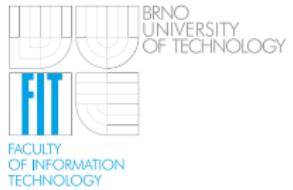


Stability and Convergence of Numerical Integration Methods

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3 Introduction

8 Stability and convergency of numerical methods

11 Dahlquis problem

Explicit numerical methods

Implicit numerical methods

Numerical solutions in MATLAB

Ordinary differential equations - Initial value problem

$$y'(t) = f(t, y(t)), \quad y(t_0) = y_0.$$

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Analytic solution

- derivations, integrations, ...
- in most cases impossible...

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Numerical methods

- Taylor series
- Euler method
- Runge-Kutta methods
- Trapezoidal method
- ...

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One-step numerical methods

$$y_{i+1} = y_i + h\Phi(t_i, y_i, y_{i+1}, h; f).$$

Condition: *The sequence of values y_i (approximation using numerical methods) must converge to exact solution $y(t_i)$.*

Existence of limit is expected

$$\lim_{h \rightarrow 0, i \rightarrow \infty} (y_i) = y(t_i).$$

Stability is connected with:

- numerical method
- ODE system (eigenvalues)
- integration step size

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- ODE system (eigenvalues)
- integration step size

The stiff systems require **large stability domain**.

I didn't like all these "strong", "perfect", "absolute", "generalized", "super", "hyper", "complete" and so on in mathematical definitions. I wanted something neutral; and having been impressed by David Young's "property A", I chose the term "A-stable".

(G. Dahlquist, in 1979)

Dahlquist test equation

$$y' = \lambda y, \quad y(0) = y_0 = 1, \quad \lambda < 0$$

Analytic solution

$$y = y_0 \cdot e^{(\lambda t)}.$$

Stability condition

$$|y_{i+1}| \leq |y_i|$$

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Stability function of numerical method

$$R(z) = \frac{y_{i+1}}{y_i}, \text{ where } z = h\lambda$$

 λ is generally a complex number, $z \in \mathcal{C}$

Stability condition

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Stability function of numerical method

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 λ is generally a complex number, $z \in \mathcal{C}$ (absolute) **Stability domain** of numerical method

$$D = \{z \in \mathcal{C}; |R(z)| \leq 1\}$$

Explicit Euler method

$$y_{i+1} = y_i + hy'_i$$

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substitution the ODE $y' = \lambda y$

$$y_{i+1} = y_i + h\lambda y_i$$

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Stability function

$$R_E(z) = \frac{y_{i+1}}{y_i} = 1 + z$$

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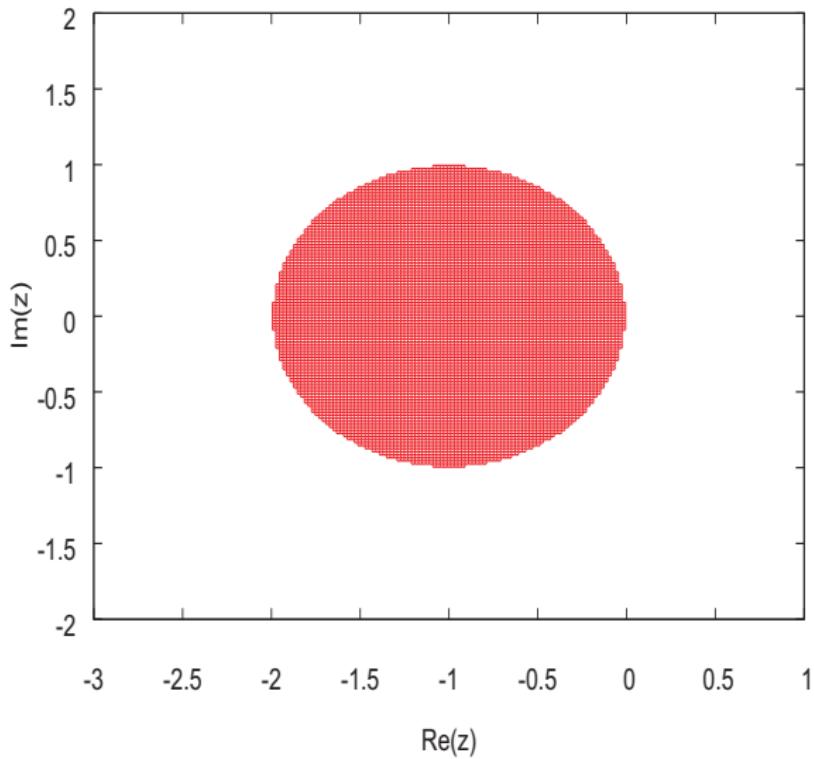
Stability domain

$$D = \{z \in \mathcal{C}; |1 + z| \leq 1\}$$

Explicit Euler method - stability domain



$$D = \{z \in \mathcal{C}; |1 + z| \leq 1\}$$



Explicit Taylor series

$$y_{i+1} = y_i + hy'_i + \frac{h^2}{2!}y''_i + \frac{h^3}{3!}y'''_i + \cdots + \frac{h^n}{n!}y_i^{(n)}$$

Explicit Taylor series

$$y_{i+1} = y_i + hy'_i + \frac{h^2}{2!}y''_i + \frac{h^3}{3!}y'''_i + \cdots + \frac{h^n}{n!}y_i^{(n)}$$

$$y_{i+1} = y_i + DY1_i + DY2_i + \cdots + DYN_i, \quad ORD = n$$

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substitution of higher derivations

$$y_{i+1} = y_i + h\lambda y_i + \frac{h^2}{2}\lambda^2 y_i + \frac{h^3}{3!}\lambda^3 y_i + \cdots + \frac{h^n}{n!}\lambda^n y_i$$

$$y_{i+1} = (1 + h\lambda + \frac{h^2}{2}\lambda^2 + \frac{h^3}{3!}\lambda^3 + \cdots + \frac{h^n}{n!}\lambda^n)y_i$$

