VNV – adjoint operators

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Version: March 30, 2020

Overview

Differential equations with time variable coefficients

Until now, all coefficients of the differential equations (terms a_i on the left side and b_j on the right side) were constant. We were just performing some transformations of the equations (creating the systems of the first order differential equations, ...).
Possible problem: what if the coefficients would change in time?
Note: all time variable coefficients will be UPPER CASE.

Differential equations with time variable coefficients

We are going to start with the simpler case: the forcing function z does not have derivatives.

$$y' + \sin(t)y = z$$

The equation above contains on time variable (dependent) coefficient

$$\boxed{A_0 = \sin(t)}$$

Problem analysis

- the values of the coefficient A_0 **change** in time,
- there are **no derivatives** of the forcing function z (the right side).

Differential equations with time variable coefficients

The input equation has to be transformed into the system of the first order differential equations. We can use the **Method of derivation order reduction** (no derivatives of z). We substitute the coefficient A_0 (to make the solution more general) and perform the Laplace transform

A_0

$$y' = z - \sin(t)y = z - A_0 y$$

LAPLACE

$$py = z - A_0 y$$

$$y = \frac{1}{p}(z - A_0 y)$$

Initial condition: $y(0) = 0$

Differential equations with time variable coefficients

Time variable coefficients

$$A_0 = \sin(t) \qquad\qquad B_0 = \cos(t)$$

have to be generated using auxiliary system of differential equations

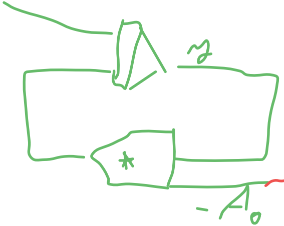
$$A_0' = \cos(t) = B_0 \qquad\qquad B_0' = -\sin(t) = -A_0$$

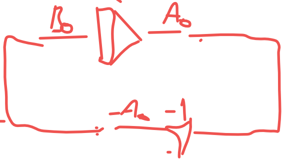
with initial conditions: $A_0(0) = \sin(0) = 0$, $\overset{B_0}{\cancel{A_{00}}}(0) = \cos(0) = 1$

Block scheme

$$py = z - A_0 y \qquad\qquad pA_0 = \cos(t) = B_0 \qquad\qquad pB_0 = -\sin(t) = -A_0$$

$$y = \frac{1}{p}(z - A_0 y) \qquad\qquad A_0 = \frac{1}{p}(B_0) \qquad\qquad \overset{B_0}{\cancel{A_{00}}} = \frac{1}{p}(-A_0)$$





Differential equations with time variable coefficients

Now, we will try to solve a more complicated problem, that has the derivatives of the forcing function z:

$$e^t y' + \sin(t) y = tz' + \cos(t) z$$

Time variable (dependent) coefficients

$$\boxed{\begin{aligned} A_1 &= e^t & \qquad B_1 &= t \\ A_0 &= \sin(t) & \qquad B_0 &= \cos(t) \end{aligned}}$$

Problem analysis

- the values of the coefficients **change** in time,
- the forcing function z has derivatives.

This problem can be solved by the method of **adjoint differential operators**.

To solve this problem, we can create a general block scheme. This block scheme is going to contain some known and one completely new one:

- coefficients (as noted previously, all time variable coefficients will be UPPER CASE and double circles)
- integrators
- **implicator** $\sum$ inputs $= 0$

