#-Rewriting Systems

Contents

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3. **Main Result**: An Infinite Hierarchy
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#-Rewriting Systems in Formal Language Theory

- Language-defining models
- Pure rewriting systems
- Between automata and grammars:
  have states but generate languages

Concept

#-Rewriting System is based on the rules of the form

$$p_m\# \rightarrow q\ x_0 \# x_1 \ldots \# x_n$$

by which the system makes a computational step $\Rightarrow$ as

m**th #**

$$(p, \ldots \# y_{m-1} \# y_m \# y_{m+1} \cdots) \Rightarrow$$

$$(q, \ldots \# y_{m-1} x_0 \# x_1 \ldots \# x_n y_m \# y_{m+1} \cdots)$$

Definition 1/2

#-Rewriting System (#***RS***) is a quadruple

$$H = (Q, \Sigma, s, R), \text{ where}$$

- Q—finite set of *states*,
- Σ—*alphabet*, $\#$ $\in$ Σ is called a *bounder*,
- $s \in Q$—*start state*,
- R—*finite set of rules* of the form

$$p_m\# \rightarrow qx$$

where p, $q \in Q$, m is a positive integer, $x \in \Sigma^*$.

Definition 2/2

Configuration: (q, x), $q \in Q$, $x \in \Sigma^*$

Computational step:

$$(p, u\#v) \Rightarrow (q, uxv)\ [p_m\# \rightarrow qx \in R],$$

where the number of #s in u is $m - 1$,

$$p, q \in Q, u, x, v \in \Sigma^*.$$

Generated language:

$$L(H) = \{w \in (\Sigma - \#)^*: (s, \#) \Rightarrow^* (q, w) \text{ in } H, q \in Q\}.$$

Example: #*RS*

#*RS H*:

***H* generates *aabbcc*:**

[1]. $s\ {}_1\# \rightarrow p\ \#\#$

[2]. $p\ {}_1\# \rightarrow q\ a\#b$

[3]. $q\ {}_2\# \rightarrow p\ \#c$

[4]. $p\ {}_1\# \rightarrow f\ ab$

[5]. $f\ {}_1\# \rightarrow f\ c$

Example: #*RS*

#*RS H*:

[1]. $s\ {}_1\# \rightarrow p\ \#\#$

[2]. $p\ {}_1\# \rightarrow q\ a\#b$

[3]. $q\ {}_2\# \rightarrow p\ \#c$

[4]. $p\ {}_1\# \rightarrow f\ ab$

[5]. $f\ {}_1\# \rightarrow f\ c$

***H* generates *aabbcc*:**

$(s, \#)$

$\Rightarrow$

Example: #*RS*

#*RS H*:

[1]. $s\ {}_1\# \rightarrow p\ \#\#$

[2]. $p\ {}_1\# \rightarrow q\ a\#b$

[3]. $q\ {}_2\# \rightarrow p\ \#c$

[4]. $p\ {}_1\# \rightarrow f\ ab$

[5]. $f\ {}_1\# \rightarrow f\ c$

***H* generates *aabbcc*:**

$(s, \underline{\#})$

$\Rightarrow$ [1]

Example: #*RS*

#*RS H*:

[1]. $s\ _1\# \rightarrow p\ \#\#$

[2]. $p\ _1\# \rightarrow q\ a\#b$

[3]. $q\ _2\# \rightarrow p\ \#c$

[4]. $p\ _1\# \rightarrow f\ ab$

[5]. $f\ _1\# \rightarrow f\ c$

***H* generates *aabbcc*:**

$(s, \#)$

$\Rightarrow (p, \#\#)$ [1]

$\Rightarrow$

Example: #*RS*

#*RS* *H*:

[1]. $s\ _1\# \rightarrow p\ \#\#$

[2]. $p\ _1\# \rightarrow q\ a\#b$

[3]. $q\ _2\# \rightarrow p\ \#c$

[4]. $p\ _1\# \rightarrow f\ ab$

[5]. $f\ _1\# \rightarrow f\ c$

***H* generates *aabbcc*:**

$(s, \#)$

$\Rightarrow (p, \underline{\#}\#)$ [1]

$\Rightarrow$ [2]

Example: #*RS*

#*RS* *H*:

[1]. $s\ {}_1\# \rightarrow p\ \#\#$

[2]. $p\ {}_1\# \rightarrow q\ a\#b$

[3]. $q\ {}_2\# \rightarrow p\ \#c$

[4]. $p\ {}_1\# \rightarrow f\ ab$

[5]. $f\ {}_1\# \rightarrow f\ c$

H generates *aabbcc*:

$(s, \#)$

$\Rightarrow (p, \#\#)$ [1]

$\Rightarrow (q, a\#b\#)$ [2]

$\Rightarrow$

Example: #*RS*

#*RS H*:

[1]. $s\ {}_1\# \rightarrow p\ \#\#$

[2]. $p\ {}_1\# \rightarrow q\ a\#b$

[3]. $q\ {}_2\# \rightarrow p\ \#c$

[4]. $p\ {}_1\# \rightarrow f\ ab$

[5]. $f\ {}_1\# \rightarrow f\ c$

_H_ generates _aabbcc_:

$(s, \#)$	
$\Rightarrow (p, \#\#)$	[1]
$\Rightarrow (q, a\#b\underline{\#})$	[2]
$\Rightarrow$	[3]

Example: #*RS*

#*RS* *H*:

[1]. $s\ {}_1\# \rightarrow p\ \#\#$

[2]. $p\ {}_1\# \rightarrow q\ a\#b$

[3]. $q\ {}_2\# \rightarrow p\ \#c$

[4]. $p\ {}_1\# \rightarrow f\ ab$

[5]. $f\ {}_1\# \rightarrow f\ c$

H generates *aabbcc*:

$(s, \#)$

$\Rightarrow (p, \#\#)$ [1]

$\Rightarrow (q, a\#b\#)$ [2]

$\Rightarrow (p, a\#b\#c)$ [3]

$\Rightarrow$

Example: #*RS*

#*RS H*:

[1]. $s\ {}_1\# \rightarrow p\ \#\#$

[2]. $p\ {}_1\# \rightarrow q\ a\#b$

[3]. $q\ {}_2\# \rightarrow p\ \#c$

[4]. $p\ {}_1\# \rightarrow f\ ab$

[5]. $f\ {}_1\# \rightarrow f\ c$

***H* generates *aabbcc*:**

$(s, \#)$	
$\Rightarrow (p, \#\#)$	[1]
$\Rightarrow (q, a\#b\#)$	[2]
$\Rightarrow (p, a\underline{\#}b\#c)$	[3]
$\Rightarrow$	[4]

Example: #*RS*

#*RS H*:

[1]. $s\ {}_1\# \rightarrow p\ \#\#$

[2]. $p\ {}_1\# \rightarrow q\ a\#b$

[3]. $q\ {}_2\# \rightarrow p\ \#c$

[4]. $p\ {}_1\# \rightarrow f\ ab$

[5]. $f\ {}_1\# \rightarrow f\ c$

***H* generates *aabbcc*:**

	$(s, \#)$	
$\Rightarrow$	$(p, \#\#)$	[1]
$\Rightarrow$	$(q, a\#b\#)$	[2]
$\Rightarrow$	$(p, a\#b\#c)$	[3]
$\Rightarrow$	$(f, aabb\#c)$	[4]
$\Rightarrow$		

Example: #*RS*

#*RS H*:

[1]. $s\ {}_1\# \rightarrow p\ \#\#$

[2]. $p\ {}_1\# \rightarrow q\ a\#b$

[3]. $q\ {}_2\# \rightarrow p\ \#c$

[4]. $p\ {}_1\# \rightarrow f\ ab$

[5]. $f\ {}_1\# \rightarrow f\ c$

***H* generates *aabbcc*:**

$(s, \#)$	
$\Rightarrow (p, \#\#)$	[1]
$\Rightarrow (q, a\#b\#)$	[2]
$\Rightarrow (p, a\#b\#c)$	[3]
$\Rightarrow (f, aabb\underline{\#}c)$	[4]
$\Rightarrow$	[5]

Example: #*RS*

#*RS H*:

[1]. $s\ {}_1\# \rightarrow p\ \#\#$

[2]. $p\ {}_1\# \rightarrow q\ a\#b$

[3]. $q\ {}_2\# \rightarrow p\ \#c$

[4]. $p\ {}_1\# \rightarrow f\ ab$

[5]. $f\ {}_1\# \rightarrow f\ c$

H generates *aabbcc*:

	$(s, \#)$	
$\Rightarrow$	$(p, \#\#)$	[1]
$\Rightarrow$	$(q, a\#b\#)$	[2]
$\Rightarrow$	$(p, a\#b\#c)$	[3]
$\Rightarrow$	$(f, aabb\#c)$	[4]
$\Rightarrow$	$(f, aabbcc)$	[5]

Example: #*RS*

#*RS H*:

[1]. $s\ {}_1\# \rightarrow p\ \#\#$

[2]. $p\ {}_1\# \rightarrow q\ a\#b$

[3]. $q\ {}_2\# \rightarrow p\ \#c$

[4]. $p\ {}_1\# \rightarrow f\ ab$

[5]. $f\ {}_1\# \rightarrow f\ c$

***H* generates *aabbcc*:**

$(s, \#)$
$\Rightarrow (p, \#\#)$ [1]
$\Rightarrow (q, a\#b\#)$ [2]
$\Rightarrow (p, a\#b\#c)$ [3]
$\Rightarrow (f, aabb\#c)$ [4]
$\Rightarrow (f, aabbcc)$ [5]

$$L(H) = \{a^n b^n c^n : n \geq 1\}$$

Finite index of #*RS*

#-Rewriting systems of ***index k***:

$\Rightarrow$ over configurations with ***k*** or fewer #s

#*RS*$_k$ – the language family generated by #*RS*s of index k

Example: Index $k = 2$:

1. (p, ***a#a#b***) $\Rightarrow$ (q, ***aa#aa#b***) [p_1# $\rightarrow$ qa#a $\in$ R]

OK

2. (p, ***a#a#b***) $\not\Rightarrow$ (q, ***a#aa##bb***) [p_2# $\rightarrow$ qa##b $\in$ R]

INCORRECT

Example: #*RS* of finite index

#*RS* *H*:

[1]. $s\ {}_1\# \rightarrow p\ \#\#$

[2]. $p\ {}_1\# \rightarrow q\ a\#b$

[3]. $q\ {}_2\# \rightarrow p\ \#c$

[4]. $p\ {}_1\# \rightarrow f\ ab$

[5]. $f\ {}_1\# \rightarrow f\ c$

***H* generates *aabbcc*:**

$(s, \#)$	
$\Rightarrow (p, \#\#)$	[1]
$\Rightarrow (q, a\#b\#)$	[2]
$\Rightarrow (p, a\#b\#c)$	[3]
$\Rightarrow (f, aabb\#c)$	[4]
$\Rightarrow (f, aabbcc)$	[5]

***H* is of index 2.**

$$L(H) = \{a^n b^n c^n : n \geq 1\} \in \#RS_2$$

Main Result: An Infinite Hierarchy

Theorem: $\#RS_k \subset \#RS_{k+1}$, for all $k \geq 1$.

Proof:

makes use of programmed grammars (*PG*) of index k

Proof: Programmed Grammars

Programmed Grammar (***PG***) is a modification of context-free grammar based on the rules of the form:

$$r: A \rightarrow x, W_r$$

- $r: A \rightarrow x$ is a context-free rule labeled by r,
- W_r—finite set of rule labels

Derivation step ($\Rightarrow$):

after the application of rule r,
a rule from W_r has to be applied

Proof: Finite index of *PG*

Programmed grammars of ***index k***:

- $\Rightarrow$ over sentential forms with ***k*** or fewer occurrences of nonterminals.