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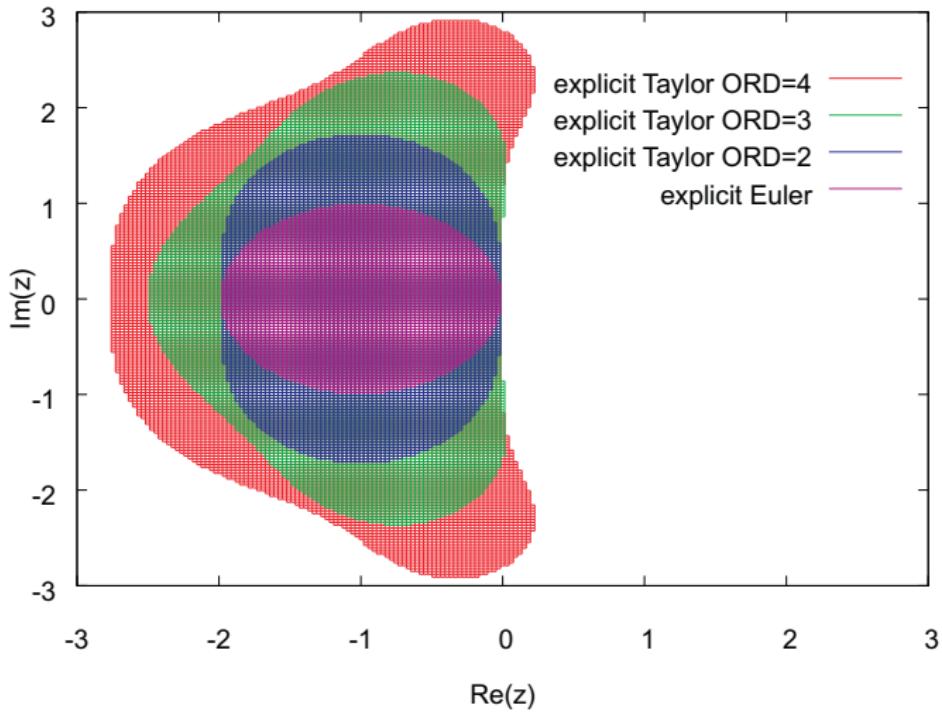
$$y_{i+1} = (1 + h\lambda + \frac{h^2}{2}\lambda^2 + \frac{h^3}{3!}\lambda^3 + \cdots + \frac{h^n}{n!}\lambda^n)y_i$$

Stability function

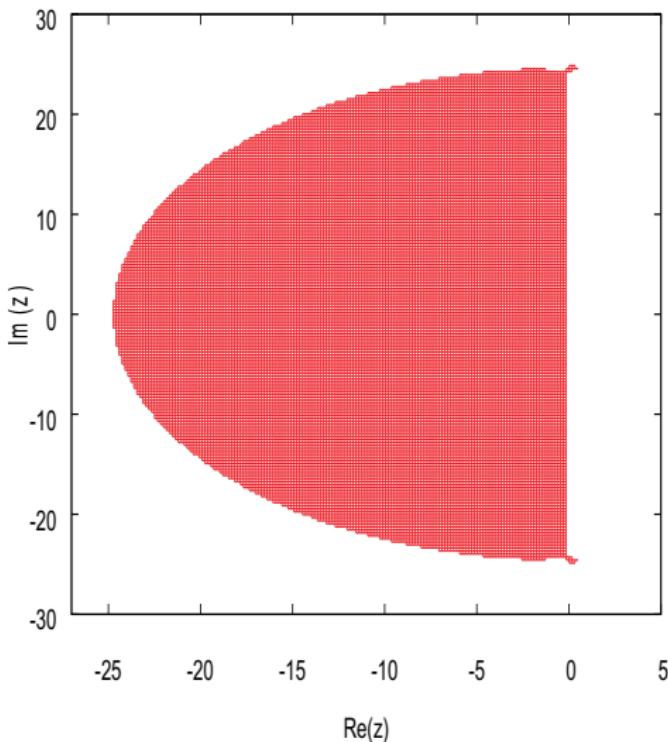
$$R_E(z) = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots + \frac{z^n}{n!}$$

Stability domains - explicit Taylor

$$\left| 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots + \frac{z^n}{n!} \right| \leq 1$$



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Explicit Taylor series

$$y_{i+1} = y_i + hy'_i + \frac{h^2}{2!}y''_i + \frac{h^3}{3!}y'''_i + \cdots + \frac{h^n}{n!}y_i^{(n)}$$

