Automatic Polynomial Transformation of Differential Equations and Derivation Closure

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Motivation

Numerical Solutions of Initial Problems

$$
\begin{aligned}
y_1' &= f_1(t, y_1, \cdots, y_n) & \qquad y_1(t_0) &= y_{10} \\
&\vdots & &\vdots \\
y_n' &= f_n(t, y_1, \cdots, y_n) & \qquad y_n(t_0) &= y_{n0}
\end{aligned}
$$

$$y_1(t) = ?, \cdots, y_n(t) = ? \quad \text{for } t \in [t_0, t_{max}]$$

Motivation

Numerical Solutions of Initial Problems

$$\begin{aligned} y_1' &= f_1(t, y_1, \cdots, y_n) & y_1(t_0) &= y_{10} \\ &\vdots & &\vdots \\ y_n' &= f_n(t, y_1, \cdots, y_n) & y_n(t_0) &= y_{n0} \end{aligned}$$

$$y_1(t) = ?, \cdots, y_n(t) = ? \quad \text{for } t \in [t_0, t_{max}]$$

Taylor Series

$$y_i(t) = y_i(t_0) + \sum_{k=1}^{m} \frac{y_i^{(k)}(t_0)}{k!}(t - t_0)^k + R_m(t)$$

Elementary Functions

Function f is elementary iff

1) $f(t) = c$ where $c \in \mathbb{R}$
2) $f(t) = a$ where $a \in \{t, y_1(t), \cdots, y_n(t)\}$
3) $f(t) = g(t)$ where $g \in \{sin, cos, exp, ^{-1}\}$
4) $f(t) = g(t) + h(t)$, $f(t) = g(t) - h(t)$, $f(t) = g(t) * h(t)$, where g, h are elementary functions
5) $f(t) = g(h(t))$ where g, h are elementary functions
6) Elementary function is only obtained by finite combination of rules 1 - 5.

Mathematical Derivations

$$
\begin{aligned}
(f(t)+g(t))' &= f'(t)+g'(t) \\
(f(t)*g(t))' &= f'(t)*g(t)+f(t)*g'(t) \\
sin(t)' &= cos(t) \\
exp(t)' &= exp(t) \\
f(g(t))' &= g(t)'*f'(g(t))
\end{aligned}
$$

Notation

$$
\begin{aligned}
\Delta =& ' \\
a + b =& plus(a, b) \\
a - b =& minus(a, b) \\
a * b =& mult(a, b) \\
a^{-1} =& over(a) \\
f(t)' =& \Delta f(t)
\end{aligned}
$$

First look at derivations

$$
\begin{aligned}
\Delta plus(x, y) \Rightarrow& plus(\Delta x, \Delta y) \\
\Delta t \Rightarrow& 1 \\
\Delta sin(exp(t)) \Rightarrow& mult(\Delta exp(t), cos(exp(t)))
\end{aligned}
$$

Regular Expression over Σ

- $\emptyset$ is RE
- ε is RE
- $\forall a \in \Sigma$ are RE

If R, S are RE then:

- $R \mid S$ is RE
- RS is RE
- R^* is RE

Extended RE over Σ

- if R is RE, then it is also Extended RE
- if R is RE, then $\%n < R > (n \in \{1, \cdots, 9\})$ is extended RE denoting R.

$\forall s \in \Sigma^*, \forall z \in \Sigma, \forall i \in \mathbb{N}$
count(z,s) - number of z occurrences in s.
sub(i,s) - first i characters from s.
len(s) - number of characters in s.

Pair Safe RE over Σ (PS-RE)

R is PS-RE over Σ considering a pair $(x, y) \in \Sigma \times \Sigma$ iff

1. R is RE
2. $\forall s \in R$: count(x,s) = count(y,s)
3. $\forall s \in R, \forall 1 < i <$len(s): count(x,sub(i,s)) $\geq$ count(y, sub(i,s))

pair = ((,)), R = sin(.*), s = sin(plus(exp(t), t))

Grammar with PS-RE

G = (T,N,P,S,A)
T - input alphabet $T \cap L = \emptyset$
N - finite set of non-terminals $N \cap L = \emptyset = N \cap T$
S $\in$ V^+
A $\in$ T $\times$ T
P - rewrite rules: $LS \rightarrow RS$, where LS is PS-RE considering A, RS $\in$ W^+

