and a capacitor are in a differential form:

$$\begin{aligned} i_C &= C\tfrac{du_C}{dt}(u_C(0) = u_{C_0}) \\ i_C &= Cu'_C \end{aligned} \tag{13}$$

Ohm's law for and alternating voltages and a induction are in a differential form:

$$\begin{aligned} u_L &= L\tfrac{di_L}{dt}, (i_L(0) = i_{L_0}) \\ u_L &= Li'_L \end{aligned} \tag{14}$$

For resistors in a series (analogy for figure 2) with source with alternating current, the following holds:

$$u = u_{R_1} + u_{R_2} \quad i = \tfrac{u}{R_1+R_2} \tag{15}$$

For RL, RC or LC in series, the situation is mathematically complicate. For example, for an RL curcuit (figure 12), we have:

$$\begin{aligned} u &= u_R + u_L \\ u &= Ri_L + Li'_C \end{aligned} \tag{16}$$

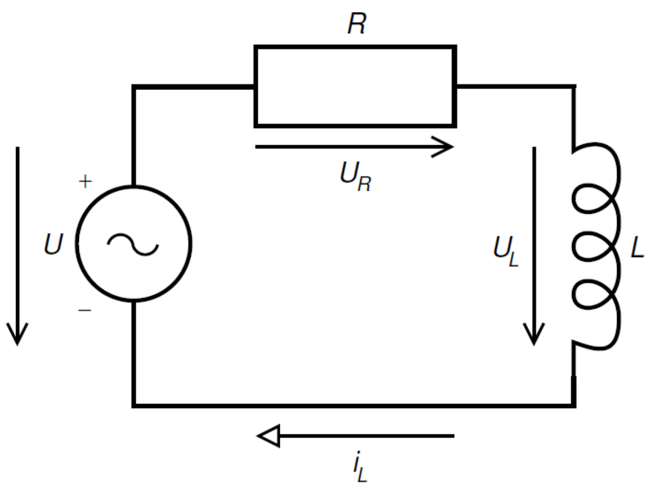

Figure 12: RL curcuit

For specified values $R = 10\Omega, L = 0.1H, \omega = 100rad/s, i_L(0) = 0, u = 200\sin(t)$, the solution is described by the following differential equation:

$$0.1i'_L + 10i_L = 200\sin(100t), i_L(0) = 0 \tag{17}$$

And this is a problem.
Then mathematicians came and created the following steps for solving this non-homogeneous linear differential equation of the first order:

1. First we compute a solution of the following homogeneous equation:

$$0.1i'_L + 10i_L = 0 \tag{18}$$

(0 on the right-hand side of an equation) and get a homogeneous solution:

$$i_{LH} = Ke^{\lambda t} \tag{19}$$

This equation is called "expected solution" of an investigated homogeneous linear differential equation of the first order.
A constant λ is derived using a characteristic equation:

$$\begin{gathered} 0.1\lambda + 10 = 0 \\ \Rightarrow \lambda = -100 \end{gathered} \tag{20}$$

So the expected homogeneous solution is:

$$i_{LH} = Ke^{-100t} \tag{21}$$

2. Now we investigate the right-hand side (function $200\sin(100t)$)
Mathematicians found out that this solution has an expected form:

$$i_{LP} = A\sin(100t) + B\cos(100t) \tag{22}$$

(P index means particular).
How can we check that this solution holds for a given equation? $\Rightarrow$ Substitute i_{LP}.
For substition, i_{LP} must be derived:

$$i'_{LP} = A100\cos(100t) - B100\sin(100t) \tag{23}$$

After substitution:

$$0.1(A100\cos(100t) - B100\sin(100t)) + 10(A\sin(100t) + B\cos(100t)) = 200\sin(100t) \tag{24}$$

After symbol manipulation we get:

$$(10A - 10B)\sin(100t) + (10A - 10B)\cos(100t) = 200\sin(100t) \tag{25}$$

Comparing coefficients on the left-hand and right-hand side:

$$\begin{gathered} 10A - 10B = 200 \\ 10A + 10B = 0 \end{gathered} \tag{26}$$

This is a system of equations for A, B
For example

$$\begin{gathered} 10A = -10B \\ -10B - 10B = 200 \Rightarrow B = -10, A = 10 \end{gathered} \tag{27}$$

so

$$i_{LP} = 10\sin(100t) - 10\cos(100t) \tag{28}$$

3. The searched general solution i_L is given by summing i_{LH} and i_{LP}:

$$i_L = Ke^{-100t} + 10\sin(100t) - 10\cos(100t) \tag{29}$$

4. The searched concrete solution is given by an initial condition $i_L(0) = 0$

$$\begin{aligned}0 &= Ke^{-100*0} + 10\sin(100*0) - 10\cos(100*0)\\ 0 &= K - 10\\ K &= 10\end{aligned} \tag{30}$$

