On *n*-Path-Controlled Grammars

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Advanced Topics of Theoretical Computer Science

FRVŠ MŠMT FR2581/2010/G1

Outline

- **Introduction**
- **Definitions**
- **Results**
- **Examples**
- **Conclusion**
- **References**

Acknowledgement

This work was partially supported by the FRVŠ MŠMT grant FR2581/2010/G1, the BUT FIT grant FIT-10-S-2, and the research plan MSM0021630528.

What's going on

- Regulated formal model.
- Model based on the restrictions on the derivation trees.
- Actual trend in today's FLT (see (1), (2), (3), (4), (5), (6), (7)).
- Simple extension of context-free grammars.
- One of the ways to increase the generative power of context-free grammar.
- Potentially applicable model.

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Motivation

Generation of *not context-free* languages of the form

- $a^n b^n c^n, a^n b^n c^n d^n, a^n b^n c^n d^n e^n, \ldots$
- $a^k b^l a^k b^l, a^k b^l c^m a^k b^l c^m, a^k b^l c^m d^n a^k b^l c^m d^n, \ldots$

Linear grammar

$G = (V, T, P, S)$, where

- V is an alphabet,
- $T \subseteq V$ is a terminal alphabet,
- P is a finite set of production rules of the form $A \rightarrow x$, where $A \in V - T$, $x \in T^*NT^*$, $N = V - T$,
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Context-free grammar

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- Let ${}_G\triangle(x)$ denote a set of the derivation trees with frontier x with respect to the grammar G starting from S.

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A path

- A path s of $t \in_G\triangle(x)$ is sequence $a_1 \dots a_n$, $n \geq 1$, of nodes of t with:
  - a_1 is the root of t,
  - a_1 is labeled by starting symbol of G,
  - a_n is a leaf of t,
  - a_n is labeled by terminal symbol of G,
  - for each $i = 1, \dots, n-1$, there is an edge from a_i to a_{i+1} in t.
- Let $path(s)$ denote the word obtained by concatenating all symbols of the path s (in order from the top).

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- Two grammars G and G':
  - G generates a language over its alphabet of terminals T.
  - G' generates a language over the total alphabet of G.

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More formal idea of *PC* grammars

A string w generated by G is accepted only if there is a derivation tree t of w with respect to G such that there exists a path in t which is described by a string from $L(G')$.

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Several types of ${}_nPC$ grammars in relation to

- Path-controlled grammars,
- The pumping lemma for linear languages.

Definition of ${}_nPC$ grammar

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- Regular paths do not increase the generative power (see (3) and (5), Prop. 2).
- Linear paths can increase the generative power (see (5)).

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Generated language

$L(G, G') = \{w \in L(G)|$ there is a set C of n different paths in $t \in_G \triangle(w)$ such that for all $p \in C$ it holds $path(p) \in L(G')$ and all $p \in C$ are divided in the common node of $t\}$.

Obvious facts about the paths

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- For a ${}_nPC$ grammar (G, G'), there is some $m_C \in \mathbb{N}$ that denotes a number of common nodes for all $p \in C$.
- For each two $p_1, p_2 \in C$ it holds that $path(p_1) = rDs_1$, $path(p_2) = rDs_2$, where $r \in N^*$, $D \in N$, $s_1, s_2 \in N^*T$ and $|rD| = m_C$.

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Pumping lemma for linear languages

If L is a linear language, then there are $p, q \in \mathbb{N}$ such that each string $z \in L$ with $|z| \geq p$ can be written in the form $z = uvwxy$ with $0 < |vx| \leq |uvxy| \leq q$, such that $uv^iwx^iy \in L$ for all $i \geq 1$.

Types of n-path-controlled grammars

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Types of ${}_nPC$ grammars

- ${}_n^{I}PC$ if C satisfies $0 \leq m_C \leq |u|$,
- ${}_n^{II}PC$ if C satisfies $|u| < m_C \leq |uv|$,
- ${}_n^{III}PC$ if C satisfies $|uv| < m_C \leq |uvw|$,
- ${}_n^{IV}PC$ if C satisfies $|uvw| < m_C \leq |uvwx|$,
- ${}_n^{V}PC$ if C satisfies $|uvwx| < m_C \leq |uvwxy|$,

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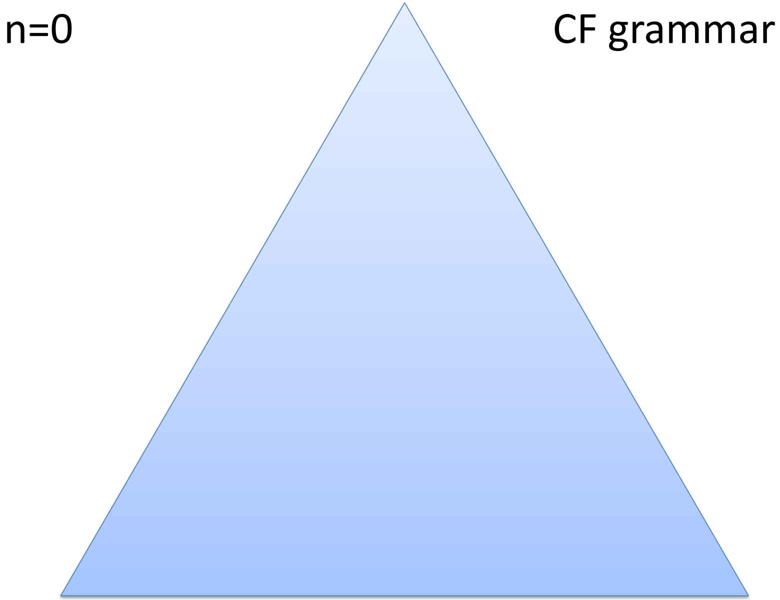
- ${}^{II}_{n}PC$ if C satisfies $|u| < m_C \leq |uv|$,
- ${}^{III}_{n}PC$ if C satisfies $|uv| < m_C \leq |uvw|$,
- ${}^{IV}_{n}PC$ if C satisfies $|uvw| < m_C \leq |uvwx|$,
- ${}^{V}_{n}PC$ if C satisfies $|uvwx| < m_C \leq |uvwxy|$,

where $uvwxy$ is the shortest path from C.

