

VNVe – numerical solution of integrals using differential calculus; function generation; simple pendulum

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Analytical solution of a differential equation

For the differential equation that represent the exponential function e^x

$$\underline{y' + ay = 0} \quad (1)$$

we expect the solution in the form $y = ce^{\lambda t}$, where

$$\underline{\lambda + a = 0} \rightarrow \underline{\lambda = -a}, \quad y = ce^{-a}, \quad y' = -ace^{-a}. \quad \text{It holds that:}$$

$$y' + ay = 0$$

$$y = ce^{-a} \quad ; \quad y' = ce^{-a} \cdot (-a) \quad ; \quad y' = -ace^{-a}$$

$$-ace^{-a} + ace^{-a} = 0$$

$$y' - a \cdot y = 0 \quad \underline{\underline{=}}$$

$$2 + a \cdot 1 = 0$$

$$2 + a = 0$$

$$y'' \rightarrow 2 \\ y' \rightarrow 2 \quad y \rightarrow 1$$

Polynomial representation

Exponential function e^x can be represented using the power series,

$$e^x = \underbrace{1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots}$$

And it can be expressed using the **Taylor polynomial**

$$f(x) = \underbrace{f(x_0)}_{\text{INT. COND. (VALUE OF } f \text{ IN } x_0\text{)}} + \frac{(x - x_0)^1}{1!} \underbrace{f'(x_0)}_{\text{DERIVATIVES}} + \frac{(x - x_0)^2}{2!} \underbrace{f''(x_0)}_{\text{INT. COND.}} + \dots + \frac{(x - x_0)^n}{n!} \underbrace{f^{[n]}(x_0)}_{\text{INT. COND.}}$$

$$\begin{aligned}y &= e^x \\y' &= e^x \\y'' &= e^x \\&\vdots\end{aligned}$$

DERIVATIVES

INT. COND.

Higher derivatives

Higher derivatives for (1) $y' = -ay$: ($y = f(x)$)

$$f'(x) = -af(x)$$

$$f''(x) = -af'(x) = \underline{(-a)(-a)}f(x)$$

⋮

$$f^{[n]}(x) = \underline{(-a)^n}f(x)$$

Using the Taylor series

$$f(x) = \underline{f(x_0)} + \sum_{i=1}^n \frac{(x-x_0)^i}{i!} f^{[i]}(x_0)$$

for our function ($y = ce^{-ax} \rightarrow y' = -ay$)

$$f(x) = f(x_0) + \sum_{i=1}^n \frac{(-a)^i(x-x_0)^i}{i!} f(x_0)$$

Higher derivatives

Differential equation (1) can be extended to the initial value problem (IVP) by adding the initial value

$$y' + ay = 0 \quad y(x_0) = 1 = c$$

$y = ce^{\lambda t}$

IVP can again be expressed using the Taylor series

$$\begin{aligned} f(x) &= 1 + \sum_{i=1}^n \frac{i(x - x_0)^i}{i!} f^{[i]}(x_0) + E(n) \\ &= 1 + \sum_{i=1}^n \frac{(-a)^i (x - x_0)^i}{i!} f(x_0) + E(n) \end{aligned}$$

where E is the error term.

System of differential equations

Consider the following ordinary differential equation (ODE) of the second order

$$\underline{y'' + ay' + by = 0} \quad y(0) = 0, y'(0) = 0$$

Using the appropriate substitution

$$\boxed{\begin{aligned} z &= y' \\ z' &= y'' \end{aligned}} \Rightarrow z - y' = 0$$

we can convert the second order ODE to the system of the first order ODEs

$$\boxed{\begin{aligned} z' + az + by &= 0 & z(0) &= 0 \\ z - y' &= 0 & y(0) &= \dots \end{aligned}} \quad (2)$$

z
y'