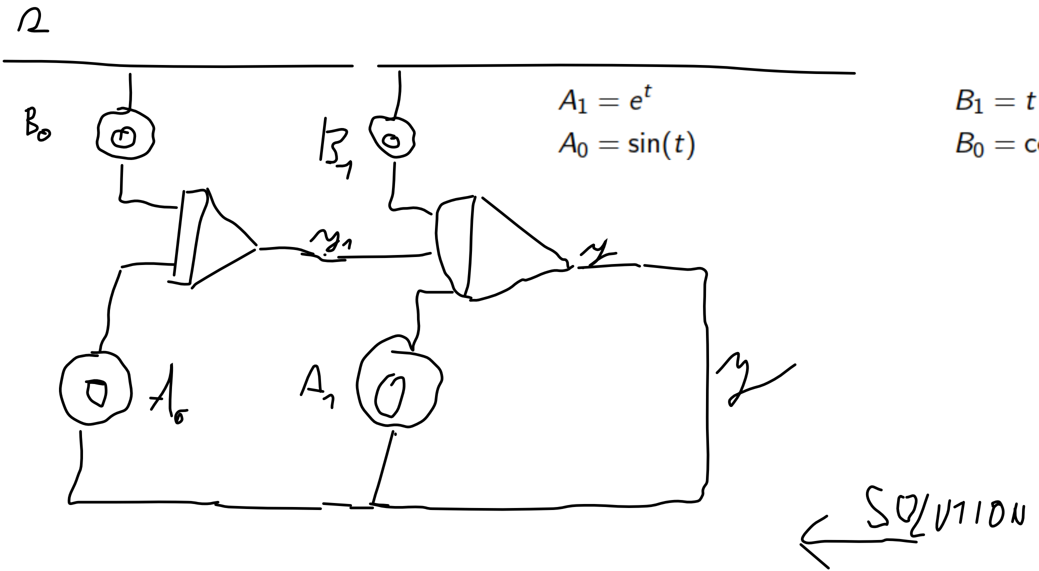
What would the block scheme look like for

$$e^t y' + \sin(t) y = t z' + \cos(t) z$$

Note: in the following examples, we are going to use inverting integrators.

Block scheme

$$e^t y' + \sin(t) y = t z' + \cos(t) z$$



$A_1 = e^t$

$A_0 = \sin(t)$

$B_1 = t$

$B_0 = \cos(t)$

General scheme – equations I

$$A_1 = e^t \qquad B_1 = t$$
$$A_0 = \sin(t) \qquad B_0 = \cos(t)$$

With the block scheme, we can start solving it by writing down the **input equations** (from right to left)

1. **implicator**

$$B_1 z + y_1 + A_1 y = 0$$
$$-y_1 = B_1 z + A_1 y$$
$$-y_1 = tz + e^t y$$

We are solving the scheme from the right, we need to create an input signal for the integrator, so we need to derive equation for y_1.

$$-y_1' = 1z + tz' + e^t y + e^t y'$$

$$-y_1' = 1 \cdot z + t z' + e^t y + e^t y'$$

General scheme – equations II

$A_1 = e^t \qquad B_1 = t$

$A_0 = \sin(t) \qquad B_0 = \cos(t)$

2. **integrator** $B_0 z + A_0 y = -y_1'$

$$B_0 z + A_0 y = z + tz' + e^t y + e^t y'$$
$$\cos(t) z + \sin(t) y = z + tz' + e^t y + e^t y'$$

After rearranging the obtained equation, we get the **system equation**

$$\boxed{-e^t y' + y(\sin(t) - e^t) = tz' + z(1 - \cos(t))}$$

Problem: the obtained coefficients **do not match** the original ones. Is the obtained solution correct?

Solution check

New (adjoint) coefficients of the obtained solution

$$\boxed{-e^t y' + y(\sin(t) - e^t) = tz' + z(1 - \cos(t))}$$

$$\boxed{\begin{aligned} A_1^* &= -e^t \\ A_0^* &= \sin(t) - e^t \end{aligned}} \qquad \boxed{\begin{aligned} B_1^* &= t \\ B_0^* &= 1 - \cos(t) \end{aligned}}$$

The solution can be checked by substituting the adjoint coefficients $(A_1^*, A_0^*, B_1^*, B_0^*)$ into the input equations of the general solution (we are adjoining again: $(A_1^*)^*, (A_0^*)^*, (B_1^*)^*, (B_0^*)^* \Rightarrow A_1, A_0, B_1, B_0$).