$$y_{i+1} = y_i + DY1_i + DY2_i + \cdots + DYn_i, \quad ORD = n$$

Note: The Taylor series terms can be calculated recurrently in the form

$$DY1_i = h\lambda y_i,$$

$$DY2_i = \frac{h}{2}\lambda DY1_i,$$

$$\vdots$$

$$DYn_i = \frac{h}{n}\lambda DY(n-1)_i.$$

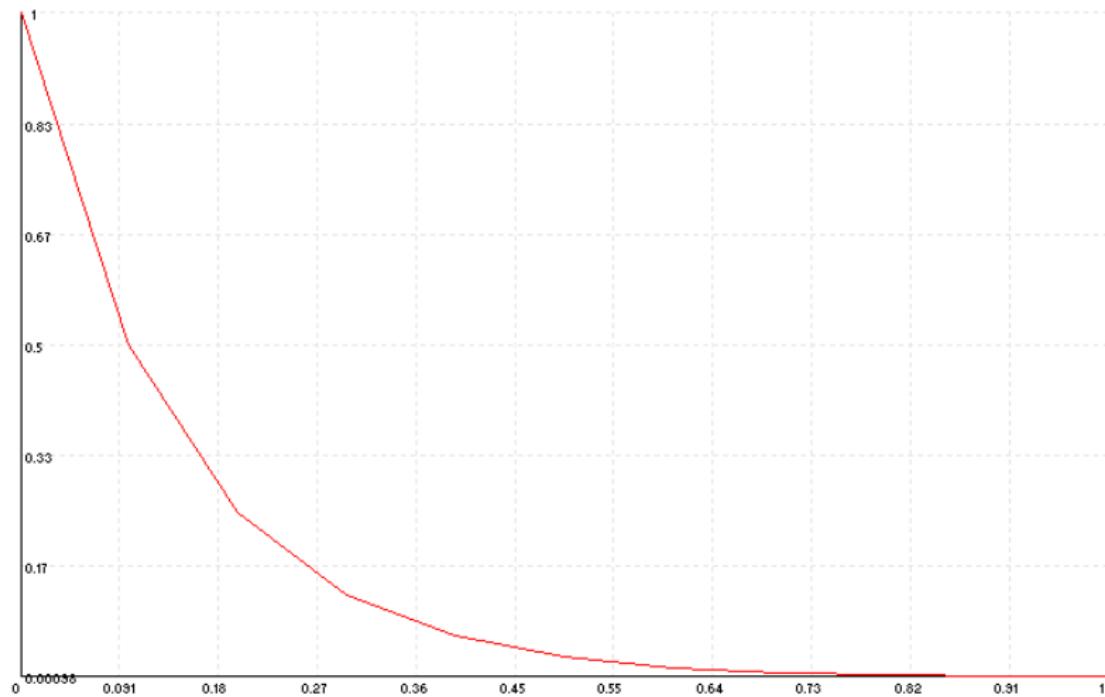
Numerical solution - explicit Taylor

$ORD = 2, h = 0.1, \lambda = -10$



Legenda: x-axis variables: t

y-axis variables: y



Taylor series terms:

DY11: -1

DY21: 0.5

DY12: -0.5

DY22: 0.25

DY13: -0.25

DY23: 0.125

DY14: -0.125

DY24: 0.0625

DY15: -0.0625

DY25: 0.03125

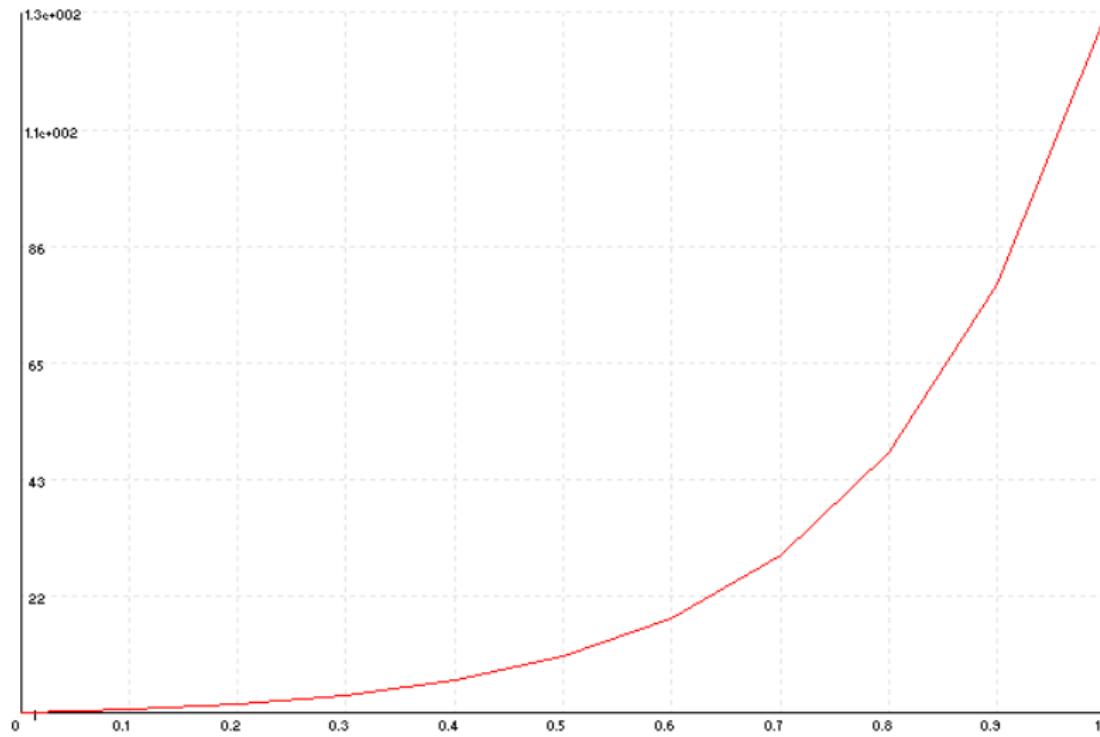
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Numerical solution - explicit Taylor

$ORD = 2, h = 0.1, \lambda = -25$



Legenda: x-axis variables: t y-axis variables: y



Taylor series terms:

DY11: -2.5

DY21: 3.125

DY12: -4.0625

DY22: 5.07813

DY13: -6.60156

DY23: 8.25195

DY14: -10.7275

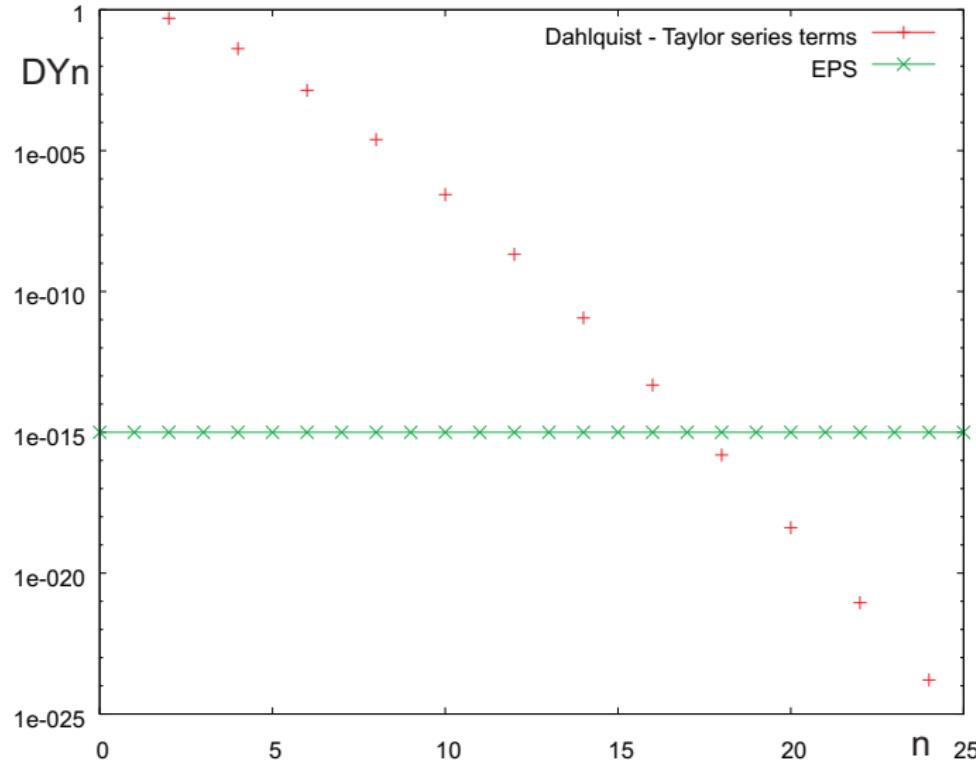
DY24: 13.4094

DY15: -17.4323

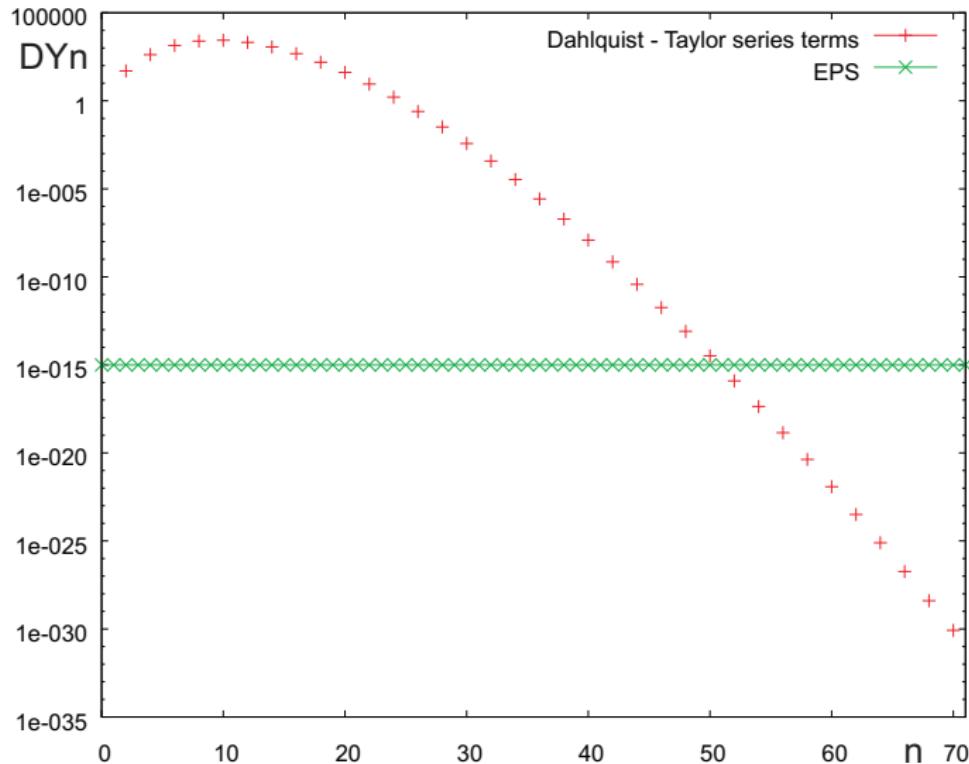
DY25: 21.7903

:

Taylor series terms ($z = -1$) :



Taylor series terms ($z = -10$) :



Taylor series terms ($z = -100$) :