System of differential equations

The system of equations can again be expressed using the Taylor polynomial ($h = \underline{x - x_0}$)

$$y(x_0 + h) = \underline{y(x_0)} + \sum_{i=1}^n \frac{h^i}{i!} y^{[i]}(x_0) \quad (3)$$

$$z(x_0 + h) = \underline{z(x_0)} + \sum_{i=1}^n \frac{h^i}{i!} z^{[i]}(x_0)$$

Higher derivatives of y and z from the system (2)

$$z' + az + by = 0$$

$$z' = -az - by$$

$$y' = z$$

$$z' = -az - by$$

$$y'' = z'$$

$$z'' = -az' - by'$$

⋮

⋮

$$y^{[i]} = z^{[i-1]}$$

$$z^{[i]} = -az^{[i-1]} - by^{[i-1]}$$

System of differential equation – sum of terms

$$y(x_0 + h) = \underline{y(x_0)} + \sum_{i=1}^n \frac{h^i}{i!} y^{[i]}(x_0)$$

h=1

Higher derivatives can be expressed using the Taylor series

Order or derivative	sum y	sum z
0.	$y(x_0)$	$z(x_0)$
1.	$\frac{h^1}{1!} y'(x_0)$	$\frac{h^1}{1!} z'(x_0)$
2.	$\frac{h^2}{2!} y''(x_0)$	$\frac{h^2}{2!} z''(x_0)$

For example the second derivative

$$z'' = -az'(x_0) - by'(x_0)$$

$$z'' = -az' - by'$$

Definite integrals

Definite integral ¹

$$F(X) = \int_0^b f(x) dx$$

can be transformed to the equivalent first order ODE (initial value problem)

$$F'(x) = f(x) \quad F(0) = 0$$

Integral in b:

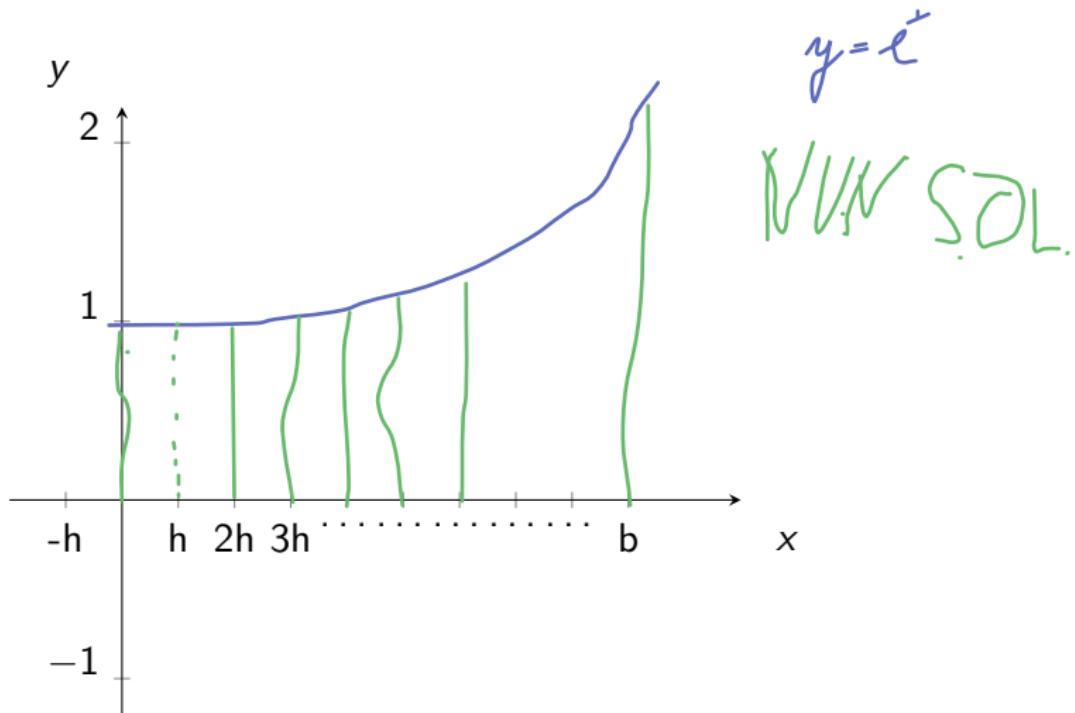
$$F(b) = \int_0^b f(x) dx + F(0)$$

For time function (t):