Solution check I

$$A_1^* = -e^t \qquad B_1^* = t$$
$$A_0^* = \sin(t) - e^t \qquad B_0^* = 1 - \cos(t)$$

New input equations (with substituted adjoint coefficients):

1. **implicator** $B_1^* z + y_1 + A_1^* y = 0$

$$-y_1 = B_1^* z + A_1^* y$$
$$-y_1 = tz - e^t y$$

We are solving the scheme from the right, we need to create an input signal for the integrator, so we need to derive equation for y_1.

$$-y_1' = 1z + tz' - e^t y - e^t y'$$

$-y_1' = 1z + tz' - e^t y - e^t y'$

Solution check II

$$A_1^* = -e^t \qquad B_1^* = t$$
$$A_0^* = \sin(t) - e^t \qquad B_0^* = 1 - \cos(t)$$

2. **integrator** $B_0^* z + A_0^* y = -y_1'$

$$\begin{aligned} B_0^* z + A_0^* y &= z + tz' - e^t y - e^t y' \\ (1-\cos(t))z + (\sin(t) - e^t)y &= z + tz' - e^t y - e^t y' \\ z - \cos(t)z + \sin(t)y - e^t y &= z + tz' - e^t y - e^t y' \end{aligned}$$

Simplifying

$$e^t y' + \sin(t)y = tz' + \cos(t)z$$

The resulting equation: $e^t y' + \sin(t)y = tz' + \cos(t)z$ has the original coefficients A_1,A_0,B_1, B_0.
The adjoint coefficients A_1^*,A_0^*,B_1^* a B_0^* are correct and can be used.

$$e^t y' + \sin(t)y = tz' + \cos(t)z$$

$A_1^* = -e^t$

$A_0^* = \sin(t) - e^t$

$B_1^* = t$

$B_0^* = 1 - \cos(t)$

The input first order differential equation

$$e^t y' + \sin(t)y = tz' + \cos(t)z$$

can be solved using **one** first order differential equation (**one integrator**)

$$\boxed{\begin{aligned} y_1' &= -B_0^* z - A_0^* y \\ \boldsymbol{y_1'} &= \boldsymbol{-(1 - \cos(t))z - (\sin(t) - e^t)y}\,, \end{aligned}} \qquad \boldsymbol{y_1(0) = 0}$$

Output of the system (**always implicator**) can be expressed as

$$B_1^* z + y_1 + A_1^* y = 0$$

IMPLICATOR

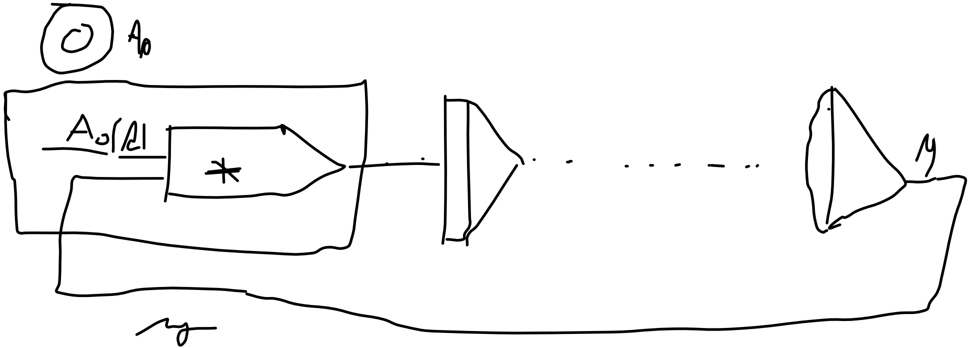
$$\boxed{\begin{aligned} y &= -\frac{B_1^* z + y_1}{A_1^*} \\ \boldsymbol{y} &= \boldsymbol{-\frac{1}{e^t}(tz + y_1)} \end{aligned}}$$

DIVIDER

HW representation of time variable coefficients

- time variable (dependent) coefficient – multiplier + generating differential equations
- implicator – divider