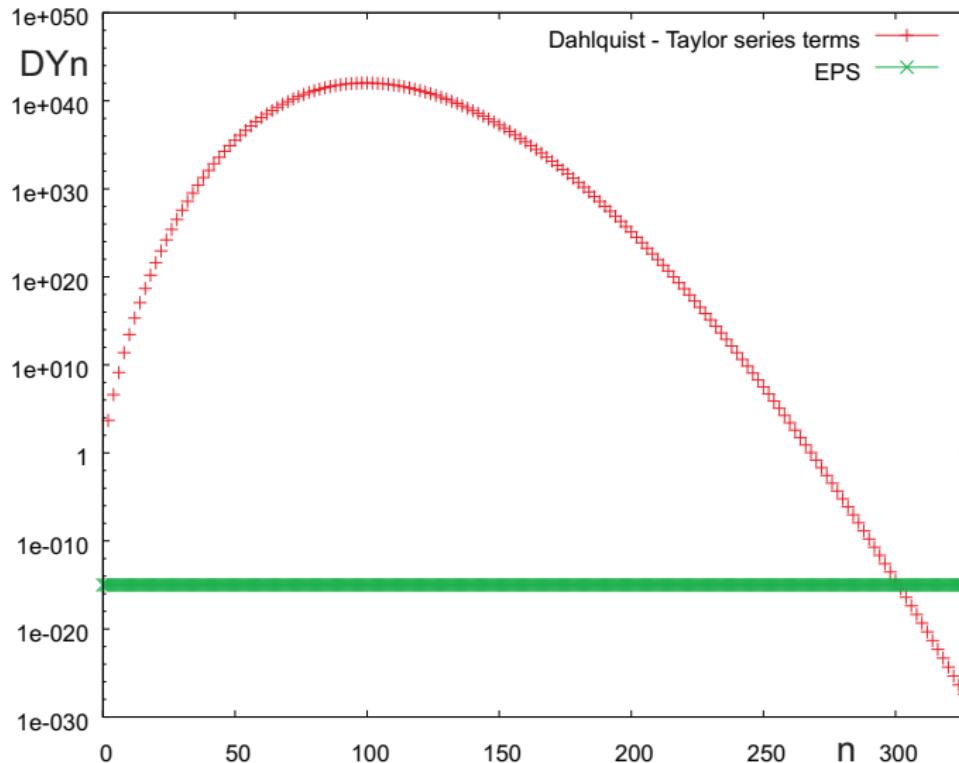


Table: Maximum integration step size h_{max} , explicit numerical methods
($\lambda = -100$, $EPS = 10^{-20}$)

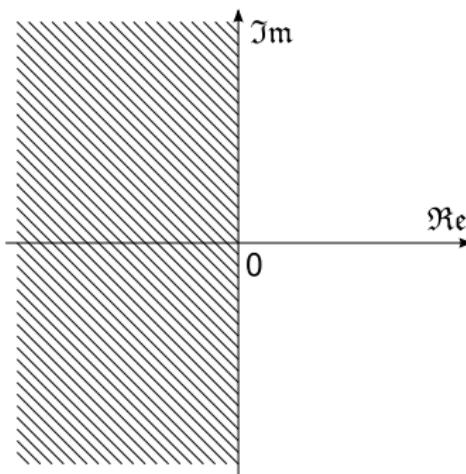
method	h_{max}
Euler method	4.22×10^{-11}
Runge-Kutta method - 2. order	4.77×10^{-8}
Runge-Kutta method - 4. order	1.52×10^{-5}
Taylor series method ($ORD = 63$)	0.1

Definition

Dahlquist 1963: A method, whose stability domain D satisfies

$$D \supset \mathcal{C}^- = \{z \in \mathcal{C}; \Re e(z) \leq 0\},$$

is called **A-stable**.



Definition

Ehle 1969: A method is called **L-stable**, if it is A-stable and if in addition

$$\lim_{\operatorname{Re}(z) \rightarrow -\infty} R(z) = 0.$$

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The L-stable numerical methods are suitable to stiff systems solution.

Implicit Trapezoidal rule

$$y_{i+1} = y_i + \frac{h}{2}(y'_i + y'_{i+1})$$

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substitution of derivations

$$y_{i+1} = y_i + \frac{h}{2}(\lambda y_i + \lambda y_{i+1}) = \frac{y_i(1 + h\lambda/2)}{1 - h\lambda/2}$$

Implicit Trapezoidal rule

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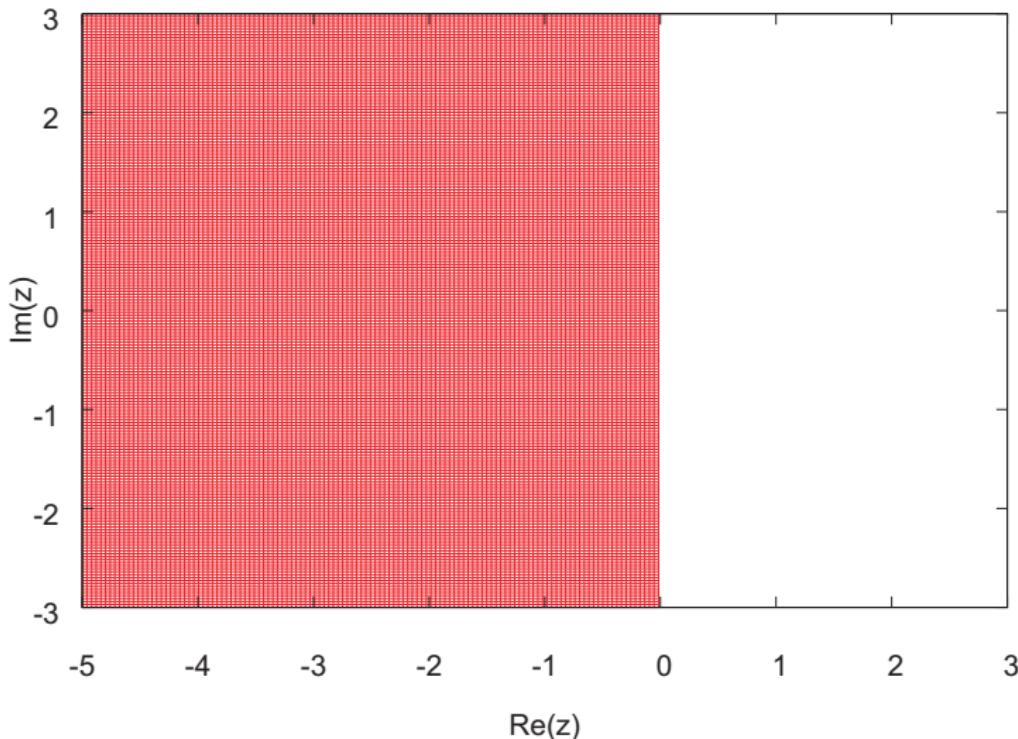
substitution of derivations

$$y_{i+1} = y_i + \frac{h}{2}(\lambda y_i + \lambda y_{i+1}) = \frac{y_i(1 + h\lambda/2)}{1 - h\lambda/2}$$