$$F(t) = \int_0^t f(x) dx$$

$${}^1 \int_a^b f(x) dx = \int_{a+n}^{b+n} f(x - n) dx$$

Graphical representation of the numerical solution



Example: Exponential function

Definite integral of the function $f(x) = e^x$

$$\int_0^1 e^x \, dx$$

has the analytical solution

$$[e^x]_0^1 = e^1 - e^0 = e - 1$$

We can transform the integral into the differential equation:

$$F(x) = \int_0^1 e^x \, dx$$
$$F'(x) = e^x \quad F(0) = 0$$

All derivatives of $f(x)$ are **the same** 
 $(f(x) = f'(x) = f''(x) = \dots = e^x)$.

Example: Exponential function

Taylor series for step size $h = 1 = t_{max} = b$

$$F(1) = F(0) + \sum_{i=1}^n \frac{1^i}{i!} (e^x)^{[i]}(0)$$

For arbitrary step size, the polynomial would look like this:

$$F(h) = F(0) + \frac{h}{1!} F^{[1]}(0) + \frac{h^2}{2!} F^{[2]}(0) + \cdots + \frac{h^n}{n!} F^{[n]}(0)$$

where $F^{[1]}(x) = e^x; F^{[2]}(x) = e^x, \dots$. Integral in the upper bound ($b = 1$) can be calculated

$$\begin{aligned} F(1) &= 0 + \sum_{i=1}^n \frac{1^i}{i!} (e^0)^{[i]} = 0 + \frac{1^1}{1!} - \boxed{\frac{1^2}{2!} + \frac{1^3}{3!} + \frac{1^4}{4!}} = \\ &= 0 + e^1 - 1 = \underline{\underline{e^1 - 1}} \end{aligned}$$

Homework: Function $\sin(x)$

Similarly, try to calculate the value of the definite integral of the function

$$\int_0^{\frac{\pi}{2}} \sin(x) dx$$

with the analytical solution

$$[-\cos(x)]_0^{\frac{\pi}{2}} = -\cos\left(\frac{\pi}{2}\right) - (-\cos(0)) = 0 + 1 = \underline{1}$$

Function $\cos(x)$ can be expressed using the power series

$$\cos(h) = \frac{h^2}{2!} - \frac{h^4}{4!} + \frac{h^6}{6!} - \frac{h^8}{8!} \dots$$

