

# Transducers and Translation Grammars

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# Finite Transducers

## Finite Transducer

A finite transducer is a quintuple

$$M = (Q, \Sigma, R, s, F)$$

where

$Q$  is a finite set of states

$\Sigma$  is an alphabet,  $\Sigma \cap Q = \emptyset$ ,  $\Sigma = I \cup O$ ,  
 $I$  and  $O$  are input and output alphabets

$R$  is a finite set of rules of the form

$$pa \rightarrow qz$$

$$p, q \in Q, a \in I \cup \{\epsilon\}, z \in O^*$$

$s \in Q$  is the start state

$F \subseteq Q$  is a set of final states

# Input Finite Automaton

## Input Finite Automaton

Let  $M = (Q, \Sigma, R, s, F)$  be a finite transducer, then

$$M_I = (Q, I, R_I, s, F)$$

where

- $I$  is the input alphabet of  $M$  and
- $R_I = \{qa \rightarrow p : qa \rightarrow px \in R, x \in O^*\}$

is the input finite automaton

# Finite Transducers – Computational Step

## Configuration

$$\chi \in QI^*\{\mid\}O^*$$

## Move

If

$$r : qa \rightarrow pz \in R,$$

$$\chi = qaw|y,$$

$$\chi' = pw|yz,$$

then

$$\chi \Rightarrow \chi' [r]$$

# Finite Transducers – Translation

## Translation of a Word

$M$  translates  $x$  into  $y$  if

$$sx| \Rightarrow^* f|y \text{ where } f \in F$$

## Translation Defined by $M$

$$T(M) = \{(x, y) \in I^* \times O^* : sx| \Rightarrow^* f|y, f \in F\}$$

■  $\Rightarrow^*$  denotes the reflexive and transitive closure of  $\Rightarrow$

# Finite Transducers – Input and Output Language

## Input Language

$$L_I(M) = \{x \in I^* : (x, y) \in T(M) \text{ for some } y \in O^*\}$$

## Output Language

$$L_O(M) = \{y \in O^* : (x, y) \in T(M) \text{ for some } x \in I^*\}$$

## Theorem

*Both input and output languages are regular.*

# Finite Transducers – Example

## Example

$$M = (\{s, q, f\}, \{!, a\}, R, s, \{f\})$$

where

$$R = \begin{array}{ll} 1 : s! \rightarrow q, & 3 : q! \rightarrow s, \\ 2 : sa \rightarrow fa, & 4 : qa \rightarrow f!a \end{array}$$

$$s!!!a| \Rightarrow q!!a| [1] \Rightarrow s!a| [3] \Rightarrow qa| [1] \Rightarrow f!a [4]$$

$$\begin{aligned} T(M) = & \{(!^i a, a) : i \geq 0, i = 2k, k \geq 0\} \\ & \cup \{(!^i a, !a) : i \geq 1, i = 2k + 1, k \geq 0\} \end{aligned}$$

$$L_I(M) = \{!^i a : i \geq 0\} \quad L_O(M) = \{a, !a\}$$

# Finite Transducers – Determinism

## Deterministic Finite Transducer

$M$  is deterministic if each rule  $r \in R$  with  $\text{lhs}(r) = pa$  satisfies

$$\{r\} = \{r' \in R : pa = \text{lhs}(r') \text{ or } p = \text{lhs}(r')\}$$

## Example

Deterministic finite transducer

$$M = (\{f\}, \{0\}, \{f \rightarrow f0\}, f, \{f\}),$$

then

$$T(M) = \{(\varepsilon, 0^i) : i \geq 0\}$$



# Pushdown Transducers

## Pushdown Transducer

A pushdown transducer is a quintuple

$$M = (Q, \Sigma, R, s, F)$$

where

$Q, s, F$  have the same meaning as in the case of finite transducers

$\Sigma$  is an alphabet,  $\Sigma \cap Q = \emptyset$ ,  $\Sigma = I \cup O \cup P_D$ ,

$I, O, P_D$  are input, output and pushdown alphabets,

$S \in P_D$  is the start pushdown symbol

$R$  is a finite set of rules of the form

$$A p a \rightarrow u q v$$

$$A \in P_D, p, q \in Q, a \in I \cup \{\varepsilon\}, u \in P_D^*, v \in O^*$$

# Input Pushdown Automaton

## Input Pushdown Automaton

Let  $M = (Q, \Sigma, R, s, F)$  be a pushdown transducer, then

$$M_I = (Q, I \cup P_D, R_I, s, F)$$

where

- $I$  and  $P_D$  are the input and the pushdown alphabets of  $M$
- $R_I = \{Aqa \rightarrow up : Aqa \rightarrow upv \in R, v \in O^*\}$

is the input pushdown automaton

# Pushdown Transducers – Computational Step

## Configuration

$$\chi \in P_D^* Q I^* \{|\} O^*$$

## Move

If

$$r : Aqa \rightarrow upv \in R,$$

$$\chi = zAqaw|y,$$

$$\chi' = zupw|yv,$$

then

$$\chi \Rightarrow \chi' [r]$$

# Pushdown Transducers – Translation

## Translation of a Word

$M$  translates  $x$  into  $y$  if

$$Ssx| \Rightarrow^* zf|y \text{ where } f \in F$$

## Translation Defined by $M$

$$T(M) = \{(x, y) \in I^* \times O^* : Ssx| \Rightarrow^* zf|y, f \in F\}$$

■  $\Rightarrow^*$  denotes the reflexive and transitive closure of  $\Rightarrow$