Stability function

$$R_T(z) = \frac{1 + z/2}{1 - z/2}$$

$$\left| \frac{1+z/2}{1-z/2} \right| < 1$$



Implicit Taylor series

$$y_{i+1} = y_i + hy'_{i+1}$$

Implicit Taylor series

$$y_{i+1} = y_i + hy'_{i+1} - \frac{h^2}{2!}y''_{i+1} + \frac{h^3}{3!}y'''_{i+1} - \cdots - \frac{(-h)^n}{n!}y_{i+1}^{(n)}$$

Implicit Taylor series

$$\begin{aligned}y_{i+1} &= y_i + hy'_{i+1} - \frac{h^2}{2!}y''_{i+1} + \frac{h^3}{3!}y'''_{i+1} - \cdots - \frac{(-h)^n}{n!}y_{i+1}^{(n)} \\y_{i+1} &= y_i - DY1_{i+1} - \cdots - DYN_{i+1}, \quad ORD = n\end{aligned}$$

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substitution of higher derivations

$$\begin{aligned}y_{i+1} &= y_i + h\lambda y_{i+1} - \frac{h^2}{2!}\lambda^2 y_{i+1} + \frac{h^3}{3!}\lambda^3 y_{i+1} - \cdots - \frac{(-h)^n}{n!}\lambda^n y_{i+1} \\y_{i+1} &= \left(\frac{1}{1-h\lambda+\frac{h^2}{2!}\lambda^2-\frac{h^3}{3!}\lambda^3+\cdots+\frac{(-h)^n}{n!}\lambda^n}\right)y_i\end{aligned}$$

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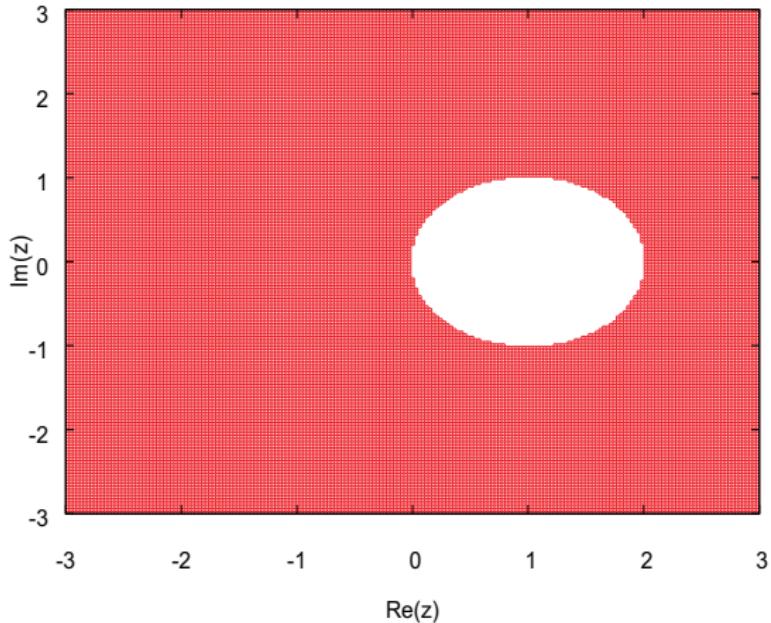
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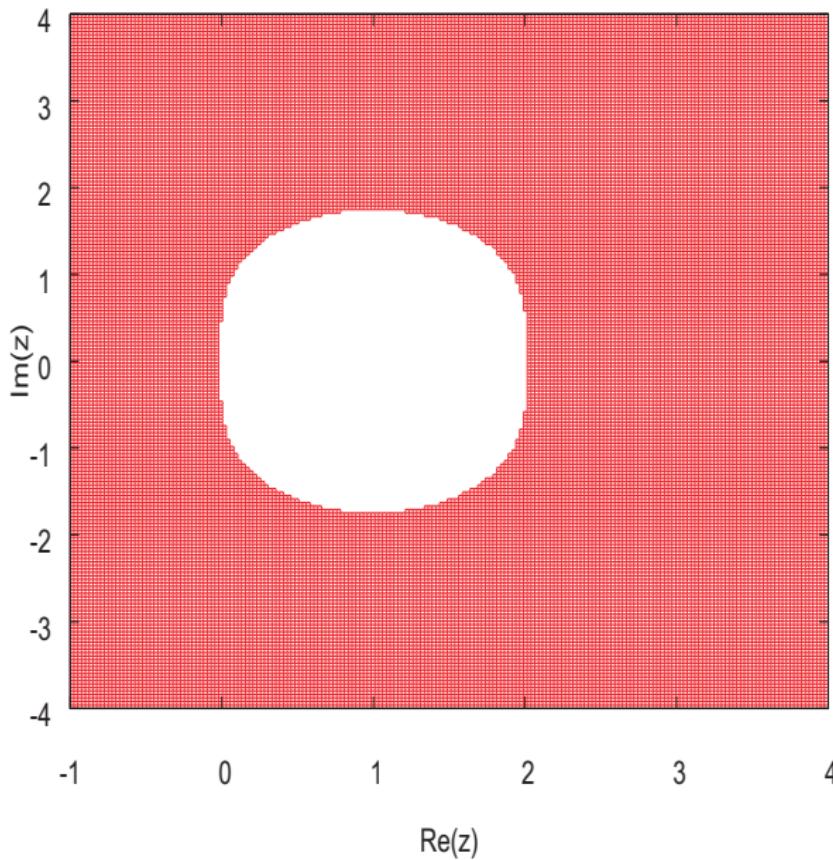
Stability function

$$R_I(z) = \frac{1}{1 - z + \frac{z^2}{2} - \frac{z^3}{3!} + \cdots + \frac{(-z)^n}{n!}}$$