HW representation of time variable coefficients – scheme



HW representation of time variable coefficients

$$A_1^* = -e^t \qquad B_1^* = t$$
$$A_0^* = \sin(t) - e^t \qquad B_0^* = 1 - \cos(t)$$

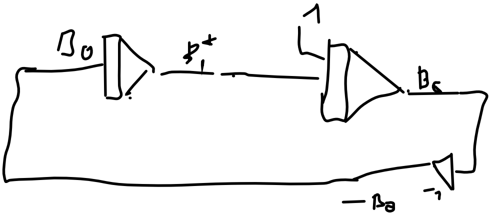
It is necessary to **generate the functions of time** $B_0^*(t)$,$B_1^*(t)$,$A_0^*(t)$ a $A_1^*(t)$. For the adjoint coefficient B_0^* is the function of time $B_0^*(t) = 1 - \cos(t)$

$$B_0^* = 1 - \cos(t) \qquad B_{00}^* = \sin(t)$$
$$B_0^{*'} = \sin(t) = B_{00}^* \qquad B_{00}^{*'} = \cos(t) = 1 - B_0^*$$

All initial conditions ($B_0^*(0)$ a $B_{00}^*(0)$) are **zero**. After Laplace transform :

$$pB_0^* = B_0 \qquad pB_{00}^* = 1 - B_0^*$$
$$B_0^* = -\frac{1}{p}(-B_0) \qquad B_{00}^* = -\frac{1}{p}-(1 - B_0^*)$$

HW representation – function generation



Alternative approach to the solution

The differential equations with time-variable coefficients can be solved using a differential operators (for example Sturm–Liouville operator [1]) It is proven, that

$$\boxed{L^*(y) = -M^*(z)}$$

where $L(y)$ is the left side, $M(z)$ is the right side of the input differential equation. Operation $*$ represents the adjoining of the operators.

[1] 2

Alternative approach – example

For the differential equation

$$A_1^* = -e^t \qquad B_1^* = t$$
$$A_0^* = \sin(t) - e^t \qquad B_0^* = 1 - \cos(t)$$

$$e^t y' + \sin(t) y = t z' + \cos(t) z$$

$$L^*(y) = (-1)^1 (e^t y)' + (-1)^0 (\sin(t) y) = -(e^t y + e^t y') + \sin(t) y$$

$$L(y) = e^t y' + \sin(t) y$$

$$L^*(y) = (-1)^1 (e^t y)' + (-1)^0 (\sin(t) y) = -(e^t y + e^t y') + \sin(t) y$$
$$= -e^t y - e^t y' + \sin(t) y = \underline{-e^t y' + y(\sin(t) - e^t)}$$

Expected coefficients of the left side:

$$A_1^* = -e^t \quad A_0^* = \sin(t) - e^t$$

To check the result: $(L^*(y))^* = L(y)$

Alternative approach – example (check of the left side)

$$\boxed{L^*(y) = -e^t y' + y(\sin(t) - e^t)}$$

$$(L^*(y))^* = (-1)^1(-e^t y)' + (-1)^0(\sin(t) - e^t)y = -1\cdot(-e^t y - e^t y') + \sin(t)y - e^t y$$

$$\begin{aligned}(L^*(y))^* &= (-1)^1(-e^t y)' + (-1)^0(\sin(t) - e^t)y \\ &= \underline{e^t y + e^t y' + \sin(t)y - e^t y} \\ &= e^t y' + \sin(t)y\end{aligned}$$

Original coefficients:

$$A_1 = e^t \quad A_0 = \sin(t)$$

Left side coefficients are **correct** $((L^*(y))^* = L(y))$.