$L = \{\%1, \cdots, \%9\}$
$V = T \cup N$
$W = V \cup L$

Mathematical Derivations

Global Rules

$\Delta plus(\%1 < .^* >, \%2 < .^* >) \rightarrow plus(\Delta\%1, \Delta\%2)$
$\Delta minus(\%1 < .^* >, \%2 < .^* >) \rightarrow minus(\Delta\%1, \Delta\%2)$
$\Delta mult(\%1 < .^* >, \%2 < .^* >) \rightarrow plus(mult(\Delta\%1, \%2), mult(\%1, \Delta\%2))$
$\Delta over(\%1 < .^* >) \rightarrow mult(\Delta\%1, mult(over(\%1), over(\%1)))$
$\Delta sin(\%1 < .^* >) \rightarrow mult(\Delta\%1, cos(\%1))$
$\Delta cos(\%1 < .^* >) \rightarrow mult(-1, mult(\Delta\%1, sin(\%1)))$
$\Delta exp(\%1 < .^* >) \rightarrow mult(\Delta\%1, exp(\%1))$
$\Delta t \rightarrow 1$
$\Delta -?[0-9]^+ \rightarrow 0$

Simplifications

$plus(0, \%1 < .^* >) \rightarrow \%1$
$plus(\%1 < .^* >, 0) \rightarrow \%1$
$mult(0, \%1 < .^* >) \rightarrow 0$
$mult(\%1 < .^* >, 0) \rightarrow 0$
$mult(1, \%1 < .^* >) \rightarrow \%1$
$mult(\%1 < .^* >, 1) \rightarrow \%1$
$over(1) \rightarrow 1$
$sin(0) \rightarrow 0$
$\vdots$

$$
\begin{aligned}
y_1' &= f_1(t, y_1, \cdots, y_n) \\
&\vdots \\
y_n' &= f_n(t, y_1, \cdots, y_n)
\end{aligned}
$$

$$\Downarrow$$

G = (T, N, P, S, A)
A = ((,))
T = {t, $y_1, \cdots, y_n$, (,), sin, cos, exp, plus, minus, mult, over, $0, \cdots, 9$}
N = {Δ}
P: global ∪ simplification∪
$\{\Delta y_i \to f_i(t, y_1, \cdots, y_n) \mid y_i' = f_i(t, y_1, \cdots, y_n)$
is in set of differential equations}

$\Delta plus(1, exp(t))$

$\Delta plus(\%1 < .^* >, \%2 < .^* >) \rightarrow plus(\Delta\%1, \Delta\%2)$

$\Delta plus(1, exp(t)) \;\Rightarrow plus(\Delta 1, \Delta exp(t))$

$\Delta -?[0-9]^+ \rightarrow 0$

$$\Delta plus(1, exp(t)) \;\Rightarrow plus(\Delta 1, \Delta exp(t)) \;\Rightarrow plus(0, \Delta exp(t))$$

$$\Delta exp(\%1 < .^* >) \rightarrow mult(\Delta\%1, exp(\%1))$$

$\Delta plus(1, exp(t)) \;\Rightarrow plus(\Delta 1, \Delta exp(t)) \;\Rightarrow plus(0, \Delta exp(t))$
$\Rightarrow plus(0, mult(\Delta t, exp(t)))$

$plus(0, \%1 < .^{*} >) \rightarrow \%1$

$\Delta plus(1, exp(t)) \;\Rightarrow plus(\Delta 1, \Delta exp(t)) \;\Rightarrow plus(0, \Delta exp(t))$
$\Rightarrow plus(0, mult(\Delta t, exp(t))) \Rightarrow mult(\Delta t, exp(t))$

$\Delta t \rightarrow 1$

$$\Delta plus(1, exp(t)) \ \Rightarrow plus(\Delta 1, \Delta exp(t)) \ \Rightarrow plus(0, \Delta exp(t))$$
$$\Rightarrow plus(0, mult(\Delta t, exp(t))) \Rightarrow mult(\Delta t, exp(t)) \Rightarrow mult(1, exp(t))$$

$$mult(1, \%1 < .^{*} >) \rightarrow \%1$$

$$\Delta plus(1, exp(t)) \;\Rightarrow plus(\Delta 1, \Delta exp(t)) \;\Rightarrow plus(0, \Delta exp(t))$$
$$\Rightarrow plus(0, mult(\Delta t, exp(t))) \Rightarrow mult(\Delta t, exp(t)) \Rightarrow mult(1, exp(t))$$
$$\Rightarrow exp(t)$$