$\boldsymbol{P_k}$ – the language family defined by programmed grammars of index k

Example: PG

***PG G*:**

1: $S \rightarrow ABC$, $\{2, 5\}$

2: $A \rightarrow aA$, $\{3\}$

3: $B \rightarrow bB$, $\{4\}$

4: $C \rightarrow cC$, $\{2, 5\}$

5: $A \rightarrow a$, $\{6\}$

6: $B \rightarrow b$, $\{7\}$

7: $C \rightarrow c$, $\emptyset$

***G* generates *aabbcc*:**

Example: *PG*

***PG G*:**

1: $S \rightarrow ABC$, {2, 5}

2: $A \rightarrow aA$, {3}

3: $B \rightarrow bB$, {4}

4: $C \rightarrow cC$, {2, 5}

5: $A \rightarrow a$, {6}

6: $B \rightarrow b$, {7}

7: $C \rightarrow c$, $\emptyset$

***G* generates *aabbcc*:**

S

$\Rightarrow$

Example: *PG*

***PG G*:**

1: $S \rightarrow ABC$, {2, 5}

2: $A \rightarrow aA$, {3}

3: $B \rightarrow bB$, {4}

4: $C \rightarrow cC$, {2, 5}

5: $A \rightarrow a$, {6}

6: $B \rightarrow b$, {7}

7: $C \rightarrow c$, $\varnothing$

***G* generates *aabbcc*:**

S

$\Rightarrow$ [**1**]

Example: *PG*

***PG G*:**

1: $S \rightarrow ABC$, {2, 5}

2: $A \rightarrow aA$, {3}

3: $B \rightarrow bB$, {4}

4: $C \rightarrow cC$, {2, 5}

5: $A \rightarrow a$, {6}

6: $B \rightarrow b$, {7}

7: $C \rightarrow c$, $\varnothing$

***G* generates *aabbcc*:**

S

$\Rightarrow ABC$ [1]

$\Rightarrow$

Example: *PG*

***PG G*:**

1: $S \rightarrow ABC, \{2, 5\}$

2: $A \rightarrow aA, \{3\}$

3: $B \rightarrow bB, \{4\}$

4: $C \rightarrow cC, \{2, 5\}$

5: $A \rightarrow a, \{6\}$

6: $B \rightarrow b, \{7\}$

7: $C \rightarrow c, \emptyset$

***G* generates *aabbcc*:**

S

$\Rightarrow ABC$ [1]

$\Rightarrow$ [2]

$\Rightarrow$

Example: *PG*

***PG G*:**

1: $S \to ABC, \{2, 5\}$

2: $A \to aA, \{3\}$

3: $B \to bB, \{4\}$

4: $C \to cC, \{2, 5\}$

5: $A \to a, \{6\}$

6: $B \to b, \{7\}$

7: $C \to c, \emptyset$

***G* generates *aabbcc*:**

S

$\Rightarrow ABC$ [1]

$\Rightarrow aABC$ [2]

$\Rightarrow$

Example: *PG*

***PG G*:**

1: $S \rightarrow ABC$, {2, 5}

2: $A \rightarrow aA$, {3}

3: $B \rightarrow bB$, {4}

4: $C \rightarrow cC$, {2, 5}

5: $A \rightarrow a$, {6}

6: $B \rightarrow b$, {7}

7: $C \rightarrow c$, $\varnothing$

***G* generates *aabbcc*:**

S

$\Rightarrow ABC$ [1]

$\Rightarrow aABC$ [2]

$\Rightarrow$ [3]

Example: *PG*

***PG G*:**

1: $S \rightarrow ABC, \{2, 5\}$

2: $A \rightarrow aA, \{3\}$

3: $B \rightarrow bB, \{4\}$

4: $C \rightarrow cC, \{2, 5\}$

5: $A \rightarrow a, \{6\}$

6: $B \rightarrow b, \{7\}$

7: $C \rightarrow c, \emptyset$

***G* generates *aabbcc*:**

S

$\Rightarrow ABC$ [1]

$\Rightarrow aABC$ [2]

$\Rightarrow aAbBC$ [3]

$\Rightarrow$

Example: *PG*

***PG G*:**

1: $S \rightarrow ABC$, {2, 5}

2: $A \rightarrow aA$, {3}

3: $B \rightarrow bB$, {4}

4: $C \rightarrow cC$, {2, 5}

5: $A \rightarrow a$, {6}

6: $B \rightarrow b$, {7}

7: $C \rightarrow c$, $\varnothing$

***G* generates *aabbcc*:**

S

$\Rightarrow ABC$ [1]

$\Rightarrow aABC$ [2]

$\Rightarrow aAbBC$ [3]

$\Rightarrow$ [4]