Alternative approach – example (right side)

$$B_1^* = t$$
$$B_0^* = 1 - \cos(t)$$

$$\boxed{M(z) = tz' + \cos(t)z}$$
$$M^*(z) = (-1)^1(tz)' + (-1)^0(\cos(t)z) = -(z + tz') + \cos(t)z$$
$$= -z - tz' + \cos(t)z = \underline{-tz' + z(-1 + \cos(t))}$$

Check:

$$M^*(z) = -tz' + z(-1 + \cos(t))$$
$$(M^*(z)^*) = (-1)^1(-tz)' + (-1)^0(-z + \cos(t)z))$$
$$= z + tz' - z + \cos(t)z = \underline{tz' + \cos(t)z}$$

Right side coefficients are correct. It is however necessary to multiply the obtained adjoint coefficients by -1 (because it holds that $L^*(y) = -M^*(z)$).

Alternative approach – summary

The obtained adjoint coefficients are:

$$A_1^* = -e^t \qquad\qquad B_1^* = t$$
$$A_0^* = \sin(t) - e^t \qquad\qquad B_0^* = 1 - \cos(t)$$

These adjoint coefficients match the coefficients obtained using the previous method.

The solution of the **second order** differential equation is going to be a little bit more "complicated":

$$t^2 y'' + \sin(t) y' + e^{-t} y = 2tz'' + \cos(t) z' + e^{3t} z$$

Again, we need to solve for y. Because, the input differential equation is second order, we need to create a system of two first order differential equations.

Coefficients:

$A_2 = t^2$ $\quad$ $B_2 = 2t$

$A_1 = \sin(t)$ $\quad$ $B_1 = \cos(t)$

$A_0 = e^{-t}$ $\quad$ $B_0 = e^{3t}$

Problem definition

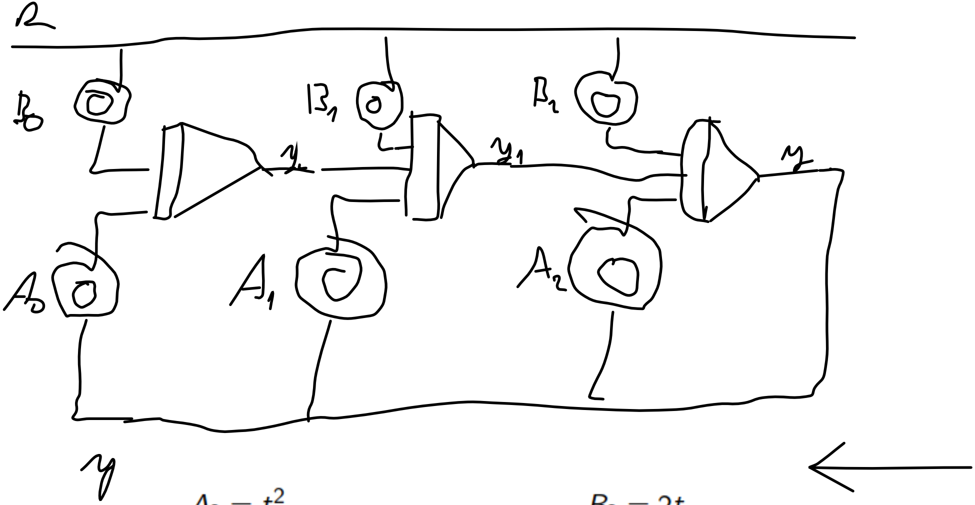
The solution of the **second order** differential equation is going to be a little bit more "complicated":

$$t^2 y'' + \sin(t) y' + e^{-t} y = 2t z'' + \cos(t) z' + e^{3t} z$$

Again, we need to solve for y. Because, the input differential equation is second order, we need to create a system of two first order differential equations.
Coefficients:

$$A_2 = t^2 \qquad B_2 = 2t$$
$$A_1 = \sin(t) \qquad B_1 = \cos(t)$$
$$A_0 = e^{-t} \qquad B_0 = e^{3t}$$

General block scheme



$$A_2 = t^2 \qquad B_2 = 2t$$
$$A_1 = \sin(t) \qquad B_1 = \cos(t)$$
$$A_0 = e^{-t} \qquad B_0 = e^{3t}$$