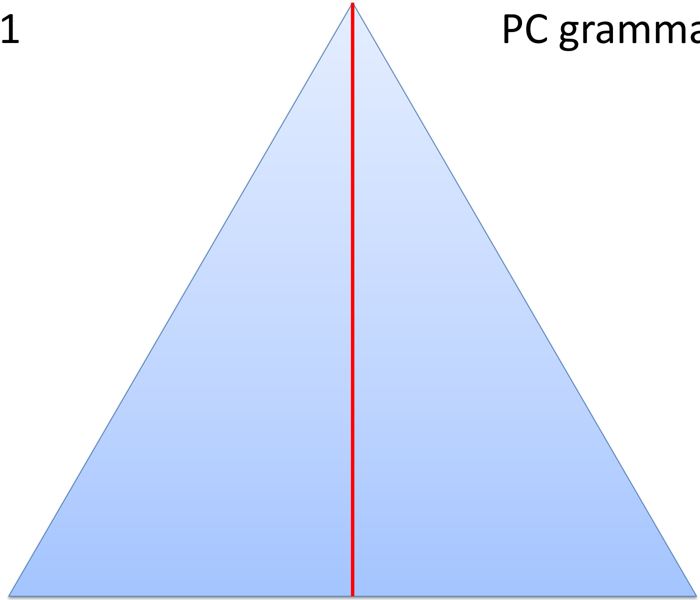
Language families

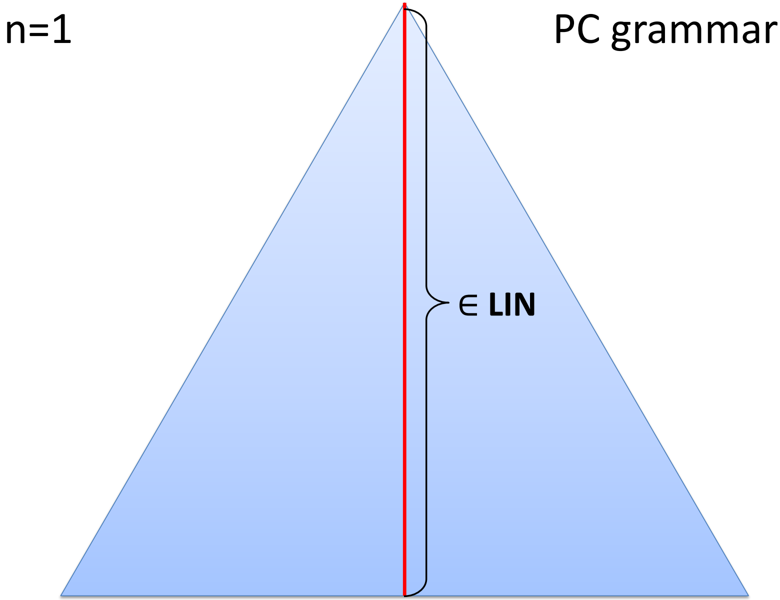
The family of the languages generated by LIN, CF, PC, ${}_nPC$, ${}^{I}_{n}PC$, ${}^{II}_{n}PC$, ${}^{III}_{n}PC$, ${}^{IV}_{n}PC$, ${}^{V}_{n}PC$ grammars is denoted by **LIN**, **CF**, **PC**, ***n*-PC**, **I-*n*-PC**, **II-*n*-PC**, **III-*n*-PC**, **IV-*n*-PC**, **V-*n*-PC**, respectively.


n=0
CF grammar

n=1

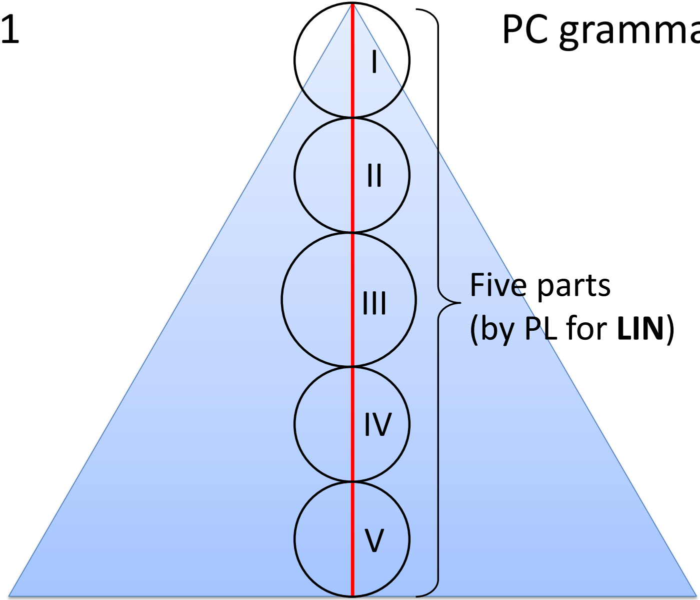
PC grammar




n=1
PC grammar
∈ LIN

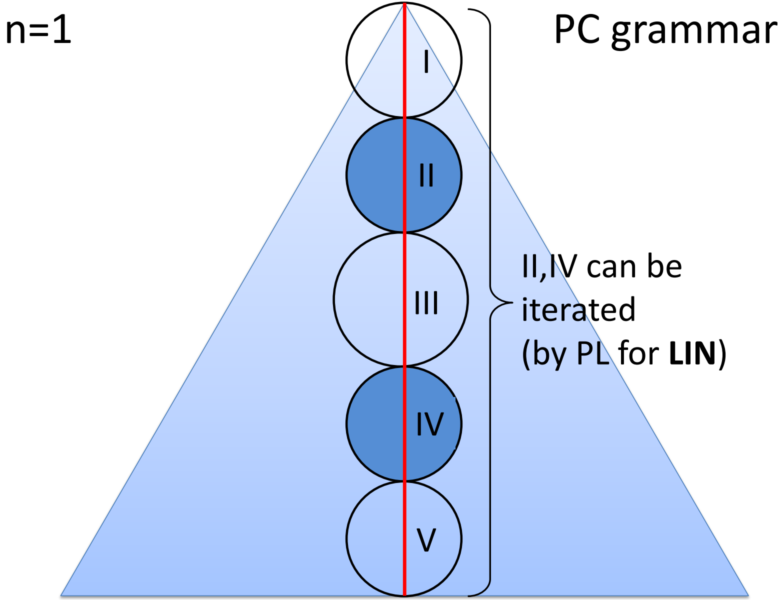
Types of *n*-path-controlled grammars

n=1



PC grammar

Five parts
(by PL for **LIN**)


n=1
PC grammar
I
II
III
IV
V
II,IV can be
iterated
(by PL for **LIN**)

Theorem 1

PC $=$ **1-PC** $=$ **I-1-PC** $=$ **II-1-PC** $=$ **III-1-PC** $=$ **IV-1-PC** $=$ **V-1-PC**.