Example: *PG*

***PG G*:**

1: $S \rightarrow ABC$, {2, 5}

2: $A \rightarrow aA$, {3}

3: $B \rightarrow bB$, {4}

4: $C \rightarrow cC$, {2, 5}

5: $A \rightarrow a$, {6}

6: $B \rightarrow b$, {7}

7: $C \rightarrow c$, $\varnothing$

***G* generates *aabbcc*:**

S

$\Rightarrow ABC$ [1]

$\Rightarrow aABC$ [2]

$\Rightarrow aAbBC$ [3]

$\Rightarrow aAbBcC$ [4]

$\Rightarrow$

Example: *PG*

***PG G*:**

1: $S \rightarrow ABC$, {2, 5}

2: $A \rightarrow aA$, {3}

3: $B \rightarrow bB$, {4}

4: $C \rightarrow cC$, {2, 5}

5: $A \rightarrow a$, {6}

6: $B \rightarrow b$, {7}

7: $C \rightarrow c$, $\emptyset$

***G* generates *aabbcc*:**

S

$\Rightarrow ABC$ [1]

$\Rightarrow aABC$ [2]

$\Rightarrow aAbBC$ [3]

$\Rightarrow aAbBcC$ [4]

$\Rightarrow$ [5]

Example: *PG*

***PG G*:**

1: ***S*** $\rightarrow$ ***ABC*, {2, 5}**

2: ***A*** $\rightarrow$ ***aA*, {3}**

3: ***B*** $\rightarrow$ ***bB*, {4}**

4: ***C*** $\rightarrow$ ***cC*, {2, 5}**

5: ***A*** $\rightarrow$ ***a*, {6}**

6: ***B*** $\rightarrow$ ***b*, {7}**

7: ***C*** $\rightarrow$ ***c*, $\emptyset$**

***G* generates *aabbcc*:**

S	
$\Rightarrow$ ***ABC***	[1]
$\Rightarrow$ ***aABC***	[2]
$\Rightarrow$ ***aAbBC***	[3]
$\Rightarrow$ ***aAbBcC***	[4]
$\Rightarrow$ ***aabBcC***	[5]
$\Rightarrow$	

Example: *PG*

***PG G*:**

1: $\boldsymbol{S \rightarrow ABC}$, {2, 5}

2: $\boldsymbol{A \rightarrow aA}$, {3}

3: $\boldsymbol{B \rightarrow bB}$, {4}

4: $\boldsymbol{C \rightarrow cC}$, {2, 5}

5: $\boldsymbol{A \rightarrow a}$, {6}

6: $\boldsymbol{B \rightarrow b}$, {7}

7: $\boldsymbol{C \rightarrow c}$, $\emptyset$

***G* generates *aabbcc*:**

S

$\Rightarrow ABC$ [1]

$\Rightarrow aABC$ [2]

$\Rightarrow aAbBC$ [3]

$\Rightarrow aAbBcC$ [4]

$\Rightarrow aabBcC$ [5]

$\Rightarrow$ [6]

Example: *PG*

***PG G*:**

1: $\boldsymbol{S \rightarrow ABC}$, {2, 5}

2: $\boldsymbol{A \rightarrow aA}$, {3}

3: $\boldsymbol{B \rightarrow bB}$, {4}

4: $\boldsymbol{C \rightarrow cC}$, {2, 5}

5: $\boldsymbol{A \rightarrow a}$, {6}

6: $\boldsymbol{B \rightarrow b}$, {7}

7: $\boldsymbol{C \rightarrow c}$, $\emptyset$

***G* generates *aabbcc*:**

S

$\Rightarrow ABC$ [1]

$\Rightarrow aABC$ [2]

$\Rightarrow aAbBC$ [3]

$\Rightarrow aAbBcC$ [4]

$\Rightarrow aabBcC$ [5]

$\Rightarrow aabbcC$ [6]