General block scheme – equations I

The equations for the inputs:

$$A_2 = t^2 \qquad B_2 = 2t$$
$$A_1 = \sin(t) \qquad B_1 = \cos(t)$$
$$A_0 = e^{-t} \qquad B_0 = e^{3t}$$

1. **implicator** $B_2 z + y_1 + A_2 y = 0$

$$-y_1 = B_2 z + A_2 y$$
$$-y_1 = 2tz + t^2 y$$

We are solving the scheme from the right, we need to create an input signal for the integrator, so we need to derive equation for y_1.

$$-y_1' = 2z + 2tz' + 2ty + t^2 y'$$

$$-y_1' = 2z + 2tz' + 2ty + t^2 y'$$

General block scheme – equations II

$A_2 = t^2$ $\quad B_2 = 2t$

$A_1 = \sin(t)$ $\quad B_1 = \cos(t)$

$A_0 = e^{-t}$ $\quad B_0 = e^{3t}$

2. **1. integrator** $B_1 z + y_2 + A_1 y = -y_1'$

$$B_1 z + \underline{y_2} + A_1 y = 2z + 2tz' + 2ty + t^2 y'$$

$$\cos(t)z + \underline{y_2} + \sin(t)y = 2z + 2tz' + 2ty + t^2 y'$$

$$y_2 = 2z + 2tz' + 2ty + t^2 y' - \sin(t)y - \cos(t)z$$

We are solving the scheme from the right, we need to create an input signal for the integrator, so we need to derive equation for y_2.

$$\begin{aligned} y_2' &= 2z' + 2z' + 2tz'' + 2y + 2ty' + 2ty' + t^2 y'' \\ &\quad - \cos(t)y - \sin(t)y' - (-\sin(t)z + \cos(t)z') \\ y_2' &= 4z' + 2tz'' + 2y + 4ty' + t^2 y'' \\ &\quad - \cos(t)y - \sin(t)y' + \sin(t)z - \cos(t)z' \\ -y_2' &= -4z' - 2tz'' - 2y - 4ty' - t^2 y'' \\ &\quad + \cos(t)y + \sin(t)y' - \sin(t)z + \cos(t)z' \end{aligned}$$

$y_2' = 0 \cdot z + 2z' + 2z' + 2tz'' + 2y + 2ty' + 2ty' + t^2 y'' \ldots$

General block scheme – equations III

3. **2. integrator** $B_0 z + A_0 y = -y_2'$

$$\begin{aligned} B_0 z + A_0 y = &-4z' - 2tz'' - 2y - 4ty' - t^2 y'' \\ &+ \cos(t)y + \sin(t)y' - \sin(t)z + \cos(t)z' \\ e^{3t} z + e^{-t} y = &-4z' - 2tz'' - 2y - 4ty' - t^2 y'' \\ &+ \cos(t)y + \sin(t)y' - \sin(t)z + \cos(t)z' \end{aligned}$$

After rearranging into the proper form, the **resulting system equation**:

$$\begin{aligned} &t^2 y'' + 4ty' - \sin(t)y' + 2y + e^{-t} y - \cos(t)y \\ = &-2tz'' - 4z' + \cos(t)z' - \sin(t)z - e^{3t} z \end{aligned}$$

$$\boxed{\begin{aligned} &t^2 y'' + y'(4t - \sin(t)) + y(2 + e^{-t} - \cos(t)) \\ = &-2tz'' + z'(-4 + \cos(t)) + z(-\sin(t) - e^{3t}) \end{aligned}}$$

Solution check

Adjoint coefficients or the resulting equation

$$t^2y'' + y'(4t - \sin(t)) + y(2 + e^{-t} - \cos(t))$$
$$= -2tz'' + z'(-4 + \cos(t)) + z(-\sin(t) - e^{3t})$$

$$A_2^* = t^2 \qquad B_2^* = -2t$$
$$A_1^* = 4t - \sin(t) \qquad B_1^* = -4 + \cos(t)$$
$$A_0^* = 2 + e^{-t} - \cos(t) \qquad B_0^* = -\sin(t) - e^{3t}$$

To check if the coefficients are correct, the obtained adjoint coefficients (A_2^*,A_1^*,A_0^*,B_2^*,B_1^*,B_0^*) can be substituted into the input equations.