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Proof: The equality clearly follows from the definitions of PC, ${}_nPC$, and ${}^i_nPC$, for $i = I, II, III, IV, V$, grammars.
Informally: One path to control means no division of the controlled paths.

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Theorem 2

If $L \in$ **III-n-PC**, for $n = card(C) \geq 0$, then there are $p, q \in \mathbb{N}$ such that each $z \in L$ with $|z| > p$ can be written in the form $z = u_1 v_1 u_2 v_2 \ldots u_{2n+2} v_{2n+2} u_{2n+3}$, such that $0 < |v_1 v_2 \ldots v_{2n+2}| \leq q$ and $u_1 v_1^i u_2 v_2^i \ldots u_{2n+2} v_{2n+2}^i u_{2n+3} \in L$ for all $i \geq 1$.

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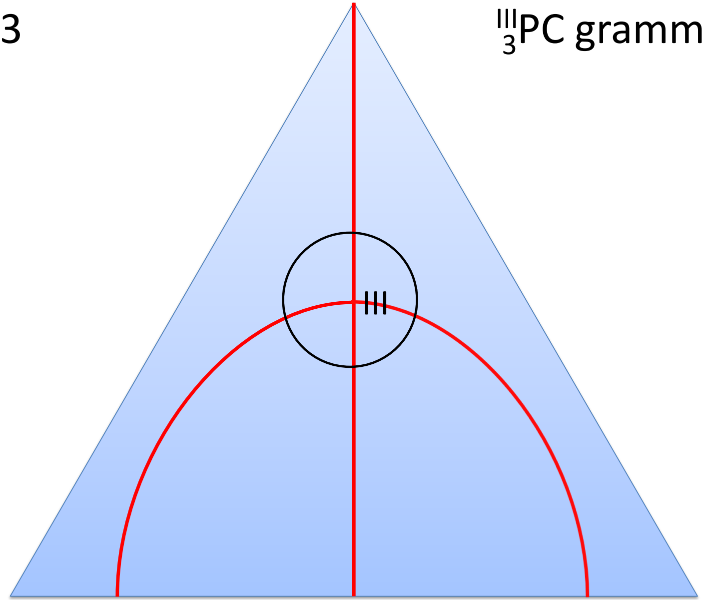
Notice that for $n = 0$, the Theorem 2 holds for context-free languages.

n=3

$^{III}_{3}$PC grammar



n=3

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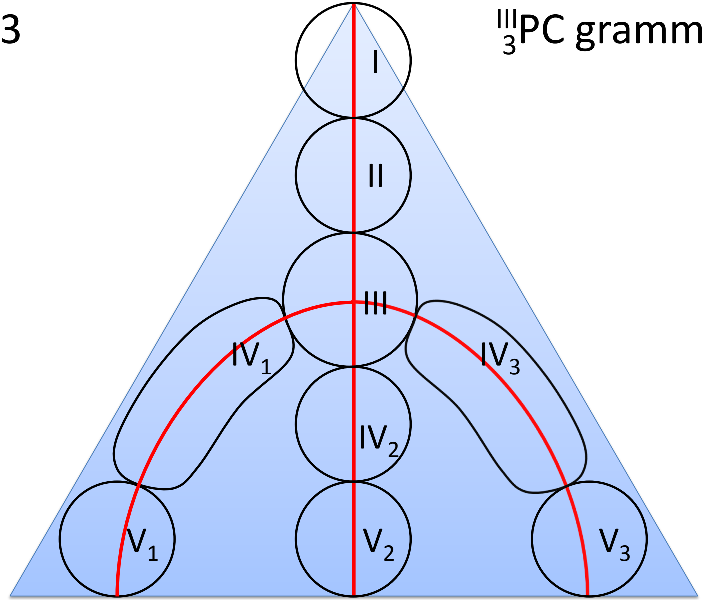
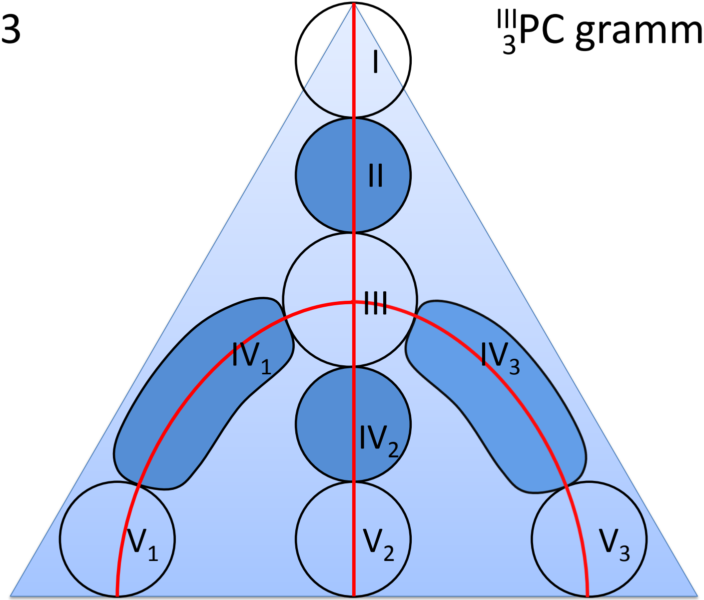
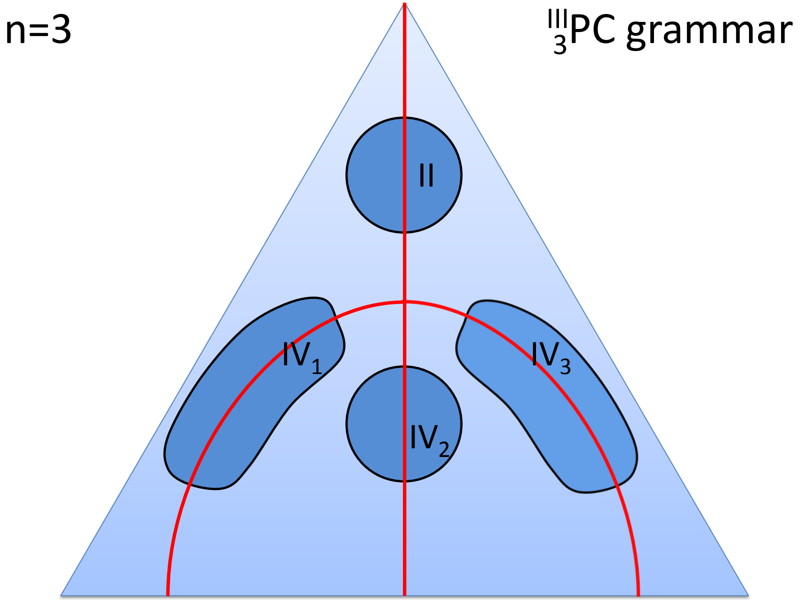