$\Rightarrow$

Example: *PG*

***PG G*:**

1: $\boldsymbol{S \rightarrow ABC}$, {2, 5}

2: $\boldsymbol{A \rightarrow aA}$, {3}

3: $\boldsymbol{B \rightarrow bB}$, {4}

4: $\boldsymbol{C \rightarrow cC}$, {2, 5}

5: $\boldsymbol{A \rightarrow a}$, {6}

6: $\boldsymbol{B \rightarrow b}$, {7}

7: $\boldsymbol{C \rightarrow c}$, $\emptyset$

***G* generates *aabbcc*:**

$$
\begin{array}{ll}
\quad S & \\
\Rightarrow ABC & [1] \\
\Rightarrow aABC & [2] \\
\Rightarrow aAbBC & [3] \\
\Rightarrow aAbBcC & [4] \\
\Rightarrow aabBcC & [5] \\
\Rightarrow aabbcC & [6] \\
\Rightarrow aabbcc & [7]
\end{array}
$$

Example: *PG*

***PG G*:**

1: $S \rightarrow ABC, \{2, 5\}$

2: $A \rightarrow aA, \{3\}$

3: $B \rightarrow bB, \{4\}$

4: $C \rightarrow cC, \{2, 5\}$

5: $A \rightarrow a, \{6\}$

6: $B \rightarrow b, \{7\}$

7: $C \rightarrow c, \emptyset$

***G* generates *aabbcc*:**

S	
$\Rightarrow ABC$	[1]
$\Rightarrow aABC$	[2]
$\Rightarrow aAbBC$	[3]
$\Rightarrow aAbBcC$	[4]
$\Rightarrow aabBcC$	[5]
$\Rightarrow aabbcC$	[6]
$\Rightarrow aabbcc$	[7]

$$L(G) = \{a^n b^n c^n : n \geq 1\} \in P_3$$

Proof: $\boldsymbol{P_k} = \#\boldsymbol{RS_k}$, $k \geq 1$

$\boldsymbol{P_k} \subseteq \#\boldsymbol{RS_k}$:

Let *G* be a *PG* of index *k*. Construct a *#RS* *H* of index *k*, so *H* simulates derivation step

$$aAbBc \Rightarrow_G adXYbBc \; [p\text{: } A \rightarrow dXY, \{q, o\}] \Rightarrow_G \ldots [q]$$

as

$$(\langle AB, p\rangle, a\#b\#c) \Rightarrow_H (\langle XYB, q\rangle, ad\#\#b\#c)$$
$$[\langle AB, p\rangle \; {}_1\# \rightarrow \langle XYB, q\rangle \; d\#\#]$$

Proof: $\#RS_k = P_k$, $k \geq 1$

$\#RS_k \subseteq P_k$:

Let H be a $\#RS$ of index k. Construct a PG G of index k, so G simulates a computational step

$$(p, a\underline{\#}b\#c) \Rightarrow_H (q, aa\#b\#b\#c)\ [p_1\# \rightarrow q\ a\#b\#]$$

as

$$a\langle p, 1, 2\rangle b\langle p, 2, 2\rangle c$$

1) Renumbering: $\Rightarrow_G a\langle q'', 1, 3\rangle b\langle p, 2, 2\rangle c$

$\Rightarrow_G a\langle q'', 1, 3\rangle b\langle q', 3, 3\rangle c$

2) Rewriting: $\Rightarrow_G aa\langle q', 1, 3\rangle b\langle q', 2, 3\rangle b\langle q', 3, 3\rangle c$

3) Finalization: $\Rightarrow_G aa\langle q, 1, 3\rangle b\langle q', 2, 3\rangle b\langle q', 3, 3\rangle c$

$\Rightarrow_G aa\langle q, 1, 3\rangle b\langle q, 2, 3\rangle b\langle q', 3, 3\rangle c$

$\Rightarrow_G aa\langle q, 1, 3\rangle b\langle q, 2, 3\rangle b\langle q, 3, 3\rangle c$

Proof: $\#RS_k \subset \#RS_{k+1}$, $k \geq 1$

Recall that:

- $P_k \subset P_{k+1}$, for all $k \geq 1$

As $P_k = \#RS_k$, for all $k \geq 1$, we have

Theorem: $\#RS_k \subset \#RS_{k+1}$, for all $k \geq 1$.

Future Investigation

- Determinism
- Unlimited index
- Other variants:
  - Right-linear
  - Context-sensitive
  - Parallel

Reference:

- **Křivka, Z., Meduna, A., Schönecker, R.:** Generation of Languages by Rewriting Systems that Resemble Automata, In: *IJFCS* Vol. 17, No. 5, 2006