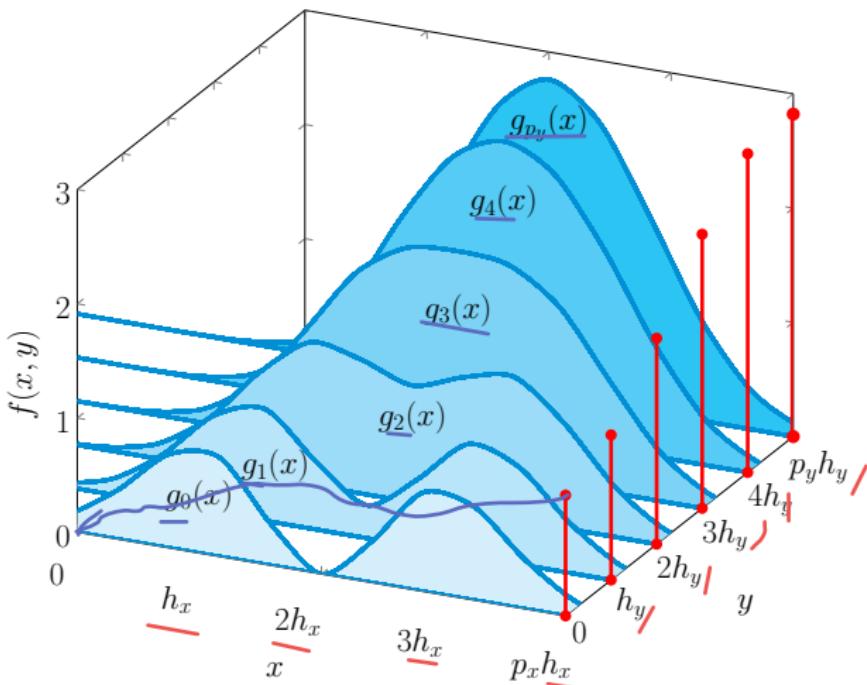
Homework: Function $\sin(x)$

$$F(X) = \int_0^{\frac{\pi}{2}} \sin(x) \, dx$$
$$F(0) = 0$$

$$F'(x) = \sin(x) \quad F(0) = 0$$

$$\mathbf{F}\left(\frac{\pi}{2}\right) = ? \quad \mathbf{h} = \frac{\pi}{2}$$

Demonstration

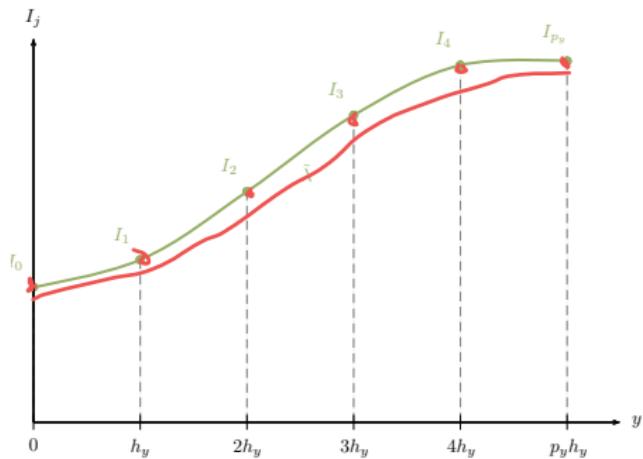


Demonstration – matrix representation

Finite integrals in 2D can again be solved using the Taylor series. For the demonstration above, we integrate continuously (solving the IVPs) over x , the values of the integrals are the – **red points**. Then, we integrate over y . Values of the red points serve as the interpolation points of the function.

When we perform the calculation in the second dimension, we know the values of the red points and the constant matrix.
However, we have to also know the higher derivatives.

Calculation in y



Demonstration – matrix representation

$$\left\{ \begin{array}{l} f(y_0 + h_y) = f(y_0) + \frac{h_y}{1!} f'(y_0) + \frac{h_y^2}{2!} f''(y_0) + \dots \\ f(y_0 + 2h_y) = f(y_0) + \frac{2h_y}{1!} f'(y_0) + \frac{(2h_y)^2}{2!} f''(y_0) + \dots \\ \vdots \\ f(y_0 + p_y h_y) = f(y_0) + \frac{p_y h_y}{1!} f'(y_0) + \frac{(p_y h_y)^2}{2!} f''(y_0) + \dots \end{array} \right.$$

Generally, the equation has the following form $\mathbf{A} \cdot \vec{d} = \vec{b}$.

$$\vec{b} = \mathbf{A} \cdot \vec{d}$$

$$\vec{b} = \begin{pmatrix} f(y_0 + h_y) - f(y_0) \\ f(y_0 + 2h_y) - f(y_0) \\ \vdots \\ f(y_0 + p_y h_y) - f(y_0) \end{pmatrix} \quad \mathbf{A} = \begin{pmatrix} \frac{1}{1!} & \frac{1^2}{2!} & \cdots & \frac{1^{p_y}}{p_y!} \\ \frac{2}{1!} & \frac{2^2}{2!} & \cdots & \frac{2^{p_y}}{p_y!} \\ \vdots & \cdots & \cdots & \vdots \\ \frac{p_y}{1!} & \frac{p_y^2}{2!} & \cdots & \frac{p_y^{p_y}}{p_y!} \end{pmatrix} \quad \vec{d} = \begin{pmatrix} f'(y_0)h_y \\ f''(y_0)h_y^2 \\ \vdots \\ f^{[p_y]}(y_0)h_y^{p_y} \end{pmatrix}$$