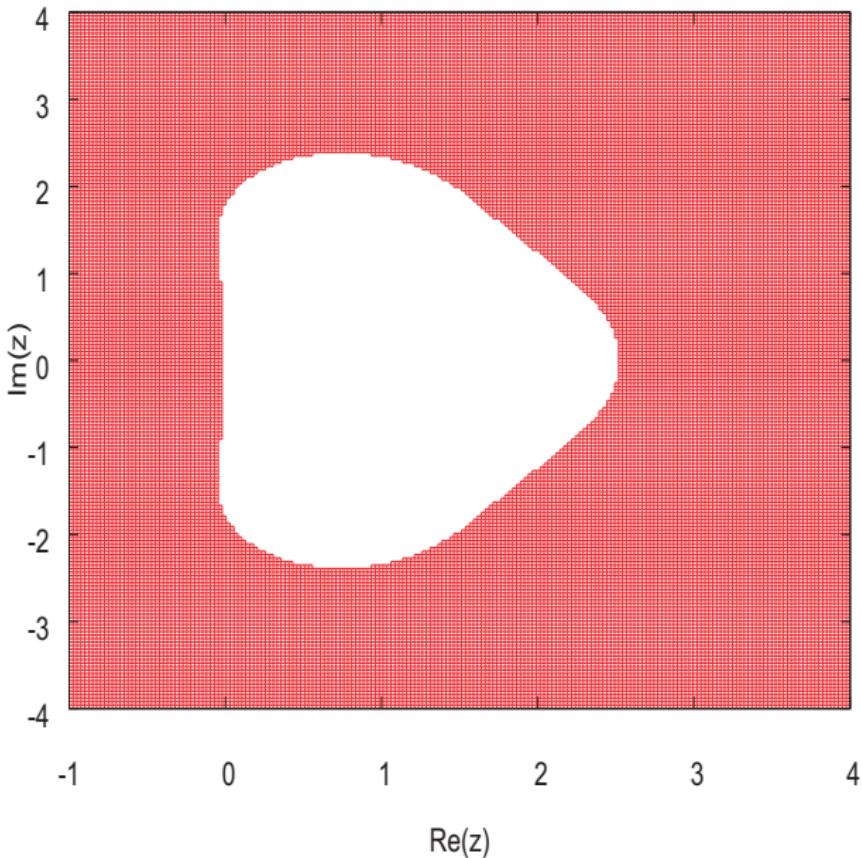
$$\left| \frac{1}{1 - z + \frac{z^2}{2} - \frac{z^3}{3!} + \dots + \frac{(-z)^n}{n!}} \right| \leq 1$$



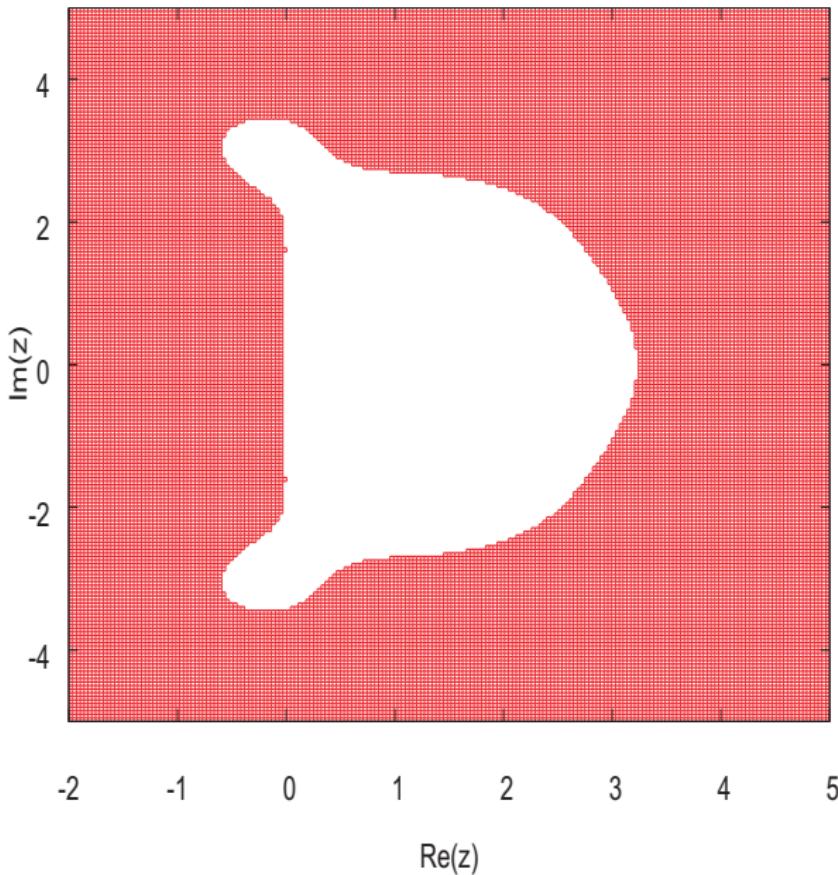
Implicit Taylor $ORD = 2$ - stability domain