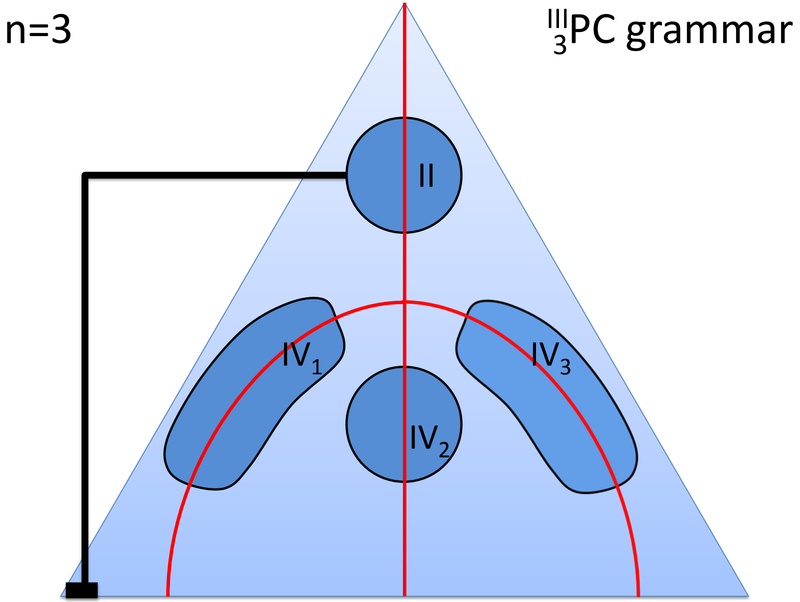
n=3

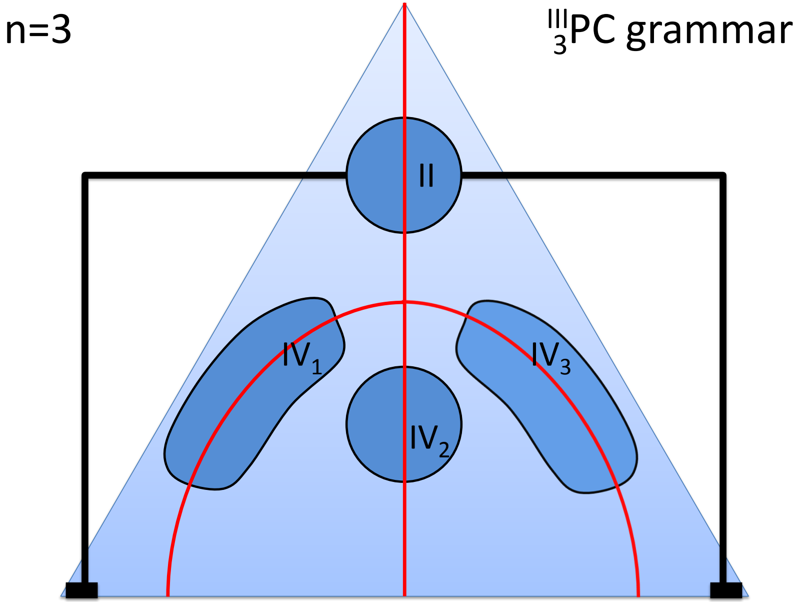
$_{3}^{III}$PC grammar




n=3
$^{III}_{3}$PC grammar
II
IV_1
IV_2
IV_3

The idea of Theorem 1




n=3
$^{III}_{3}$PC grammar
II
IV$_1$
IV$_2$
IV$_3$

The idea of Theorem 1

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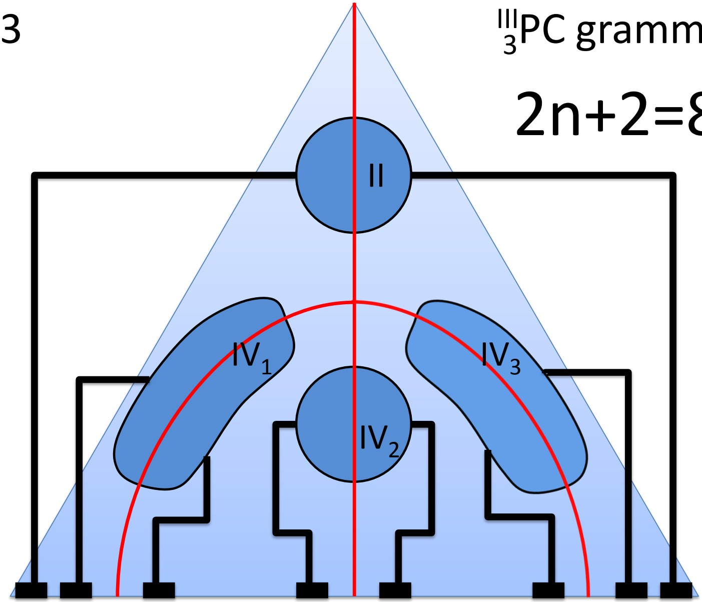
$^{III}_{3}$PC grammar

2n+2=8



Theorem 2

If $L \in$ **III-n-PC**, for $n = card(C) \geq 0$, then there are $p, q \in \mathbb{N}$ such that each $z \in L$ with $|z| > p$ can be written in the form $z = u_1 v_1 u_2 v_2 \dots u_{2n+2} v_{2n+2} u_{2n+3}$, such that $0 < |v_1 v_2 \dots v_{2n+2}| \leq q$ and $u_1 v_1^i u_2 v_2^i \dots u_{2n+2} v_{2n+2}^i u_{2n+3} \in L$ for all $i \geq 1$.

