

Maximal and Minimal Scattered Context Rewriting

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Scattered Context Grammar

Scattered context grammar (SC grammar)

$G = (V, T, P, S)$, where

V is a finite alphabet

T is a set of terminals, $T \subset V$

S is the start symbol, $S \in V - T$

P is a finite set of productions of the form

$$(A_1, \dots, A_n) \rightarrow (x_1, \dots, x_n),$$

where $A_1, \dots, A_n \in V - T$, $x_1, \dots, x_n \in V^*$

Propagating scattered context grammar (PSC grammar)

- each $(A_1, \dots, A_n) \rightarrow (x_1, \dots, x_n)$ satisfies $x_1, \dots, x_n \in V^+$

SC Grammar—Derivation Step

Derivation step

For $(A_1, \dots, A_n) \rightarrow (x_1, \dots, x_n) \in P$ and

$$u = u_1 A_1 \dots u_n A_n u_{n+1}$$

$$v = u_1 x_1 \dots u_n x_n u_{n+1}$$

we write $u \Rightarrow v [(A_1, \dots, A_n) \rightarrow (x_1, \dots, x_n)]$

Generated language

$$L(G) = \{x \in T^* : S \Rightarrow^* x\}$$

Generative power

- $\mathcal{L}(SC) = \mathcal{L}(RE)$
- $\mathcal{L}(CF) \subset \mathcal{L}(PSC) \subseteq \mathcal{L}(CS)$

Example

SC grammar $G = (\{A, B, C, S, a, b, c\}, \{a, b, c\}, P, S)$ with

$$P = \{(S) \rightarrow (ABC), \\ (A, B, C) \rightarrow (aA, bB, cC), \\ (A, B, C) \rightarrow (\varepsilon, \varepsilon, \varepsilon)\}$$

$S \Rightarrow ABC \Rightarrow aAbBcC \Rightarrow aaAbbBccC \Rightarrow aabbcc$

$L(G) = \{a^n b^n c^n : n \geq 0\}$

Maximal and Minimal Derivation

Production length

$$\blacksquare \text{len}((A_1, \dots, A_n) \rightarrow (x_1, \dots, x_n)) = |A_1 \dots A_n| = n$$

Maximal derivation step

$u \xrightarrow{\text{max}} v [p]$, $p \in P$ if and only if

- 1 $u \Rightarrow v [p]$,
- 2 there is no $r \in P$, $\text{len}(r) > \text{len}(p)$, such that $u \Rightarrow w [r]$

Minimal derivation step

$u \xrightarrow{\text{min}} v [p]$, $p \in P$ if and only if

- 1 $u \Rightarrow v [p]$,
- 2 there is no $r \in P$, $\text{len}(r) < \text{len}(p)$, such that $u \Rightarrow w [r]$

Maximal and minimal languages

- $L_{\max}(G) = \{x \in T^* : S \xrightarrow{\max}^* x\}$
- $L_{\min}(G) = \{x \in T^* : S \xrightarrow{\min}^* x\}$
- language families denoted by $\mathcal{L}(PSC, \max)$ and $\mathcal{L}(PSC, \min)$

Example

SC grammar $G = (\{A, B, C, S, a, b, c\}, \{a, b, c\}, P, S)$ with

$$P = \{(S) \rightarrow (ABC), \\ (A) \rightarrow (a), (B) \rightarrow (b), (C) \rightarrow (c), \\ (A, B, C) \rightarrow (aA, bB, cC), \\ (A, B, C) \rightarrow (\varepsilon, \varepsilon, \varepsilon)\}$$

$$L_{\max}(G) = \{a^n b^n c^n : n \geq 0\},$$

$$L_{\min}(G) = \{abc\}$$

Theorem

$$\mathcal{L}(CS) = \mathcal{L}(PSC, \text{max})$$

State grammar (ST grammar)

$G = (V, T, K, P, S, p_0)$, where

V is a finite alphabet

T is a set of terminals, $T \subset V$

K is a finite set of states

S is the start symbol, $S \in V - T$

$p_0 \in K$

P is a finite set of productions of the form

$$(A, p) \rightarrow (x, q),$$

where $p, q \in K$, $A \in V - T$, $x \in V^+$

State Grammar—Derivation Step

Derivation step

For $(A, p) \rightarrow (x, q) \in P$,

$$u = (rAs, p),$$

$$v = (rxs, q),$$

where $r, s \in V^*$, and for every $(B, p) \rightarrow (y, t) \in P$, $B \notin \text{alph}(r)$, we write

$$u \Rightarrow v [(A, p) \rightarrow (x, q)]$$

Generated language

$$L(G) = \{x \in T^* : (S, p_0) \Rightarrow^* (x, q) \text{ for some } q \in K\}$$

Generative power

$$\mathcal{L}(ST) = \mathcal{L}(CS)$$

CS language representation

- let $L \in \mathcal{L}(CS)$
- let $\bar{G} = (\bar{V}, T, K, \bar{P}, \bar{S}, p_0)$ be a state grammar such that $L = L(\bar{G})$
- let $(x_1 B x_2 A x_3 C x_4, p)$ be a sentential form of \bar{G}
- let $(x_1 B x_2 A x_3 C x_4, p) \Rightarrow (x_1 B x_2 x x_3 C x_4, q) [(A, p) \rightarrow (x, q)]$ in \bar{G}