Solution is

$$i_L = 10(e^{-100t} + 10\sin 100t - 10\cos 100t) \tag{31}$$

What about a voltage in the curcuit? For resistor it is obvious:

$$\begin{aligned}u_R &= Ri_L = 10*10(e^{-100t} + 10\sin 100t - 10\cos 100t)\\ u_R &= 100e^{-100t} + 100\sin 100t - 100\cos 100t\\ u_L &= u - u_R = 200\sin 100t - 100e^{-100t} - 100\sin 100t + 100\cos 100t\\ u_L &= -100e^{-100t} + 100\sin 100t + 100\cos 100t\end{aligned} \tag{32}$$

More illustrative than equations are graphs (for example using TKSL). Individual progresses are very illustrative and show what is the resulting behaviour of the RL curcuit: amplitudes and phase ratios of currents and voltages are changing according to an input harmonic signal. What about an RS in a series (figure 13)?

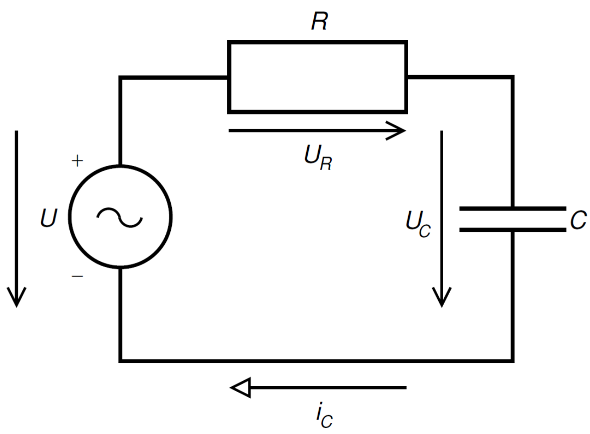

Figure 13: RC curcuit

$$\begin{aligned}u'_C &= \tfrac{1}{C}i_C & i_C &= \tfrac{u-u_C}{R}\\ u'_C &= \tfrac{1}{CR}(u-u_C) & u_C(0) &= 0\end{aligned} \tag{33}$$

For specified values $C = 0.002F, R = 20\Omega, u = 5\sin 100t$ we have:

$$\begin{aligned}u'_C &= \tfrac{1}{0.002*20}(5\sin 100t - u_c)\\ 25u'_C + u_C &= 5\sin 100t\end{aligned} \tag{34}$$

Repeating the process above we get:

$$\begin{aligned}&25\lambda + 1 = 0\\ &\lambda = -\tfrac{1}{25}\\ &\lambda = -0.04\\ &\Rightarrow U_{CH} = Ke^{-0.04t}\end{aligned} \tag{35}$$

We are expecting:

$$
\begin{aligned}
&u_{CP} = A\sin 100t + B\cos 100t \\
&\dots \\
&u_C = u_{CH} + u_{CP} \\
&u_C(0) = 0 \Rightarrow K = \dots
\end{aligned}
\tag{36}
$$

Results in TKSL, for example.
What about a little more complicated curcuits?