Proof Idea:

- Let (G, G') be a ${}^{III}_{n}PC$-grammar, where
    - $G = (V, T, P, S)$,
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- Let (G, G') be a ${}_n^{III}PC$-grammar, where
  - $G = (V, T, P, S)$,
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- Consider $t \in_{(G,G')} \triangle(z)$. For each $path(s) = SA_1 \ldots A_k a$ of t, where $s \in C$, consider
  - the rules $A_i \to x_i A_{i+1} y_i$ used when passing from A_i to A_{i+1} on this path,
  - the rule $A_k \to x_k a y_k$ used in the last step of the derivation in G corresponding to the path s.

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- Consider that any $x_i y_i$, $i = 1, \ldots, k$, contains a nonterminal B that do not belong on any path $s \in C$. Clearly, there is substring z' of z derived from B.

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- By the pumping lemma for context-free languages, z'_1, z'_2 are bounded in length.

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Proof Idea:

- If $L(G)$ is infinite, the string $path(s) \in L(G')$ is potentially arbitrarily long. Thus, if $path(s) = u_s v_s x_s y_s z_s$ with $|u_s v_s x_s y_s z_s| \geq k_2$, for some $k_2 \geq 0$, then $u_s v_s x_s y_s z_s$ satisfies $u_s v_s^i x_s y_s^i z_s \in L(G')$, for $i \geq 1$.

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- The derivations starting from the symbols of v and y can be repeated in G.

Theorem 2

If $L \in$ **III-n-PC**, for $n = card(C) \geq 0$, then there are $p, q \in \mathbb{N}$ such that each $z \in L$ with $|z| > p$ can be written in the form $z = u_1 v_1 u_2 v_2 \dots u_{2n+2} v_{2n+2} u_{2n+3}$, such that $0 < |v_1 v_2 \dots v_{2n+2}| \leq q$ and $u_1 v_1^i u_2 v_2^i \dots u_{2n+2} v_{2n+2}^i u_{2n+3} \in L$ for all $i \geq 1$.

Proof Idea:

- If $L(G)$ is infinite, the string $path(s) \in L(G')$ is potentially arbitrarily long. Thus, if $path(s) = u_s v_s x_s y_s z_s$ with $|u_s v_s x_s y_s z_s| \geq k_2$, for some $k_2 \geq 0$, then $u_s v_s x_s y_s z_s$ satisfies $u_s v_s^i x_s y_s^i z_s \in L(G')$, for $i \geq 1$.
- The derivations starting from the symbols of v and y can be repeated in G.
- Since (G, G') is ${}_n^{III}PC$ grammar, it follows that:
    - the derivations starting from the symbols of v in G are common for all $s \in C$,
    - the derivations starting from the symbols of y in G are potentially unique for each $s \in C$.

Theorem 2

If $L \in$ **III-n-PC**, for $n = card(C) \geq 0$, then there are $p, q \in \mathbb{N}$ such that each $z \in L$ with $|z| > p$ can be written in the form $z = u_1 v_1 u_2 v_2 \dots u_{2n+2} v_{2n+2} u_{2n+3}$, such that $0 < |v_1 v_2 \dots v_{2n+2}| \leq q$ and $u_1 v_1^i u_2 v_2^i \dots u_{2n+2} v_{2n+2}^i u_{2n+3} \in L$ for all $i \geq 1$.

Proof Idea:

- Consider the derivations starting from v in G. This leads to the pumping of two substrings v_1, v_{2n+2} of z—one in the left-hand side, one in the right-hand side controlled by the common part of all $s \in C$.

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Proof Idea:

- Consider the derivations starting from v in G. This leads to the pumping of two substrings v_1, v_{2n+2} of z—one in the left-hand side, one in the right-hand side controlled by the common part of all $s \in C$.
- Consider the derivations starting from y in G. This leads to the pumping of two substrings of z—one in the left-hand side, one in the right-hand side corresponding to each $s \in C$. For each $s_{i+1} \in C$, denote this two substrings v_{2i+2}, v_{2i+3}, $i = 0, 1, \ldots, n-1$. Since (G, G') is ${}^{III}_{n}PC$ grammar, we obtain $2n$ pumped substrings of z.

Theorem 2

If $L \in$ **III-n-PC**, for $n = card(C) \geq 0$, then there are $p, q \in \mathbb{N}$ such that each $z \in L$ with $|z| > p$ can be written in the form $z = u_1 v_1 u_2 v_2 \dots u_{2n+2} v_{2n+2} u_{2n+3}$, such that $0 < |v_1 v_2 \dots v_{2n+2}| \leq q$ and $u_1 v_1^i u_2 v_2^i \dots u_{2n+2} v_{2n+2}^i u_{2n+3} \in L$ for all $i \geq 1$.

Proof Idea:

- By the pumping lemma for context-free languages, the substrings $v_1, v_2, \dots, v_{2n+2}$ are bounded in length.

Theorem 2

If $L \in$ **III-n-PC**, for $n = card(C) \geq 0$, then there are $p, q \in \mathbb{N}$ such that each $z \in L$ with $|z| > p$ can be written in the form $z = u_1 v_1 u_2 v_2 \ldots u_{2n+2} v_{2n+2} u_{2n+3}$, such that $0 < |v_1 v_2 \ldots v_{2n+2}| \leq q$ and $u_1 v_1^i u_2 v_2^i \ldots u_{2n+2} v_{2n+2}^i u_{2n+3} \in L$ for all $i \geq 1$.