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$$\Rightarrow plus(0, mult(\Delta t, exp(t))) \Rightarrow mult(\Delta t, exp(t)) \Rightarrow mult(1, exp(t))$$
$$\Rightarrow exp(t)$$

$$\Delta^{20} mult(sin(t), t) = ?$$

Derivation Closure

Let $f : \mathbb{R} \to \mathbb{R}$ be an elementary function and let $F_0 = \{f\}$. We define the set $F_i, 0 < i$ by

$$F_i = \{f, f_1, \cdots, f_n | \forall f \in F_{i-1}, f'(t) = p(f_1(t), \cdots, f_n(t)), p \in P\}$$

The derivation closure F (if it exists) is then defined by

$$F = \bigcup_{i=1}^{\infty} F_i$$

We will use a notation $f \xrightarrow{\partial} F$.

$e^{\sin(t)} \xrightarrow{\partial} \{e^{\sin(t)}, \sin(t), \cos(t)\}$

Property of Derivation Closure

$| F_i | = | F_{i+1} | \Rightarrow F = F_i$

Generating Derivation Closure from Initial Problem

Input: Set of differential equations
Output: Derivation closure of the input set

1) $F_0 = \{y_1, \cdots, y_n\}$
2) $i = 1$
3) **repeat**
   Generate F_i from F_{i-1};
   i = i + 1;
   until $| F_i | = | F_{i-1} |$;

Polynomial Transformation

Input set of equations

$$y_1' = e^{\sin(t)}$$

$$y_1(t_0) = y_{1,0}$$

$e^{\sin(t)} \xrightarrow{\partial} \{y_1(t), e^{\sin(t)}, \sin(t), \cos(t)\}$

Transformed set of equations

$$y_1' = y_2$$
$$y_2' = y_3 y_2$$
$$y_3' = -y_4$$
$$y_4' = y_3$$

$$y_1(t_0) = y_{1,0}$$
$$y_2(t_0) = e^{y_4(t_0)}$$
$$y_3(t_0) = \cos(t_0)$$
$$y_4(t_0) = \sin(t_0)$$

Higher Order Derivations

Derivations

$$y_1^{(n)} = y_2^{(n-1)}$$

Transformed set of equations

$$y_1' = y_2$$
$$y_2' = y_3 y_2$$
$$y_3' = -y_4$$
$$y_4' = y_3$$

$$y_1(t_0) = y_{1,0}$$
$$y_2(t_0) = e^{y_4(t_0)}$$
$$y_3(t_0) = \cos(t_0)$$
$$y_4(t_0) = \sin(t_0)$$

Higher Order Derivations

Derivations

$$y_1^{(n)} = y_2^{(n-1)}$$
$$y_2' = y_2 y_3$$

Transformed set of equations

$$y_1' = y_2$$
$$y_2' = y_3 y_2$$
$$y_3' = -y_4$$
$$y_4' = y_3$$

$$y_1(t_0) = y_{1,0}$$
$$y_2(t_0) = e^{y_4(t_0)}$$
$$y_3(t_0) = \cos(t_0)$$
$$y_4(t_0) = \sin(t_0)$$

Higher Order Derivations

Derivations

$$y_1^{(n)} = y_2^{(n-1)}$$
$$y_2' = y_2 y_3$$
$$y_2'' = y_2' y_3 + y_2 y_3'$$

Transformed set of equations

$$y_1' = y_2$$
$$y_2' = y_3 y_2$$
$$y_3' = -y_4$$
$$y_4' = y_3$$

$$y_1(t_0) = y_{1,0}$$
$$y_2(t_0) = e^{y_4(t_0)}$$
$$y_3(t_0) = \cos(t_0)$$
$$y_4(t_0) = \sin(t_0)$$

Higher Order Derivations

Derivations

$$y_1^{(n)} = y_2^{(n-1)}$$
$$y_2' = y_2 y_3$$
$$y_2'' = y_2' y_3 + y_2 y_3'$$
$$y_2''' = y_2'' y_3 + 2 y_2' y_3' + y_2 y_3''$$

Transformed set of equations

$$y_1' = y_2$$
$$y_2' = y_3 y_2$$
$$y_3' = -y_4$$
$$y_4' = y_3$$

$$y_1(t_0) = y_{1,0}$$
$$y_2(t_0) = e^{y_4(t_0)}$$
$$y_3(t_0) = \cos(t_0)$$
$$y_4(t_0) = \sin(t_0)$$