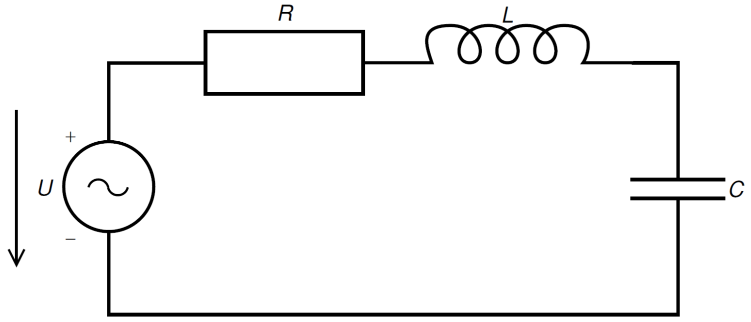

Figure 14: RCL curcuit

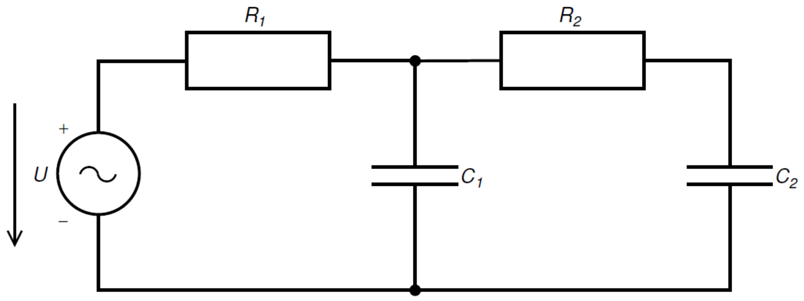

Figure 15: A little more complicated curcuit

4 A stabilized harmonic state of a curcuit

Mathematical deductions are not very pleasant so the following technical and practical simplifacation was introduced:
After finishing a temporary changes in a curcuit (change of currents and voltages caused by inductance and capacitance) and neglicting very small changes (from certain time t, expression e^{-100t} is very small), stabilized harmonic state is created for a harmonic signal, where aplitudes and phases of individual quantities stay the same.
So Ohm's law for capacitor

$$i_C = C\frac{du_C}{dt} \tag{37}$$

and

$$u_C = U_M \sin \omega t \tag{38}$$

can be rewritten for a stabilized harmonic state into:

$$\begin{aligned} i_C &= CU_M\omega \cos \omega t \\ i_C &= I_M \cos \omega t \\ i_C &= I_M \sin(\omega t + \tfrac{\pi}{2}) \\ I_M &= \tfrac{U_M}{X_C}, X_C = \tfrac{1}{\omega C} \end{aligned} \tag{39}$$

and actually a current in a capacitor is overrunning a voltage on a capacitor for 90°:

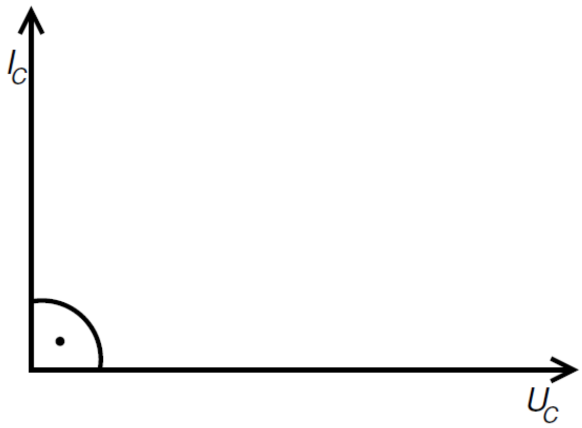

Figure 16: A phase diagram for RC curcuit

Similarly for

$$u_L = L\frac{di_L}{dt} \tag{40}$$

and

$$i_L = I_M \sin \omega t \tag{41}$$

we get

$$\begin{aligned} u_L &= LI_M\omega \sin \omega t \\ u_L &= U_M \cos \omega t \\ u_L &= U_M \sin(\omega t + \tfrac{\pi}{2}) \\ U_M &= I_M X_L, X_L \omega L \end{aligned} \tag{42}$$

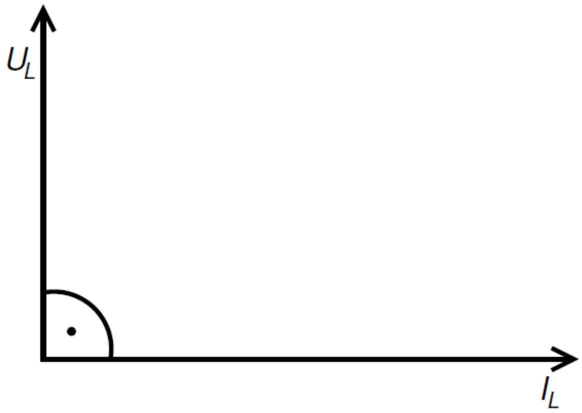

Figure 17: A phase diagram for RL curcuit

So the situation for RLC (figure 14) is - figure 19.