Proof Idea:

- By the pumping lemma for context-free languages, the substrings $v_1, v_2, \ldots, v_{2n+2}$ are bounded in length.
- Thus, the total length of the $2n+2$ pumped substrings of z is bounded by a constant q.

Corollary 3

III-n-PC cannot count to $2n + 3$, but can count to $2n + 2$.

Proof: $L = \{a^i b^i c^i d^i e^i f^i g^i \mid i \geq 1\} \notin$ **III-2-PC**, but $L \in$ **III-3-PC**.

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Corollary 4

There is an infinite hierarchy of $\bigcup_{i=0}^{n}$ **III-i-PC** languages.

Proof: $\bigcup_{i=0}^{n}$ **III-i-PC** $\subset \bigcup_{i=0}^{n+1}$ **III-i-PC**, for $n \geq 0$, is proper.

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Corollary 5

III-n-PC is not closed under concatenation.

Proof: $L = \{a^i a^i a^i a^i a^i a^i \mid i \geq 1\} \in$ **III-2-PC**, but $LL \notin$ **III-2-PC**.

Example 1

Consider ${}_2^{lll}PC$ grammar (G, G'), where

$$G = (\{S, X, Y, U, V, a, b, c, d, e, f\}, \{a, b, c, d, e, f\}, P, S)$$
$$P = \{S \to aSf, \quad S \to aXYf, \quad X \to bXc, \quad Y \to dYe,$$
$$X \to U, \quad U \to bc, \quad Y \to V, \quad V \to de\}$$
$$L(G') = \{S^n X^n Ub \cup S^n Y^n Vd \mid n \geq 1\}$$

$$L(G, G') = \{a^i b^i c^i d^i e^i f^i \mid i \geq 1\}$$

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$$L(G, G') = \{a^i b^i c^i d^i e^i f^i \mid i \geq 1\}$$

Example of the derivation:

$S \Rightarrow aSf \Rightarrow aaSff \Rightarrow aaaSfff \Rightarrow aaaaXYffff \Rightarrow aaaabXcYffff \Rightarrow aaaabbXccYffff \Rightarrow aaaabbbXcccYfffff \Rightarrow aaaabbbUcccYffff \Rightarrow aaaabbbbccccYffff \Rightarrow aaaabbbbccccdYeffff \Rightarrow aaaabbbbccccddYeeffff \Rightarrow aaaabbbbccccdddYeeeffff \Rightarrow aaaabbbbccccdddVeeeffff \Rightarrow aaaabbbbccccddddeeeeffff = a^4b^4c^4d^4e^4f^4$

Example 2

Let us have ${}^{III}_{n}PC$ grammar (G, G'), $n \geq 0$, where

$$G_1 = (\{S\} \cup \{A_i, B_i \mid i = 1, \ldots, n\} \cup \{a_i \mid i = 1, \ldots, 2n+2\},$$
$$\{a_i \mid i = 1, \ldots, 2n+2\}, P, S)$$
$$P = \{S \to a_1 S a_{2n+2}, \quad S \to a_1 A_1 A_2 \ldots A_n a_{2n+2}\} \cup$$
$$\{A_{i+1} \to a_{2i+2} A_{i+1} a_{2i+3}, \quad A_{i+1} \to B_{i+1},$$
$$B_{i+1} \to a_{2i+2} a_{2i+3} \mid i = 0, \ldots, n-1\}$$
$$L(G') = \bigcup_{i=1}^{n} \{S^k A_i^k B_i a_{2i} \mid k \geq 1\}$$

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Let us have ${}^{III}_{n}PC$ grammar (G, G'), $n \geq 0$, where

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$$\{A_{i+1} \to a_{2i+2} A_{i+1} a_{2i+3}, \quad A_{i+1} \to B_{i+1},$$
$$B_{i+1} \to a_{2i+2} a_{2i+3} |\ i = 0, \dots, n-1\}$$
$$L(G') = \bigcup_{i=1}^{n} \{S^k A_i^k B_i a_{2i} |\ k \geq 1\}$$

Consider a derivation in (G, G'):

$$S \Rightarrow^k a_1^k S a_{2n+2}^k$$
$$\Rightarrow a_1^k a_1 A_1 \dots A_n a_{2n+2} a_{2n+2}^k$$
$$\Rightarrow^{n \times k} a_1^{k+1} a_2^k B_1 a_3^k \dots a_{2n}^k B_n a_{2n+1}^k a_{2n+2}^{k+1}$$
$$\Rightarrow^n a_1^{k+1} a_2^{k+1} a_3^{k+1} \dots a_{2n}^{k+1} a_{2n+1}^{k+1} a_{2n+2}^{k+1}$$

Example 2

Let us have ${}^{III}_{n}PC$ grammar (G, G'), $n \geq 0$, where

$$G_1 = (\{S\} \cup \{A_i, B_i |\ i = 1, \ldots, n\} \cup \{a_i |\ i = 1, \ldots, 2n+2\},$$
$$\{a_i |\ i = 1, \ldots, 2n+2\}, P, S)$$
$$P = \{S \to a_1 S a_{2n+2}, \quad S \to a_1 A_1 A_2 \ldots A_n a_{2n+2}\} \cup$$
$$\{A_{i+1} \to a_{2i+2} A_{i+1} a_{2i+3}, \quad A_{i+1} \to B_{i+1},$$
$$B_{i+1} \to a_{2i+2} a_{2i+3} |\ i = 0, \ldots, n-1\}$$
$$L(G') = \bigcup_{i=1}^{n} \{S^k A_i^k B_i a_{2i} |\ k \geq 1\}$$