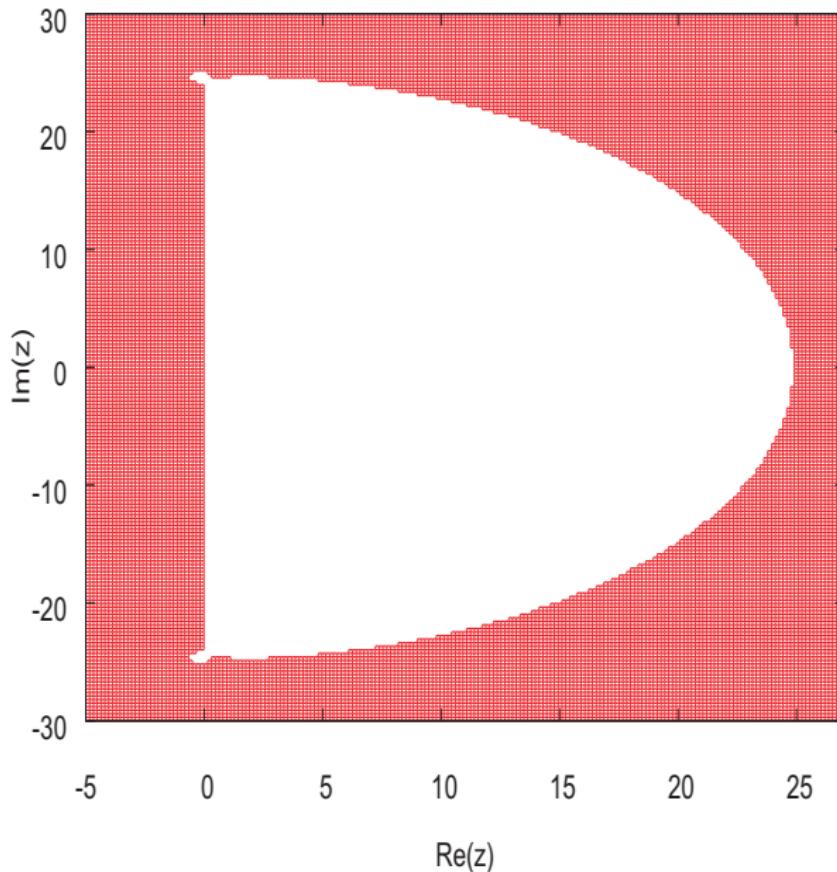


Implicit Taylor $ORD = 3$ - stability domain



Implicit Taylor $ORD = 5$ - stability domain





Implicit Taylor series

$$\begin{aligned}y_{i+1} &= y_i + hy'_{i+1} - \frac{h^2}{2!}y''_{i+1} + \frac{h^3}{3!}y'''_{i+1} - \cdots - \frac{(-h)^n}{n!}y_{i+1}^{(n)} \\y_{i+1} &= y_i - DY1_{i+1} - \cdots - DYn_{i+1}, \quad ORD = n\end{aligned}$$

The Taylor series terms can be calculated recurrently in the form

$$DY1_{i+1} = -h\lambda y_{i+1},$$

$$DY2_{i+1} = -\frac{h}{2}\lambda DY1_{i+1},$$

$$\vdots$$

$$DYn_{i+1} = -\frac{h}{n}\lambda DY(n-1)_{i+1}.$$

The Newton's iteration is in the form

$$y_{i+1,j+1} = y_{i+1,j} - \frac{f(y_{i+1,j})}{f'_{y_{i+1}}(y_{i+1,j})},$$

where

$$f(y_{i+1,j}) = -y_{i+1,j} + y_i - DY1_{i+1,j} - DY2_{i+1,j} - \cdots - DYN_{i+1,j}.$$

The Newton's iteration is in the form

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where

$$f(y_{i+1,j}) = -y_{i+1,j} + y_i - DY1_{i+1,j} - DY2_{i+1,j} - \cdots - DYN_{i+1,j}.$$

$$f'_{y_{i+1}}(y_{i+1,j}) = ?$$

Symbolic computation of the derivative $f'_{y_{i+1}}(y_{i+1,j})$

$$f'_{y_{i+1}}(y_{i+1,j}) = -1 - DY1'_{i+1,j} - DY2'_{i+1,j} - \cdots - DYn'_{i+1,j},$$

where derivation of Taylor series terms are in the form

$$DY1'_{i+1,j} = -h\lambda,$$

$$DY2'_{i+1,j} = -\frac{h}{2}\lambda DY1'_{i+1,j},$$

$$\vdots$$

$$DYn'_{i+1,j} = -\frac{h}{n}\lambda DY(n-1)'_{i+1,j}.$$

Differential formulae used for computation of derivative

$$f'_{y_{i+1}}(y_{i+1,j})$$

$$f'_{y_{i+1}}(y_{i+1,j}) = \frac{f(y_{i+1,j} + h_N) - f(y_{i+1,j})}{h_N},$$

where

$$f(y_{i+1,j} + h_N) = -(y_{i+1,j} + h_N) + y_i - DY1_{N,i+1,j} - DY2_{N,i+1,j} - \cdots - DYn_{N,i+1,j}.$$

Taylor series terms are in the form

$$DY1_{N,i+1,j} = -h\lambda(y_{i+1,j} + h_N),$$

$$DY2_{N,i+1,j} = \frac{(-h)^2}{2!}\lambda^2(y_{i+1,j} + h_N) = \frac{-h}{2}\lambda DY1_{N,i+1,j},$$

$$\vdots \qquad \qquad \vdots$$

$$DYn_{N,i+1,j} = \frac{(-h)^n}{n!}\lambda^n(y_{i+1,j} + h_N) = \frac{-h}{n}\lambda DY(n-1)_{N,i+1,j}.$$

Let's see MATLAB...