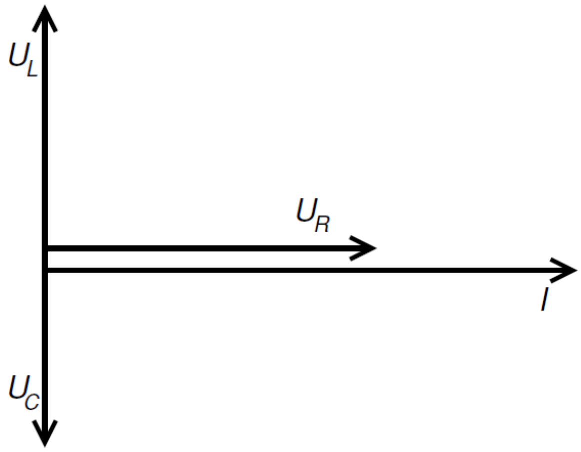

Figure 18: Phase diagram for RLC curcuit

But difficult for more complicated curcuits:

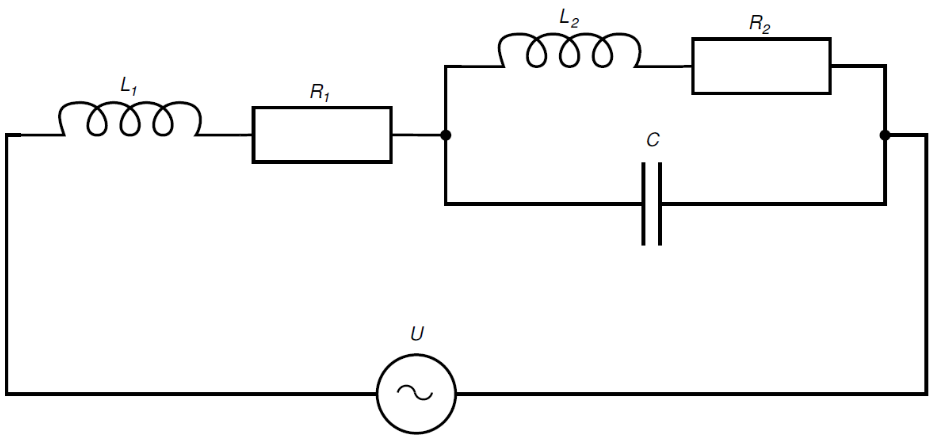

Figure 19: A complex curcuit

⇒ trasformation into complex numbers.

But is it legit to use complex numbers? In RL curcuit, for example (figure 12) ...

With help of "overrunning" and "slowing" we can build a famous representation of Pythagoras's theorem (on x, y-axis) - figure 20.

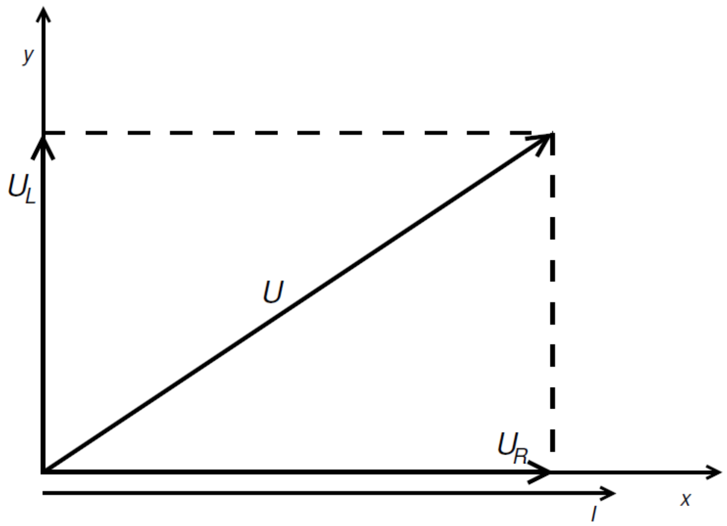

Figure 20: Ilustrutation of transmission to complex numbers

If we take a horizontal line as a "real" axis and a vertical one as a "imaginary", we have:

$$\begin{aligned} &U_R = RI \\ &U_L = j\omega LI \\ &\vec{U} = \vec{U_R} + \vec{U_L} = RI = j\omega LI \end{aligned} \tag{43}$$

It is a little more better than to "draw" Pythagoras's theorem, but if a curcuit is complex (and possibly nonlinear), we will make more computations again.

So we go back to differential equations: RL curcuit (figure 12) was described by a differential equation:

$$0.1 i_L' + 10 i_L = 200 \sin 100t \tag{44}$$

This kind of equations can be solved by efficient programs (e.g. TKSL) - this equation is rewritten into:

$$i_L' = \frac{1}{0.1}(200 \sin 100t - 10 i_L) \tag{45}$$

Similarly for the rest of curcuits $\Rightarrow$ just be able to describe a curcuit by a system of differential equations of the first order.

So what is easier: mathematics or TKSL?