Construction of maximal PSC grammar

- $(x_1 B x_2 A x_3 C x_4, p)$ corresponds to $x_1 B x_2 \langle A, p \rangle x_3 C x_4$ in G
- simulate $(A, p) \rightarrow (x, q)$ by
 - 1 $(B, \langle A, p \rangle, C) \rightarrow (X, X, X)$ for every $(B, p) \rightarrow (y, r) \in P$
 - 2 $(B, \langle A, p \rangle) \rightarrow (\langle B, q \rangle, x)$

$$x_1 B x_2 \langle A, p \rangle x_3 C x_4 \Rightarrow x_1 \langle B, q \rangle x_2 x x_3 C x_4$$

Definitions

- 1 let $\Delta(t)$ be the set of all permutations of $\{1, \dots, t\}$
- 2 let $\text{permute}(n, m) = \{(i_1, \dots, i_{n+m}) \in \Delta(n+m) : 1 \leq i_k < i_l \leq n \text{ implies } k < l\}$
- 3 let $\text{reorder}((x_1, \dots, x_n), (i_1, \dots, i_n)) = (x_{i_1}, \dots, x_{i_n})$
for $x_1, \dots, x_n \in V^*$, $(i_1, \dots, i_n) \in \Delta(n)$
- 4 let $\Pi((x_1, \dots, x_n), i) = x_i$

Example

- 1 $\Delta(3) = \{(1, 2, 3), (1, 3, 2), (2, 1, 3), (2, 3, 1), (3, 1, 2), (3, 2, 1)\}$
- 2 $\text{permute}(2, 1) = \{(1, 2, 3), (1, 3, 2), (3, 1, 2)\}$
- 3 $\text{reorder}((x_1, x_2, x_3), (3, 1, 2)) = (x_3, x_1, x_2)$
- 4 $\Pi((x_3, x_1, x_2), 3) = x_2$

Construction I

- let $\bar{G} = (\bar{V}, T, K, \bar{P}, \bar{S}, p_0)$
- set $Y = \{\langle A, q \rangle : A \in \bar{V} - T, q \in K\}$
- set $Z = \{\bar{a} : a \in T\}$
- define the homomorphism α , from \bar{V}^* to $((\bar{V} - T) \cup Z)^*$ as
 - $\alpha(A) = A$ for all $A \in \bar{V} - T$ and
 - $\alpha(a) = \bar{a}$ for all $a \in T$
- set $V = \bar{V} \cup Y \cup Z \cup \{S, X\}$
- define the propagating scattered context grammar $G = (V, T, P, S)$ with P defined as follows:

Step 1

For every $x \in L(\bar{G})$, $|x| \leq 2$, add
 $(S) \rightarrow (x)$

Step 2

For every

$$(\mathbf{x}, q) \in \{(\mathbf{x}, q) : (\bar{S}, p_0) \Rightarrow_G^+ (\mathbf{x}, q) \text{ for some } q \in K \\ \text{and } 3 \leq |\mathbf{x}| \leq \min\{3, \max\{|\Pi(\text{rhs}(p), 1)| : p \in \bar{P}\}\}\},$$

where

- 1 $\mathbf{x} \in T^*$, add
 $(S) \rightarrow (\mathbf{x})$
- 2 $\mathbf{x} = x_1 A x_2$, $A \in \bar{V} - T$, $x_1, x_2 \in \bar{V}^*$, add
 $(S) \rightarrow (\alpha(x_1)\langle A, q \rangle\alpha(x_2))$

Step 3

For every $(A, p) \rightarrow (x, q)$, $(B, p) \rightarrow (y, r) \in \bar{P}$, $C \in \bar{V}$,
 $\Gamma_{21} \in \text{permute}(2, 1)$,

$$z = \text{reorder}((B, \langle A, p \rangle, \alpha(C)), \Gamma_{21}),$$

add

$$z \rightarrow (X, X, X)$$

Step 4

For every $(A, p) \rightarrow (x, q) \in \bar{P}$, $B \in \bar{V} - T$, $C \in \bar{V}$, $\Gamma_{11} \in \text{permute}(\mathbf{1}, \mathbf{1})$,

$$y = \text{reorder}(\langle A, p \rangle, \alpha(C), \Gamma_{11}),$$

add

- 1 $(B, \langle A, p \rangle) \rightarrow (\langle B, q \rangle, \alpha(x))$,
- 2 $(\langle A, p \rangle, B) \rightarrow (\alpha(x), \langle B, q \rangle)$
- 3 If $x = vBw$, $v, w \in \bar{V}^*$, for every

$$z = \text{reorder}(\alpha(v)\langle B, q \rangle\alpha(w), \alpha(C), \Gamma_{11}),$$

add $y \rightarrow z$

- 4 For every $u = \text{reorder}(\alpha(x), \alpha(C), \Gamma_{11})$,
add $y \rightarrow u$

Step 5

For every $a \in T$, add
 $(\bar{a}) \rightarrow (a)$

Theorem

$$\mathcal{L}(CS) = \mathcal{L}(PSC, \text{min})$$

CS language representation

- let $L \in \mathcal{L}(CS)$
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 - 1 $(B, \langle A, p \rangle) \rightarrow (X, X)$ for every $(B, p) \rightarrow (y, r) \in P$
 - 2 $(B, \langle A, p \rangle, C) \rightarrow (\langle B, q \rangle, x, C)$

$$x_1 B x_2 \langle A, p \rangle x_3 C x_4 \Rightarrow x_1 \langle B, q \rangle x_2 x x_3 C x_4$$

$$\mathcal{L}(CF) \subset \mathcal{L}(PSC) \subseteq \mathcal{L}(CS)$$

By restricting derivations to

- the **longest** applicable productions, we obtain

$$\mathcal{L}(PSC, \text{max}) = \mathcal{L}(CS)$$

- the **shortest** applicable productions, we obtain

$$\mathcal{L}(PSC, \text{min}) = \mathcal{L}(CS)$$