# Pushdown Transducers – Input and Output Language

## Input Language

$$L_I(M) = \{x \in I^* : (x, y) \in T(M) \text{ for some } y \in O^*\}$$

## Output Language

$$L_O(M) = \{y \in O^* : (x, y) \in T(M) \text{ for some } x \in I^*\}$$

# Pushdown Transducers – Example

## Example

$$M = (\{s, q, f\}, \{S, A, +, *, a\}, R, s, \{f\})$$

where

$$R = \left\{ \begin{array}{ll} 1 : Ss \rightarrow SAq, & 4 : Aq * \rightarrow * AAq, \\ 2 : Aqa \rightarrow qa, & 5 : +q \rightarrow q+, \\ 3 : Aq + \rightarrow + AAq, & 6 : *q \rightarrow q*, \end{array} \right. \quad 7 : Sq \rightarrow f \}$$

$$\begin{aligned} Ss + *aaa &\mid \Rightarrow SAq + *aaa \mid [1] \Rightarrow S + AAq * aaa \mid [3] \\ &\Rightarrow S + A * AAqaaa \mid [4] \Rightarrow S + A * Aqaa \mid a [2] \Rightarrow S + A * qa \mid aa [2] \\ &\Rightarrow S + Aqa \mid aa * [6] \Rightarrow S + q \mid aa * a [2] \Rightarrow Sq \mid aa * a + [5] \\ &\Rightarrow f \mid aa * a + [7] \end{aligned}$$

$$T(M) = \{(pre, post) : pre = \text{prefix expression}, post = \text{postfix expression}\}$$

# Pushdown Transducers – Determinism

## Deterministic Pushdown Transducer

$M$  is deterministic if each rule  $r \in R$  with  $\text{lhs}(r) = Apa$  satisfies

$$\{r\} = \{r' \in R : Apa = \text{lhs}(r') \text{ or } Ap = \text{lhs}(r')\}$$

## Extended Pushdown Transducer

$M = (Q, \Sigma, R, s, F)$  is extended if  $R$  is a finite set of productions of the form

$$zpa \rightarrow uqv$$

$$z \in P_D^*, p, q \in Q, a \in I \cup \{\varepsilon\}, u \in P_D^*, v \in O^*$$

# Translation Grammars

## Translation Grammar

A translation grammar is a quadruple

$$G = (N, T, P, S)$$

where

$N, S$  are defined as usual,  $S \in N$

$T$  is a terminal alphabet,  $T \cap N = \emptyset$ ,  $T = I \cup O$

$I$  and  $O$  are input and output alphabets

$P$  is a finite set of productions of the form

$$A \rightarrow u_0 B_1 u_1 \dots B_n u_n | v_0 B_1 v_1 \dots B_n v_n$$

$|$  is a special symbol,  $B_i \in N$ ,  $u_j \in I^*$ ,  $v_j \in O^*$ ,

$i = 1, \dots, n$ ,  $j = 0, \dots, n$



# Translation Grammars – Direct Derivation

## Notation

For

$$p = A \rightarrow u_0 B_1 u_1 \dots B_n u_n | v_0 B_1 v_1 \dots B_n v_n,$$

- $\text{lhs}(p) = A$
- $\text{irhs}(p) = u_0 B_1 u_1 \dots B_n u_n$
- $\text{orhs}(p) = v_0 B_1 v_1 \dots B_n v_n$

## Direct Derivation

For  $G = (N, T, P, S)$ ,  $p \in P, x, y, u, v \in (N \cup T)^*$ , then

$$x \text{lhs}(p) y | u \text{lhs}(p) v \Rightarrow x \text{irhs}(p) y | u \text{orhs}(p) v$$

# Translation Grammars – Translation

## Translation Defined by $G$

$$T(G) = \{u|v : S|S \Rightarrow^* u|v, u \in I^*, v \in O^*\}$$

## Input Grammar

$$G_I = (N, I, P_I, S)$$

where  $P_I = \{A \rightarrow x : A \twoheadrightarrow x|y \in P\}$

## Output Grammar

$$G_O = (N, O, P_O, S)$$

where  $P_O = \{A \rightarrow y : A \twoheadrightarrow x|y \in P\}$

# Translation Grammars – Example

## Example

Let the translation grammar  $G$  be defined by the following productions:

$$\begin{aligned}\langle expr \rangle &\rightarrow \langle expr \rangle + \langle term \rangle \mid \langle expr \rangle \langle term \rangle + \\ \langle expr \rangle &\rightarrow \langle term \rangle \mid \langle term \rangle \\ \langle term \rangle &\rightarrow \langle term \rangle * \langle factor \rangle \mid \langle term \rangle \langle factor \rangle * \\ \langle term \rangle &\rightarrow \langle factor \rangle \mid \langle factor \rangle \\ \langle factor \rangle &\rightarrow (\langle expr \rangle) \mid \langle expr \rangle \\ \langle factor \rangle &\rightarrow a \mid a\end{aligned}$$

$$\langle expr \rangle \mid \langle expr \rangle \Rightarrow^* (a + a) * a \mid aa + a *$$

$G$  translates an infix expression with  $+$  and  $*$  to the corresponding postfix expression.

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