Solution check I

New input equations (with substituted adjoint coefficients):

1. **implicator** $B_2^* z + y_1 + A_2^* y = 0$

$$-y_1 = B_2^* z + A_2^* y$$
$$-y_1 = -2tz + t^2 y$$

We are solving the scheme from the right, we need to create an input signal for the integrator, so we need to derive equation for y_1.

$$-y_1' = -2z - 2tz' + 2ty + t^2 y'$$

Solution check II

2. **1. integrátor** $B_1^* z + y_2 + A_1^* y = -y_1'$

$$B_1^* z + y_2 + A_1^* y = -2z - 2tz' + 2ty + t^2 y'$$
$$(-4 + \cos(t))z + y_2 + (4t - \sin(t))y = -2z - 2tz' + 2ty + t^2 y'$$

Simplifying, solving for $-y_2$, and derivative:

$$\begin{aligned}
-y_2 &= -2z + 2tz' + 2ty - t^2 y' + \cos(t)z - \sin(t)y \\
-y_2' &= -2z' + 2z' + 2tz'' + 2y + 2ty' - 2ty' - t^2 y'' \\
&\quad - \sin(t)z + \cos(t)z' - \cos(t)y - \sin(t)y' \\
-y_2' &= 2tz'' + 2y - t^2 y'' - \sin(t)z + \cos(t)z' - \cos(t)y - \sin(t)y'
\end{aligned}$$

Solution check III

3. **2. integrator** $B_0^* z + A_0^* y = -y_2'$

$$
\begin{aligned}
&B_0^* z + A_0^* y = -y_2' \\
&(-\sin(t) - e^{3t})z + (2 + e^{-t} - \cos(t))y \\
&= 2tz'' + 2y - t^2 y'' - \sin(t)z + \cos(t)z' - \cos(t)y - \sin(t)y'
\end{aligned}
$$

Rearranging into the proper form

$$
t^2 y'' + \sin(t)y' + e^{-t} y = 2tz'' + \cos(t)z' + e^{3t} z
$$

The adjoint coefficients A_2^*,A_1^*,A_0^*,B_2^*,B_1^* a B_0^* match the original coefficients A_2,A_1,A_0,B_2,B_1 a B_0 and can be used.

Summary – 2nd order I

The input **second order** differential equation

$$t^2 y'' + \sin(t)y' + e^{-t}y = 2tz'' + \cos(t)z' + e^{3t}z$$

can be solved by the system of **two** first order differential equations

$$\begin{aligned}
\boldsymbol{y_2'} &= \underline{-B_0^* z - A_0^* y =} \\
&\quad \boldsymbol{-(-\sin(t) - e^{3t})z - (2 + e^{-t} - \cos(t))y}\,, \quad \boldsymbol{y_2(0) = 0} \\
\boldsymbol{y_1'} &= \underline{-B_1^* z - A_1^* y - y_2 =} \\
&\quad \boldsymbol{-(-4 + \cos(t))z - (4t - \sin(t))y - y_2}\,, \quad \boldsymbol{y_1(0) = 0}
\end{aligned}$$

Summary – 2nd order II

The output of the system (**always implicator**) can be expressed as

$$
\begin{aligned}
B_2^* z + y_1 + A_2^* y &= 0 \\
y &= -\frac{B_2^* z + y_1}{A_2^*} \\
\boldsymbol{y} &= \boldsymbol{-\frac{1}{t^2}(-2tz + y_1)}
\end{aligned}
$$

Homework

Solve:

$$e^{-t}y'' + \sin(t)y' + \cos(t)y = 2tz'' + \cos(t)z' + t^2 z$$

Use both approaches, compare the time needed for solution. Do the solution check.

Thank you for your attention!