

Matrix Grammars

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Regulated Rewriting

Example

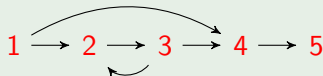
1 : $S \rightarrow AB$

2 : $A \rightarrow aA$

3 : $B \rightarrow bBc$

4 : $A \rightarrow a$

5 : $B \rightarrow bc$



Example of derivation

$$\begin{aligned} S &\Rightarrow AB \text{ [1]} \Rightarrow aAB \text{ [2]} \Rightarrow aAbBc \text{ [3]} \\ &\Rightarrow aaAbBc \text{ [2]} \Rightarrow aaAbbBcc \text{ [3]} \\ &\Rightarrow aaabbBcc \text{ [4]} \Rightarrow aaabbbccc \text{ [5]} \end{aligned}$$

Generated language

$$L(G) = \{a^n b^n c^n : n \geq 1\}$$

Matrix Grammar

Matrix Grammar

A **matrix grammar** is a pair

$$H = (G, M),$$

where

- $G = (N, T, P, S)$ is a context-free grammar
- M is a **finite language over P** ($M \subseteq P^*$)

Notation

- Let $N = \{A_1, \dots, A_m\}$ for some $m \geq 1$
- For some $m_i = p_{i_1} \dots p_{i_j} \dots p_{i_{k_i}} \in M$,

$$p_{i_j} : A_{i_j} \rightarrow x_{i_j}$$

Generated Language

Derivation Step

For $x, y \in (N \cup T)^*$, $m \in M$,

$$x \Rightarrow y [m]$$

in H if there are x_0, \dots, x_n such that $x = x_0$, $x_n = y$, and

1 $x_0 \Rightarrow x_1 [p_1] \Rightarrow x_2 [p_2] \Rightarrow \dots \Rightarrow x_n [p_n]$ in G , and

2 $m = p_1 \dots p_n$

Generated Language

$$L(H) = \{x \in T^* : S \Rightarrow^* x\}$$

Example

$$H = (G, M),$$

where

■ $G = (N, T, P, S)$, where

$$N = \{S, A, B\}$$

$$T = \{a, b, c\}$$

$$P = \{\begin{array}{l} \textcolor{red}{1} : S \rightarrow AB, \\ \textcolor{red}{2} : A \rightarrow aA, \\ \textcolor{red}{3} : B \rightarrow bBc, \\ \textcolor{red}{4} : A \rightarrow a, \\ \textcolor{red}{5} : B \rightarrow bc \end{array}\}$$

■ $M = \{\textcolor{red}{1}, \textcolor{red}{23}, \textcolor{red}{45}\}$

Example II

Example

1 In G ,

$$\begin{aligned} aAbBc &\Rightarrow aaAbBc \quad [2] \\ &\Rightarrow aaAbbBcc \quad [3] \end{aligned}$$

As $23 \in M$,

$$aAbBc \Rightarrow aaAbbBcc \quad [23]$$

in H

2 In G ,

$$\begin{aligned} aAbBc &\Rightarrow aaAbBc \quad [2] \\ &\Rightarrow aaAbbcc \quad [5] \end{aligned}$$

As $25 \notin M$

$$aAbBc \not\Rightarrow aaAbbcc \quad [25]$$

in H

Example III

Example

$S \Rightarrow AB$ [1]

$\Rightarrow aAbBc$ [23]

$\Rightarrow aaAbbbBcc$ [23]

$\Rightarrow aaabbbbccc$ [45]

in H

$S \Rightarrow AB$ [1]

$\Rightarrow aAB$ [2]

$\Rightarrow aAbBc$ [3]

$\Rightarrow aaAbBc$ [2]

$\Rightarrow aaAbbbBcc$ [3]

$\Rightarrow aaabbbBcc$ [4]

$\Rightarrow aaabbbbccc$ [5]

in G

Example IV

Example

By using $23 \in M$ n -times, $n \geq 0$

$$\begin{aligned} S &\Rightarrow AB && [1] \\ &\Rightarrow aAbBc && [23] \\ &\Rightarrow aaAbbbBcc && [23] \\ &\vdots \\ &\Rightarrow a^n Ab^n Bc^n && [23] \\ &\Rightarrow a^{n+1} b^{n+1} c^{n+1} && [45] \end{aligned}$$

Generated language

$$L(H) = \{a^m b^m c^m : m \geq 1\}$$

Example V

Example

Claim A

If $AB \Rightarrow^n x$, where $n \geq 1$, then $x \in \{a^n b^n c^n, a^n Ab^n Bc^n\}$.

Proof by Induction on $n \geq 1$

■ Basis: $n = 1$.

$$AB \Rightarrow aAbBc$$

$$AB \Rightarrow abc$$

■ Induction Hypothesis:

Assume Claim A holds for all $n = 1, \dots, k$, where k is a positive integer.

Example

Proof by Induction on $n \geq 1$

■ Induction Step:

$$S \Rightarrow^{k+1} x$$

can be rewritten as

$$S \Rightarrow^k y \Rightarrow x$$

By Induction Hypothesis, $y \in \{a^k Ab^k Bc^k, a^k b^k c^k\}$. As $y \Rightarrow x$,
 $y = a^k Ab^k Bc^k$,

$$y \Rightarrow x [23] \text{ and } x = a^{k+1} Ab^{k+1} Bc^{k+1}$$

$$y \Rightarrow x [45] \text{ and } x = a^{k+1} b^{k+1} c^{k+1}$$

so $x \in \{a^{k+1} Ab^{k+1} Bc^{k+1}, a^{k+1} b^{k+1} c^{k+1}\}$



Example VII

Example

Claim B

$$L(H) \subseteq \{a^k b^k c^k : k \geq 1\}.$$

Proof

$L(H) = \{x \in T^* : S \Rightarrow^* x\}$. Every $S \Rightarrow^* x$ with $x \in T^*$ has the form

$$S \Rightarrow AB \Rightarrow^* x$$

From Claim A,

$$L(H) \subseteq \{a^k b^k c^k : k \geq 1\}$$



Example VIII

Example

Claim C

For every $a^n b^n c^n$, where $n \geq 1$, $S \Rightarrow^* a^n b^n c^n$.