Consider a derivation in (G, G'):

$$S \Rightarrow^k a_1^k S a_{2n+2}^k$$
$$\Rightarrow a_1^k a_1 A_1 \ldots A_n a_{2n+2} a_{2n+2}^k$$
$$\Rightarrow^{n \times k} a_1^{k+1} a_2^k B_1 a_3^k \ldots a_{2n}^k B_n a_{2n+1}^k a_{2n+2}^{k+1}$$
$$\Rightarrow^n a_1^{k+1} a_2^{k+1} a_3^{k+1} \ldots a_{2n}^{k+1} a_{2n+1}^{k+1} a_{2n+2}^{k+1}$$

$$L(G_1, G') = \{a_1^k \ldots a_{2n+2}^k |\ k \geq 1\}.$$

Examples

Example 3

Let $m \geq 0$ with $m \bmod 2 = 0$. Let us have ${}^{III}_{n}PC$ grammar (G, G'), $n \geq 0$, where

$$
\begin{aligned}
G = (&\{A_j, B_j, a_j \mid j = 1, \ldots, m\} \cup \{C\}, \{a_j \mid j = 1, \ldots, m\}, P, A_1) \\
P = &\{A_1 \to a_1A_1, \quad A_1 \to a_1A_2, \quad B_1 \to B_1a_1, \quad B_1 \to C, \quad C \to a_1\} \cup \\
&\{A_m \to A_ma_m, \quad A_m \to \{B_m\}^n\} \cup \\
&\{A_i \to A_ia_i, \quad A_i \to A_{i+1} \mid i = 2, \ldots, m-1 \text{ with } i \bmod 2 = 0\} \cup \\
&\{A_i \to a_iA_i, \quad A_i \to A_{i+1} \mid i = 3, \ldots, m-1 \text{ with } i \bmod 2 = 1\} \cup \\
&\{B_i \to a_iB_i, \quad B_i \to B_{i-1} \mid i = 2, \ldots, m \text{ with } i \bmod 2 = 0\} \cup \\
&\{B_i \to B_ia_i, \quad B_i \to B_{i-1} \mid i = 3, \ldots, m \text{ with } i \bmod 2 = 1\}
\end{aligned}
$$

$$L(G') = \{A_1^{k_1}A_2^{k_2} \ldots A_m^{k_m}B_m^{k_m}B_{m-1}^{k_{m-1}} \ldots B_2^{k_2}B_1^{k_1}Ca_1 \mid k_i \geq 0, i = 1, \ldots, m\}$$

Consider a derivation in (G, G')*:*

$$
\begin{aligned}
A_1 &\Rightarrow^{k_1} a_1^{k_1} A_1 \Rightarrow a_1^{k_1+1} A_2 \Rightarrow^{k_2} a_1^{k_1+1} A_2 a_2^{k_2} \Rightarrow a_1^{k_1+1} A_3 a_2^{k_2} \\
&\Rightarrow^* a_1^{k_1+1} a_3^{k_3} a_5^{k_5} \dots a_{m-1}^{k_{m-1}} A_m a_m^{k_m} \dots a_6^{k_6} a_4^{k_4} a_2^{k_2} \\
&\Rightarrow a_1^{k_1+1} a_3^{k_3} a_5^{k_5} \dots a_{m-1}^{k_{m-1}} \{B_m\}^n a_m^{k_m} \dots a_6^{k_6} a_4^{k_4} a_2^{k_2} \\
&\Rightarrow^{n \times k_m} a_1^{k_1+1} a_3^{k_3} a_5^{k_5} \dots a_{m-1}^{k_{m-1}} \{a_m^{k_m} B_m\}^n a_m^{k_m} \dots a_6^{k_6} a_4^{k_4} a_2^{k_2} \\
&\Rightarrow^{n} a_1^{k_1+1} a_3^{k_3} a_5^{k_5} \dots a_{m-1}^{k_{m-1}} \{a_m^{k_m} B_{m-1}\}^n a_m^{k_m} \dots a_6^{k_6} a_4^{k_4} a_2^{k_2} \\
&\Rightarrow^{n \times k_{m-1}} a_1^{k_1+1} a_3^{k_3} a_5^{k_5} \dots a_{m-1}^{k_{m-1}} \{a_m^{k_m} B_{m-1} a_{m-1}^{k_{m-1}}\}^n a_m^{k_m} \dots a_6^{k_6} a_4^{k_4} a_2^{k_2} \\
&\Rightarrow^* a_1^{k_1+1} a_3^{k_3} a_5^{k_5} \dots a_{m-1}^{k_{m-1}} \{a_m^{k_m} a_{m-2}^{k_{m-2}} \dots a_2^{k_2} B_1 a_1^{k_1} \dots a_{m-3}^{k_{m-3}} a_{m-1}^{k_{m-1}}\}^n \\
&\quad a_m^{k_m} \dots a_6^{k_6} a_4^{k_4} a_2^{k_2} \\
&\Rightarrow^{n} a_1^{k_1+1} a_3^{k_3} a_5^{k_5} \dots a_{m-1}^{k_{m-1}} \{a_m^{k_m} a_{m-2}^{k_{m-2}} \dots a_2^{k_2} C a_1^{k_1} \dots a_{m-3}^{k_{m-3}} a_{m-1}^{k_{m-1}}\}^n \\
&\quad a_m^{k_m} \dots a_6^{k_6} a_4^{k_4} a_2^{k_2} \\
&\Rightarrow^{n} a_1^{k_1+1} a_3^{k_3} a_5^{k_5} \dots a_{m-1}^{k_{m-1}} \{a_m^{k_m} a_{m-2}^{k_{m-2}} \dots a_2^{k_2} a_1^{k+1} \dots a_{m-3}^{k_{m-3}} a_{m-1}^{k_{m-1}}\}^n \\
&\quad a_m^{k_m} \dots a_6^{k_6} a_4^{k_4} a_2^{k_2}
\end{aligned}
$$