Higher Order Derivations

Derivations

$$
\begin{aligned}
y_1^{(n)} &= y_2^{(n-1)} \\
y_2' &= y_2 y_3 \\
y_2'' &= y_2' y_3 + y_2 y_3' \\
y_2''' &= y_2'' y_3 + 2 y_2' y_3' + y_2 y_3'' \\
y_2^{(n+1)} &= \sum_{k=0}^{n} \binom{n}{k} y_2^{(n-k)} y_3^{(k)}
\end{aligned}
$$

Transformed set of equations

$$
\begin{aligned}
y_1' &= y_2 \\
y_2' &= y_3 y_2 \\
y_3' &= -y_4 \\
y_4' &= y_3
\end{aligned}
$$

$$
\begin{aligned}
y_1(t_0) &= y_{1,0} \\
y_2(t_0) &= e^{y_4(t_0)} \\
y_3(t_0) &= \cos(t_0) \\
y_4(t_0) &= \sin(t_0)
\end{aligned}
$$

Multidimensional Pascal's Triangles

$(y_1 y_2 \cdots y_m)^{(n)}$

$$\sum_{k_1,\cdots,k_m \in \mathbb{N}} \frac{n!}{\prod_{j=1}^{m} k_j!} \prod_{j=1}^{m} y_j^{(k_j)}$$

where $\sum_{j=1}^{m} k_j = n$

$(y_1 y_2 \cdots y_m)^{(n)}$

$$\sum_{k_1,\cdots,k_m \in \mathbb{N}} \frac{n!}{\prod_{j=1}^{m} k_j!} \prod_{j=1}^{m} y_j^{(k_j)}$$

where $\sum_{j=1}^{m} k_j = n$

$y_i^{m(n)}$

$$\sum_{\mathbf{k} \in \mathbf{V}(i,n)} \frac{i!}{(i - s(\mathbf{k})) \prod_{j=1}^{max(\mathbf{k})} \left(\sum_{k \in \mathbf{k}} \delta_{k,j}\right)!} \frac{n!}{\prod_{j=1}^{s(\mathbf{k})} \mathbf{k}(j)!} y^{i-s(\mathbf{k})} \prod_{j=1}^{s(\mathbf{k})} y_i^{(\mathbf{k}(j))}$$

where $s(\mathbf{k}) = \sum_{k \in \mathbf{k}} (1 - \delta_{k,0})$

Input set of equations

$$y_1' = y_3$$
$$y_2' = y_1$$
$$y_3' = y_2$$

Derivations

$$y_1' = y_3$$

Input set of equations

$$
\begin{aligned}
y_1' &= y_3 \\
y_2' &= y_1 \\
y_3' &= y_2
\end{aligned}
$$

Derivations

$$
\begin{aligned}
y_1' &= y_3 \\
y_1'' &= y_2
\end{aligned}
$$

Input set of equations

$$y_1' = y_3$$
$$y_2' = y_1$$
$$y_3' = y_2$$

Derivations

$$y_1' = y_3$$
$$y_1'' = y_2$$
$$y_1''' = y_1$$

Input set of equations

$$y_1' = y_3$$
$$y_2' = y_1$$
$$y_3' = y_2$$

Derivations

$$y_1' = y_3$$
$$y_1'' = y_2$$
$$y_1''' = y_1$$
$$y_1^{(4)} = y_3$$

Input set of equations

$$y_1' = y_3$$
$$y_2' = y_1$$
$$y_3' = y_2$$

Derivations

$$y_1' = y_3$$
$$y_1'' = y_2$$
$$y_1''' = y_1$$
$$y_1^{(4)} = y_3$$

(3, 1, 2)

Input set of equations

$$y_1' = y_3$$
$$y_2' = y_1$$
$$y_3' = y_2$$

Derivations

$$y_1' = y_3$$
$$y_1'' = y_2$$
$$y_1''' = y_1$$
$$y_1^{(4)} = y_3$$

(3, 1, 2)'

Input set of equations

$$y_1' = y_3$$
$$y_2' = y_1$$
$$y_3' = y_2$$

Derivations

$$y_1' = y_3$$
$$y_1'' = y_2$$
$$y_1''' = y_1$$
$$y_1^{(4)} = y_3$$

(3, 1, 2)' = (2, 3, 1)

The End

Thank you for your attention!