Proof by Induction on $n \geq 1$

- Basis: $n = 1$, $abc = x$.

$$\begin{aligned} S &\Rightarrow AB [1] \\ &\Rightarrow abc [45] \end{aligned}$$

- Induction Hypothesis:

Assume Claim C holds for all $n = 1, \dots, k$, where k is a positive integer.

Example IX

Example

Proof by Induction on $n \geq 1$

■ Induction Step:

$$x = a^{k+1}b^{k+1}c^{k+1}$$

Consider $a^k b^k c^k$. By Induction Hypothesis, $S \Rightarrow^* a^k b^k c^k$. Express this derivation as

$$\begin{aligned} S &\Rightarrow^* a^{k-1} A b^{k-1} B c^{k-1} \\ &\Rightarrow a^k b^k c^k \end{aligned} \quad [45]$$

Then,

$$\begin{aligned} S &\Rightarrow^* a^{k-1} A b^{k-1} B c^{k-1} \\ &\Rightarrow a^k A b^k B c^k \quad [23] \\ &\Rightarrow a^{k+1} b^{k+1} c^{k+1} = x \end{aligned}$$

Example X

Example

From Claim B,

$$L(H) \subseteq \{a^k b^k c^k : k \geq 1\}$$

From Claim C,

$$\{a^k b^k c^k : k \geq 1\} \subseteq L(H)$$

Thus,

$$L(H) = \{a^k b^k c^k : k \geq 1\}$$

Matrix Grammar with Appearance Checking

A **matrix grammar with appearance checking** is a pair

$$H = (G, M)$$

where

- $G = (N, T, P, S)$ is a context-free grammar
- M is a finite language over $P \times \{-, +\}$

Derivation Step

Derivation Step

For $x, y \in (N \cup T)^*$, $m = (p_1, q_1) \dots (p_n, q_n) \in M$, $p_i \in P$, $q_i \in \{-, +\}$, $i = 1, \dots, n$,

$$x \Rightarrow y [m]$$

in H if there are x_0, \dots, x_n such that $x = x_0$, $y = x_n$, and for $i = 1, \dots, n$

- either $x_{i-1} \Rightarrow x_i [p_i]$ in G
- or $q_i = +$, $x_{i-1} = x_i$, and p_i is not applicable to x_{i-1}

Example I

Example

1 : $S \rightarrow a$

2 : $S \rightarrow aa$

3 : $S \rightarrow AB$

4 : $A \rightarrow A, B \rightarrow CC$

5 : $A \rightarrow A'C, \underline{B \rightarrow X}$

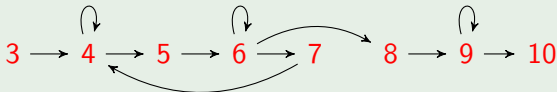
6 : $A' \rightarrow A', C \rightarrow B$

7 : $A' \rightarrow A, \underline{C \rightarrow X}$

8 : $A' \rightarrow A'', \underline{C \rightarrow X}$

9 : $A'' \rightarrow A'', B \rightarrow a$

10 : $A'' \rightarrow a$



Notes:

Underlined productions are in appearance checking mode (+)

X is a “block” symbol

Example II

Example

Derivation example

$$\begin{aligned} \dots AB\textcolor{red}{B}B &\Rightarrow A\textcolor{red}{B}CCB [4] \Rightarrow ACCCC\textcolor{red}{B} [4] \Rightarrow \textcolor{red}{A}CCCCC [4] \\ &\Rightarrow A'C^7 [5] \Rightarrow A'CCBCCCC [6] \Rightarrow^6 A'B^7 [\textcolor{red}{6} \dots \textcolor{red}{6}] \Rightarrow AB^7 [\textcolor{red}{7} \dots \textcolor{red}{7}] \\ &\Rightarrow \dots \\ &\Rightarrow A''BBBBBBB [8] \Rightarrow^7 A''\textcolor{red}{a}aaaaa [9] \Rightarrow \textcolor{red}{a}aaaaaaa [\textcolor{red}{10}] \end{aligned}$$

The generated language is $L(G) = \{a^{2^i} : i \geq 0\}$

■ for $i \geq 2$, the derivation can be expressed as

$$\begin{aligned} S &\Rightarrow_3 AB \Rightarrow_{4,5}^* A'C^3 \Rightarrow_{6,7}^* AB^3 \Rightarrow^* AC^7 \\ &\vdots \\ &\Rightarrow A'B^{2^i-1} \Rightarrow_8 A''B^{2^i-1} \Rightarrow_{9,10}^* a^{2^i} \end{aligned}$$

■ for $i = 1, 2$, $S \Rightarrow a [1]$ and $S \Rightarrow aa [2]$



S. Abraham.

Some questions of phrase-structure grammars.

Computational Linguistics, 4:61–70, 1965.



J. Dassow and Gh. Păun.

Regulated Rewriting in Formal Language Theory.

Akademie-Verlag, Berlin, 1989.