Consider a derivation in (G, G')*:*

$$A_1 \Rightarrow^{k_1} a_1^{k_1} A_1 \Rightarrow a_1^{k_1+1} A_2 \Rightarrow^{k_2} a_1^{k_1+1} A_2 a_2^{k_2} \Rightarrow a_1^{k_1+1} A_3 a_2^{k_2}$$
$$\Rightarrow^* a_1^{k_1+1} a_3^{k_3} a_5^{k_5} \dots a_{m-1}^{k_{m-1}} A_m a_m^{k_m} \dots a_6^{k_6} a_4^{k_4} a_2^{k_2}$$
$$\Rightarrow a_1^{k_1+1} a_3^{k_3} a_5^{k_5} \dots a_{m-1}^{k_{m-1}} \{B_m\}^n a_m^{k_m} \dots a_6^{k_6} a_4^{k_4} a_2^{k_2}$$
$$\Rightarrow^{n \times k_m} a_1^{k_1+1} a_3^{k_3} a_5^{k_5} \dots a_{m-1}^{k_{m-1}} \{a_m^{k_m} B_m\}^n a_m^{k_m} \dots a_6^{k_6} a_4^{k_4} a_2^{k_2}$$
$$\Rightarrow^n a_1^{k_1+1} a_3^{k_3} a_5^{k_5} \dots a_{m-1}^{k_{m-1}} \{a_m^{k_m} B_{m-1}\}^n a_m^{k_m} \dots a_6^{k_6} a_4^{k_4} a_2^{k_2}$$
$$\Rightarrow^{n \times k_{m-1}} a_1^{k_1+1} a_3^{k_3} a_5^{k_5} \dots a_{m-1}^{k_{m-1}} \{a_m^{k_m} B_{m-1} a_{m-1}^{k_{m-1}}\}^n a_m^{k_m} \dots a_6^{k_6} a_4^{k_4} a_2^{k_2}$$
$$\Rightarrow^* a_1^{k_1+1} a_3^{k_3} a_5^{k_5} \dots a_{m-1}^{k_{m-1}} \{a_m^{k_m} a_{m-2}^{k_{m-2}} \dots a_2^{k_2} B_1 a_1^{k_1} \dots a_{m-3}^{k_{m-3}} a_{m-1}^{k_{m-1}}\}^n a_m^{k_m} \dots a_6^{k_6} a_4^{k_4} a_2^{k_2}$$
$$\Rightarrow^n a_1^{k_1+1} a_3^{k_3} a_5^{k_5} \dots a_{m-1}^{k_{m-1}} \{a_m^{k_m} a_{m-2}^{k_{m-2}} \dots a_2^{k_2} C a_1^{k_1} \dots a_{m-3}^{k_{m-3}} a_{m-1}^{k_{m-1}}\}^n a_m^{k_m} \dots a_6^{k_6} a_4^{k_4} a_2^{k_2}$$
$$\Rightarrow^n a_1^{k_1+1} a_3^{k_3} a_5^{k_5} \dots a_{m-1}^{k_{m-1}} \{a_m^{k_m} a_{m-2}^{k_{m-2}} \dots a_2^{k_2} a_1^{k+1} \dots a_{m-3}^{k_{m-3}} a_{m-1}^{k_{m-1}}\}^n a_m^{k_m} \dots a_6^{k_6} a_4^{k_4} a_2^{k_2}$$

$$L(G, G') = \{(a_1^{k_1+1} a_3^{k_3} \dots a_{m-1}^{k_{m-1}} a_m^m a_{m-2}^{k_{m-2}} a_{m-4}^{k_{m-4}} \dots a_2^{k_2})^{n+1} \mid k_i \geq 0, i = 1, \dots, m\}$$

Example 4

Consider $m = 4$ and ${}_{3}^{lll}PC$ grammar (G, G'), where

$$G = (\{A, B, C, D, E, F, G, H, I, a, b, c, d\}, \{a, b, c, d\}, P, A)$$
$$\begin{aligned} P = \{ & A \to aA, \quad A \to aB, \quad B \to Bb, \quad B \to C, \\ & C \to cC, \quad C \to D, \quad D \to Dd, \quad D \to HHH, \\ & E \to Ea, \quad E \to I, \quad F \to bF, \quad F \to E, \\ & G \to Gc, \quad G \to F, \quad H \to dH, \quad H \to G, \quad I \to a\} \end{aligned}$$
$$L(G') = \{A^r B^s C^t D^u H^u G^t F^s E^r Ia \mid r, s, t, u \geq 0\}$$

$$L(G, G') = \{a^v c^w d^x b^y a^v c^w d^x b^y a^v c^w d^x b^y a^v c^w d^x b^y \mid v > 0, w, x, y \geq 0\}$$

Example of the derivation:

$A \Rightarrow aA \Rightarrow aaB \Rightarrow aaBb \Rightarrow aaCb \Rightarrow aacCb \Rightarrow aacDb \Rightarrow$
$aacDdb \Rightarrow aacHHHdb \Rightarrow aacdHHHdb \Rightarrow aacdGHHdb \Rightarrow$
$aacdGcHHdb \Rightarrow aacdFcHHdb \Rightarrow aacdbFcHHdb \Rightarrow$
$aacdbEcHHdb \Rightarrow aacdbEacHHdb \Rightarrow aacdbIacHHdb \Rightarrow$
$aacdbaacHHdb \Rightarrow aacdbaacdHHdb \Rightarrow aacdbaacdGHdb \Rightarrow$
$aacdbaacdGcHdb \Rightarrow aacdbaacdFcHdb \Rightarrow$
$aacdbaacdbFcHdb \Rightarrow aacdbaacdbEcHdb \Rightarrow$
$aacdbaacdbEacHdb \Rightarrow aacdbaacdbIacHdb \Rightarrow$
$aacdbaacdbaacHdb \Rightarrow aacdbaacdbaacdHdb \Rightarrow$
$aacdbaacdbaacdGdb \Rightarrow aacdbaacdbaacdGcdb \Rightarrow$
$aacdbaacdbaacdFcdb \Rightarrow aacdbaacdbaacdbFcdb \Rightarrow$
$aacdbaacdbaacdbEcdb \Rightarrow aacdbaacdbaacdbEacdb \Rightarrow$
$aacdbaacdbaacdbIacdb \Rightarrow aacdbaacdbaacdbaacdb$

Future research ideas

Investigation of **III-n-PC**

${}_{n}^{III}PC$ grammars are potentially usable.

- Generative power?
- Closure properties?
- Decidability properties?
- Parsing properties?
- Descriptional complexity?

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Investigation of **I-n-PC** and **V-n-PC**

- ${}^{n}_{I}PC$ grammars are equal to concatenation of n independent PC grammars?
- ${}^{n}_{V}PC$ grammars are equal to CF grammars?

Future research ideas

Investigation of **III-n-PC**

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- ${}_{I}^{n}PC$ grammars are equal to concatenation of n independent PC grammars?
- ${}_{V}^{n}PC$ grammars are equal to CF grammars?

Investigation of **II-n-PC** and **IV-n-PC**

${}_{II}^{n}PC$ grammars and ${}_{IV}^{n}PC$ grammars are unusable?

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Thank you for your attention!