Matrix \mathbf{A} is the same in all dimensions. Unknown higher derivatives y can be expressed as

$$\vec{d} = A^{-1} \cdot \vec{b}$$

and after simplification

$$f^{[i]}(y_0) = \frac{d_i}{h_y^i}$$

Function generation – exponential function

$$\underline{z = e^{\lambda t}}$$

$$z' = \lambda e^{\lambda t} = \underline{\lambda z} \quad z(0) = e^0 = 1$$



Function generation – sine and cosine

$$z = \sin(t)$$

$$z' = \cos(t)$$

$\approx n$

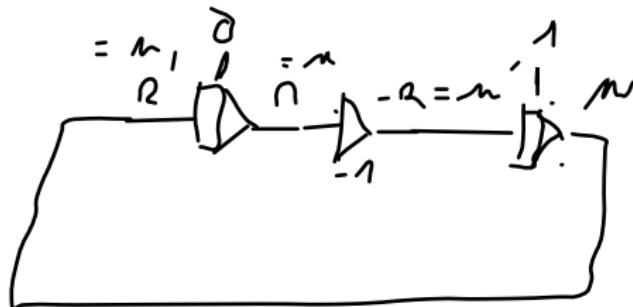
$$u = \underline{\cos(t)}$$

$$u' = -\sin(t)$$

$\approx -n$

$$z' = u \quad z(0) = \sin(0) = 0$$

$$u' = -z \quad u(0) = \cos(0) = 1$$



Function generation: $z = \underline{e^{-2t}} \cdot \sin(t)$

$$x = e^{-2t}$$

$$x' = -2e^{-2t}$$

$$= -\cancel{2} \cancel{e^{-2t}}$$

$$y = \sin(t)$$

$$y' = \cos(t)$$

$$w = \cos(t)$$

$$w' = -\sin(t)$$

$$x' = -2x$$

$$x(0) = e^{-2 \cdot 0} = 1$$

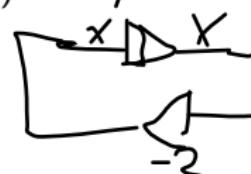
$$y' = w$$

$$y(0) = \sin(0) = 0$$

$$w' = -y$$

$$w(0) = \cos(0) = 1$$

$$\underline{\underline{z = x \cdot y}}$$



Three integrators and multiplication. Is there any other solution?

Function generation:

$$\begin{aligned} z &= e^{-2t} \cdot \sin(t) \\ z' &= e^{-2t} \cdot (-2\sin(t)) + e^{-2t} \cdot \cos(t) \\ &\equiv -2e^{-2t} \sin(t) + e^{-2t} \cos(t) = -2z + u \\ u &= e^{-2t} \cdot \cos(t) \\ u' &= e^{-2t} \cdot (-2) \cdot \cos(t) + e^{-2t} \cdot (-\sin(t)) \\ &\equiv -2e^{-2t} \cos(t) - e^{-2t} \sin(t) = -2u - z \end{aligned}$$

Initial conditions

$$z(0) = e^{-2 \cdot 0} \sin(0) = 1 \cdot 0 = 0$$

$$u(0) = e^{-2 \cdot 0} \cos(0) = 1 \cdot 1 = 1$$

Two integrators and addition.

Simple pendulum

Simple pendulum can be modeled using second order IVP

$$\varphi'' + \frac{d}{m}\varphi' + \frac{g}{L}\sin(\varphi) = 0$$

where

d – damping

m – mass

g – gravitational constant

L – length of the pendulum

Initial conditions

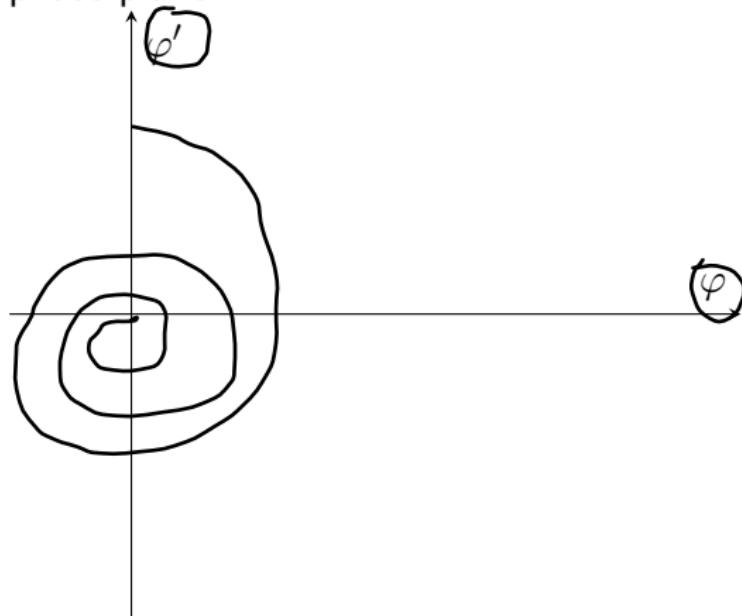
$$\varphi(0) = 0 \quad \text{initial position}$$

$$\varphi'(0) = ? \quad \text{initial velocity}$$

The objective is to find the initial velocity $\varphi'(0)$, which causes the pendulum to cross the upper fixed point once.

Simple pendulum

When we plot the behavior of the simple pendulum, we will use the phase plane.



Simple pendulum

To numerically solve the equation for the simple pendulum

$$\boxed{\varphi''} + \frac{d}{m}\varphi' + \frac{g}{L}\sin(\varphi) = 0$$

the equation has to be transformed into the system of the first order IVPs. We can use Method of Derivative order reduction (inverting components):

$$\begin{aligned} \boxed{p^2\varphi} &= -\left(\frac{d}{m}p\varphi + \frac{g}{L}\sin(\varphi)\right) \\ p\varphi &= -\frac{1}{p}(-p^2\varphi) = -\frac{1}{p}\left(\frac{d}{m}p\varphi + \frac{g}{L}\sin(\varphi)\right) \quad p\varphi(0) = ??? \\ \varphi &= -\frac{1}{p}(-p\varphi) \quad \varphi(0) = 0 \end{aligned}$$

We are missing the equations for $\sin(\varphi)$.

Simple pendulum – equations for $\sin(\varphi)$

$$\begin{cases} u = \sin(\underline{\varphi}) & v = \cos(\varphi) \\ u' = \cos(\underline{\varphi})\underline{\varphi'} & v' = -\sin(\varphi)\varphi' \\ u' = v\varphi' & v' = -u\varphi' \\ pu = vp\varphi & pv = -up\varphi \\ u = -\frac{1}{p}(-vp\varphi) & v = -\frac{1}{p}(up\varphi) \end{cases}$$

Initial conditions

$$u(0) = \sin(\varphi(0)) = \sin(0) = 0$$

$$v(0) = \cos(\varphi(0)) = \cos(0) = 1$$

$$y = \sin(\omega \cdot t)$$
$$y' = \cos(\omega \cdot t) \cdot \omega$$
$$= r \cdot y$$
$$r = \cos(\omega \cdot t)$$
$$r' = -\sin(\omega \cdot t) \cdot \omega$$
$$= -y \cdot \omega$$

Thank you for your attention